

Functional Variable Method for conformable fractional modified KdV-ZK equation and Maccari system

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Abstract

Modeling the motion and propagation characteristics of waves have importance in coastal, ocean and maritime engineering. Especially, waves are the major source of environmental actions on beaches or on man-made fixed or floating structures in most geographical areas. So Maccari system has great application in mentioned areas. The modified KdV is ion acoustic perturbations evolution model in a plasma with two negative ion components which have different temperatures. As for the KdV equation, the modified ZK (mZK) equation arises naturally as weakly two-dimensional variations of the mKdV equation. In this paper authors used functional variable method(FVM) for the first time to obtain exact travelling wave and soliton solutions of conformable fractional modified KdV-Zakharov-Kuznetsov(mKdv-ZK) equation and Maccari system. As a consequence, new solutions are obtained and it is seen that FVM is an valuable and efficient tool for solving nonlinear equations and systems where the derivatives defined by means of conformable fractional derivative.

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1 Introduction

Fractional Calculus can be called the calculus of this century. Since the letter from the L'hospital in 1695, this topic has gained great attention. There are many application areas for the fractional differential equations (FDEs) like, applied mathematics, engineering, physics, biology, chemistry, economics [21, 5, 30, 31, 9] and etc. FDEs have a role by the modeling of many physical phenomena and processes. Because of the variety of the application area, several analytic and numeric methods have been proposed for the solution of linear, nonlinear, partial or systems of FDEs. Laplace transform method [26], Adomian Decomposition Method [29], (G'/G) expansion method [28], fractional sub equation method [27], Homotopy Analysis Method [10], exp function method [18], first integral method [22] are the examples of the commonly used methods in the literature.

The Korteweg-de Vries (KdV) equation has an important role in the applied sciences. This equation is a model for one dimensional long wavelet surface waves propagating in weakly nonlinear dispersive media. Also this equation has different variations. One of the well-known variation is Zakharov-Kuznetsov (ZK) equation which is about the ionic-acoustic waves in magnetized plasmas. It is a study of coastal waves in ocean and arises as a model for one dimensional long wavelength. If higher terms in KdV equation has nonlinearity, it is called modified KdV (mKdV) equation which describes the ion acoustic waves [19]. Maccari system also has an application in science. This system is about the motion of the isolated waves and using in many fields such as quantum field

theory, hydrodynamics, plasma physics, non-linear optics and etc. [6]. There are many analytic and approximate solution methods proposed to solve fractional mKdV-ZK equation and Maccari system [2, 13, 15, 17, 8].

In the fractional calculus there are many fractional derivative definitions can be found. The most popular ones are Riemann-Liouville and Liouville-Caputo fractional derivative [23]. Riemann-Liouville gave the fractional definition by the help of the generalized Cauchy formula as;

$${}^{RL}D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx, \quad \alpha \in [n-1, n),$$

where n is a positive ineteger and α is the order of the derivative of the function. Caputo developed the Riemann-Liouville definition and gave the fractional derivative definition as;

$${}^CD_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx, \quad \alpha \in [n-1, n).$$

It is seen that Liouville-Caputo's definition has some advantages in applications like initial value problems. With the development of the fractional calculus, researchers found an insufficiency at these definitions. For example;

1. The RiemannLiouville derivative does not satisfy $D_a^\alpha 1 = 0$ (Liouville-Caputo derivative satisfies), if α is not a natural number.
2. All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions.

$$D_a^\alpha(fg) = gD_a^\alpha(f) + fD_a^\alpha(g)$$

3. All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions.

$$D_a^\alpha \left(\frac{f}{g} \right) = \frac{fD_a^\alpha(f) - gD_a^\alpha(g)}{g^2}$$

4. All fractional derivatives do not satisfy the chain rule.

$$D_a^\alpha(fog)(t) = f^\alpha(g(t))g^\alpha(t)$$

5. All fractional derivatives do not satisfy $D^\alpha D^\beta = D^{\alpha+\beta}$ in general.
6. The Caputo definition assumes that the function f is differentiable.

Therefore, researchers need new fractional derivative definitions for the solutions of fractional differential equations. Modified Riemann-Liouville and Liouville-Caputo derivative, Weyl derivative, Marchaud derivative, Davidson-Essex derivative, Coimbra derivative, Jumarie derivative, Erdelyi-Kober derivative and etc. [25]. Generally all the definitions are using integral operator for the fractional derivative definition. Finally Khalil et al. [20] gave very simple definition for fractional derivative with limit operator.

