Operator on Hilbert space and its application to certain multivalent functions with fixed point associated with hypergeometric function

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Abstract

By applying hypergeometric operator on Hilbert space, the author introduces a new class of meromorphic multivalent functions with an arbitrary fixed point omega. Properties such as coefficient inequalities, distortion bounds and extreme points were derived. Furthermore, the effect of this operator on functions in this class was also investigated.

1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized with $f(0) = f'(0) - 1 = 0$. Several reseachers have studied various aspects of this function and they came out with many interesting results as it contained in many existing literatures.

In the recent past, Kanas and Ronning [11], precisely in 1999 introduced and studied a new subclass of A denoted by $A(\omega)$ with the function of the form

$$f(z) = (z - \omega) + \sum_{k=2}^{\infty} a_k (z - \omega)^k \tag{2}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized with $f(\omega) = f'(\omega) - 1 = 0$ where ω is an arbitrary fixed point in U. Authors like Acu and Owa [1], Oladipo et al [13], Oladipo [14,15], Aouf et al [3] have studied various aspect of this class of functions and they obtained many interesting results.The works of previous authors mentioned above serve as our motivation to introduce and study a new class of multivalent function in this direction. We denote by $S(\omega)$ all functions in $A(\omega)$ that are univalent, where $A(\omega) \subset A$ and $S(\omega) \subset S$.

Let ω be an arbitrary fixed point in U and $\Phi(\omega)$ denotes the class of functions $f(z)$ of the form

$$f(z) = \frac{B}{(z - \omega)^p} + \sum_{k=p}^{\infty} a_k (z - \omega)^k, (p \in N = 1, 2, ..) \tag{3}$$

where B is the residue of $f(z)$ in $(z - \omega)$, $0 < B \leq 1$.
Also, let $T_\omega(p)$ denotes the subclass of $\Phi_\omega(p)$ consisting of functions of the form

$$f(z) = \frac{B}{(z-\omega)^p} - \sum_{k=p}^{\infty} a_k(z-\omega)^k, (z \in U, p \in N = 1, 2, ..) \tag{4}$$

For the function $f(z) \in T_\omega(p)$, the author consider the operator I_p^n as follows

$$I_p^0 f(z) = f(z)$$

$$I_p^1 f(z) = (z - \omega)f'(z) + \frac{2Bp}{(z-\omega)^p} \tag{5}$$

$$I_p^n f(z) = (z - \omega)(I_p^{n-1}f(z))' + \frac{2Bp^n}{(z-\omega)^p}$$

and thus

$$I_p^n f(z) = \frac{Bp^n}{(z-\omega)^p} + \sum_{k=p}^{\infty} k^n a_k(z-\omega)^k \tag{6}$$

Now, if $f(z)$ is of the form (4), we define the basic hypergeometric function for complex parameters $a_1, ..., a_r$ and $b_1, ..., b_s$ by

$$_rF_s(a_1, ..., a_r; b_1, ..., b_s; q, z) = \sum_{k=o}^{\infty} \frac{(a_1, q)_k ... (a_r, q)_k}{(q, q)_k (b_1, q)_k ... (b_s, q)_k}(z-\omega)^k \tag{7}$$

$(r = s + 1; r, s \in N_o = 0, 1, 2..; z \in U)$, where N denotes the set of positive integers and $(a, q)_k$ is the q-shifted factorial defined by

$$(a, q)_k = \left\{ \begin{array}{l} 1, k = 0. \\ (1 - a)(1 - aq)(1 - aq^2)...(1 - aq^{k-1}), k \in N. \end{array} \right.$$

where $_rF_s(a_1, ..., a_r; b_1, ..., b_s, z)$ is the well known generalized hypergeometric function. For further background information on basic hypergeometric function, see [2], [5], [8].
Corresponding to the function $_rF_s(a_1, ..., a_r; b_1, ..., b_s, q, z)$, consider

$$_r\Omega_s(a_1, ..., a_r; b_1, ..., b_s, q, z, \omega) = \frac{B}{(z-\omega)^p} {}_rF_s(a_1, ..., a_r; b_1, ..., b_s, q, z)$$

$$= \frac{B}{(z-\omega)^p} - \sum_{k=p}^{\infty} \frac{(a_1, q)_k ... (a_r, q)_k}{(q, q)_k ... (b_1, q)_k ... (b_s, q)_k}(z-\omega)^k \tag{8}$$

where $p \in N = 1, 2, ..$ and $0 < B \leq 1$

The author wish to define a linear operator $T_s^r(\omega)(a_1, ..., a_r; b_1, ..., b_s, q, \omega)$ $f : A(\omega) \to A(\omega)$ by

$$\rho(z) = T_s^r(a_1, ...a_r; b_1, ...b_s; q, \omega)f =_r \Omega_s(a_1, ..., a_