Definition. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a function. The α^{th} order "conformable fractional derivative" of f is defined by,

$$T_\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0, \alpha \in (0, 1)$. If f is α -differentiable in some $(0, a), a > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ and the "conformable fractional integral" of a function f starting from $a \geq 0$ is defined as:

$$I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$. The following theorem points out some properties which are satisfied by the conformable fractional derivative [20].

Theorem 1. Let $\alpha \in (0, 1]$ and suppose f, g are α -differentiable at point $t > 0$. Then

1. $T_\alpha(cf + dg) = cT_\alpha(f) + dT_\alpha(g)$ for all $a, b \in \mathbb{R}$.
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
3. $T_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$.
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
5. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
6. If f , in addition to f differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}$.

There are great number of studies have done till the definiton of the Conformable fractional definition. For instance; T. Abdeljawad [1] has presented fractional versions of the chain rule, exponential functions, Gronwalls inequality, integration by parts, Taylor power series expansions and Laplace transform. Conformable time-scale fractional calculus has been expressed by N. Benkhettou et al. [4], M.A. Hammad and R. Khalil [16] introduced the solution for the conformable fractional heat equation and W.S. Chung [7] discuss fractional Newtonian mechanics by using the conformable fractional derivative and integral. In addition to these studies A. Gökdoğan et al.[14] gave the existence and uniqueness theorems for sequential linear conformable fractional differential equations, M. Eslami and H. Rezazadeh [11] used first integral method to give the analytical solutions of conformable fractional WuZhang system. M. Eslami [12] obtain the solitary wave solutions of space-time fractional $(2 + 1)$ - dimensional dispersive long wave equations by using the G'/G -expansion method, A. Neirameh [24] applied sub-equation method to obtain the new exact solitary wave solutions of the fractional perturbed nonlinear Schrodinger equation with power law nonlinearity. Finally A. Atangana et al. [3] presented the new properties of conformable fractional derivative. Hence, it is clearly deduced that further studies and explanations can be made on this new subject area.

2 Description of Functional Variable Method

Consider the following nonlinear conformable fractional derivative

$$F\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \dots\right) = 0. \quad (2.1)$$

Firstly we determine the new wave variable as

$$u(x, y, t) = U(\xi), \xi = mx + ny + r \frac{t^\alpha}{\alpha} \quad (2.2)$$

where m, n, r are arbitrary constants to be determined later. Applying wave transformation to nonlinear conformable fractional equation (2.1) by using chain rule [1],

$$\frac{\partial^\alpha(\cdot)}{\partial t^\alpha} = r \frac{d(\cdot)}{d\xi}, \quad \frac{\partial(\cdot)}{\partial x} = m \frac{d(\cdot)}{d\xi}, \quad \frac{\partial(\cdot)}{\partial y} = n \frac{d(\cdot)}{d\xi}, \dots \quad (2.3)$$

Eq.(2.1) turns into ordinary differential equation as follows.

$$Q(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, U_{\xi\xi\xi\xi}, \dots) = 0. \quad (2.4)$$

Let determine a functional variable such as

$$U_\xi = F(U) \quad (2.5)$$

for making a transformation to unknown function U . Some derivatives of unknown function are

$$\begin{aligned} U_{\xi\xi} &= \frac{1}{2}(F^2)', \\ U_{\xi\xi\xi} &= \frac{1}{2}(F^2)'' \sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \frac{1}{2}[(F^2)''' F^2 + (F^2)'' (F^2)'], \\ &\vdots \end{aligned} \quad (2.6)$$

where ''' prime denotes the derivative $\frac{d}{dU}$. Using the expressions (2.6) into ordinary differential equation (2.4), the ODE can be reduced in terms of U , F and the derivatives of F upon U as

$$R(U, F, F', F'', F''', F^{(4)}, \dots) = 0. \quad (2.7)$$

After integration, Eq.(2.7) gives the expression of F . Using this expression into (2.5), it can provides us the appropriate solutions of the original problem.

In order the show how the method works, we give some special nonlinear conformable fractional derivatives in Section 3 and Section 4.

3 New Exact Solution of The (3 + 1)-Dimensional Fractional Modified KdV-Zakharov-Kuznetsov Equation

Let us consider (3 + 1) dimensional fractional mKdVZK equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \lambda u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial x \partial z^2} = 0 \quad (3.1)$$

in which $\frac{\partial^\alpha}{\partial t^\alpha}$ implies the conformable fractional derivative. By taking the new wave variable $\xi = mx + ny + rz + p \frac{t^\alpha}{\alpha}$ using the transformation given in Eq.(2.2) and applying the chain rule(2.3) [1], we can rewrite the Eq.(3.1) as follows

$$pU_\xi + m\lambda U^2 U_\xi + m^3 U_{\xi\xi\xi} + mn^2 U_{\xi\xi\xi} + mr^2 U_{\xi\xi\xi} = 0 \quad (3.2)$$

When Eq. (3.3) is integrated once with respect to ξ ; the constants of integration are set to zero, Eq. (3.3) can be rewritten in a practical form as follows

$$pU + m\lambda \frac{U^3}{3} + (m^3 + mn^2 + mr^2)U_{\xi\xi} = 0 \quad (3.3)$$

By using $U_{\xi\xi} = \frac{1}{2}(F^2)'$ in Eq. (2.6), $F(U)$ can be obtained as follows

$$F(U) = \sqrt{-\frac{p}{m^3 + mn^2 + mr^2}} U \sqrt{1 - \frac{-m\lambda U^2}{6p}} \quad (3.4)$$

Then using Eqs.(2.5), (2.2) and (3.4), soliton and traveling wave solutions of the Eq.(3.1) can be obtained respectively as

$$\begin{aligned} u_1(x, y, z, t) &= \left[\left(\frac{-6p}{\lambda m} \right) \operatorname{sech}^2 \left(\left(\frac{\sqrt{-p} (mx + ny + rz + p \frac{t^\alpha}{\alpha})}{\sqrt{m^3 + mn^2 + mr^2}} \right) \right) \right]^{\frac{1}{2}}, \\ u_2(x, y, z, t) &= \left[\left(\frac{6p}{\lambda m} \right) \operatorname{csch}^2 \left(\left(\frac{\sqrt{-p} (mx + ny + rz + p \frac{t^\alpha}{\alpha})}{\sqrt{m^3 + mn^2 + mr^2}} \right) \right) \right]^{\frac{1}{2}}, \\ u_3(x, y, z, t) &= \left[\left(\frac{-6p}{\lambda m} \right) \sec^2 \left(\left(\frac{\sqrt{p} (mx + ny + rz + p \frac{t^\alpha}{\alpha})}{\sqrt{m^3 + mn^2 + mr^2}} \right) \right) \right]^{\frac{1}{2}}, \\ u_4(x, y, z, t) &= \left[\left(\frac{-6p}{\lambda m} \right) \csc^2 \left(\left(\frac{\sqrt{p} (mx + ny + rz + p \frac{t^\alpha}{\alpha})}{\sqrt{m^3 + mn^2 + mr^2}} \right) \right) \right]^{\frac{1}{2}}. \end{aligned}$$

4 New Exact Solution of 2-Dimensional Fractional Maccari System

Consider the Maccari nonlinear evolution system

$$\begin{aligned} i \frac{\partial^\alpha q}{\partial t^\alpha} + \frac{\partial^2 q}{\partial x^2} + qH &= 0, \\ \frac{\partial^\alpha H}{\partial t^\alpha} + \frac{\partial H}{\partial y} + \frac{\partial |q|^2}{\partial x} &= 0 \end{aligned} \quad (4.1)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the conformable fractional derivative. Looking for the solution in the form

$$q(x, y, t) = U(\xi)e^{i(\lambda x + \beta y + \gamma \frac{t^\alpha}{\alpha})} \quad (4.2)$$

$$H(x, y, t) = W(\xi) \quad (4.3)$$

with the wave variable $\xi = m(x + ny - 2\lambda \frac{t^\alpha}{\alpha})$ and where $\lambda, \beta, \gamma, m, n$ are arbitrary non-zero constants. Inserting Eqs. (4.2) and (4.3) into Eq.(4.1) yields

$$m^2 U_{\xi\xi} - (\gamma + \lambda^2)U + WU = 0, \quad (4.4)$$

$$(n - 2\lambda)W_\xi + (U^2)_\xi = 0. \quad (4.5)$$

Integrating Eq.(4.5) with respect to the variable ξ and taking the integration constant to zero yields

$$W = \frac{U^2}{2\lambda - n}. \quad (4.6)$$

Substituting Eq.(4.6) into (4.4), we have

$$U_{\xi\xi} - \frac{(\gamma + \lambda^2)}{m^2}U + \frac{U^3}{m^2(2\lambda - n)} = 0. \quad (4.7)$$

Using Eqs. (2.6), $F(U)$ can be deduced easily as follows

$$F(U) = \sqrt{\frac{(\gamma + \lambda^2)}{m^2}}U \sqrt{1 - \frac{U^2}{(2\lambda - n)(\gamma + \lambda^2)}}. \quad (4.8)$$

Consequently regarding (4.8), (2.5), (4.6),(4.2) and (4.3), the bell-shaped and singular soliton solutions and traveling wave solutions can be obtained successively as

$$q_1(x, y, t) = \left[(\gamma + \lambda^2)(4\lambda - 2n) \operatorname{sech}^2 \left(\left(\frac{\sqrt{(\gamma + \lambda^2)}}{m} \right) \left(m \left(x + ny - 2\lambda \frac{t^\alpha}{\alpha} \right) \right) \right) \right]^{\frac{1}{2}} e^{i(\lambda x + \beta y + \gamma \frac{t^\alpha}{\alpha})},$$

$$H_1(x, y, t) = \frac{(\gamma + \lambda^2)(4\lambda - 2n) \operatorname{sech}^2 \left(\left(\frac{\sqrt{(\gamma + \lambda^2)}}{m} \right) \left(m \left(x + ny - 2\lambda \frac{t^\alpha}{\alpha} \right) \right) \right)}{2\lambda - n},$$

$$q_2(x, y, t) = \left[-(\gamma + \lambda^2)(4\lambda - 2n) \operatorname{csch}^2 \left(\left(\frac{\sqrt{(\gamma + \lambda^2)}}{m} \right) \left(m \left(x + ny - 2\lambda \frac{t^\alpha}{\alpha} \right) \right) \right) \right]^{\frac{1}{2}} e^{i(\lambda x + \beta y + \gamma \frac{t^\alpha}{\alpha})},$$

$$H_2(x, y, t) = \frac{-(\gamma + \lambda^2)(4\lambda - 2n) \operatorname{csch}^2 \left(\left(\frac{\sqrt{(\gamma + \lambda^2)}}{m} \right) \left(m \left(x + ny - 2\lambda \frac{t^\alpha}{\alpha} \right) \right) \right)}{2\lambda - n},$$

$$q_3(x, y, t) = \left[(\gamma + \lambda^2)(4\lambda - 2n) \sec^2 \left(\left(\frac{\sqrt{-(\gamma + \lambda^2)}}{m} \right) \left(m \left(x + ny - 2\lambda \frac{t^\alpha}{\alpha} \right) \right) \right) \right]^{\frac{1}{2}} e^{i(\lambda x + \beta y + \gamma \frac{t^\alpha}{\alpha})},$$

$$H_3(x, y, t) = \frac{(\gamma + \lambda^2)(4\lambda - 2n) \sec^2 \left(\left(\frac{\sqrt{-(\gamma + \lambda^2)}}{m} \right) \left(m \left(x + ny - 2\lambda \frac{t^\alpha}{\alpha} \right) \right) \right)}{2\lambda - n},$$

$$q_4(x, y, t) = \left[(\gamma + \lambda^2)(4\lambda - 2n) \csc^2 \left(\left(\frac{\sqrt{-(\gamma + \lambda^2)}}{m} \right) \left(m \left(x + ny - 2\lambda \frac{t^\alpha}{\alpha} \right) \right) \right) \right]^{\frac{1}{2}} e^{i(\lambda x + \beta y + \gamma \frac{t^\alpha}{\alpha})},$$

$$H_4(x, y, t) = \frac{(\gamma + \lambda^2)(4\lambda - 2n) \csc^2 \left(\left(\frac{\sqrt{-(\gamma + \lambda^2)}}{m} \right) \left(m \left(x + ny - 2\lambda \frac{t^\alpha}{\alpha} \right) \right) \right)}{2\lambda - n}.$$

5 Conclusions

In this paper (3+1) dimensional fractional modified KdV-ZK equation and 2 dimensional fractional Maccari system is solved with Functional Variable Method. New fractional derivative definition called Conformable fractional derivative is used in the solution procedure. This new derivative definition has simple definition and gives researchers simple algorithms in the solution procedure. mKdV-ZK equation and Maccari system are models for wave and soliton solutions and have large application area in applied sciences, especially in coastal, ocean and maritime engineering. Obtained results showed that Functional Variable Method is applicable for the solution of these type equations and powerful when it is used with conformable fractional derivative definition.

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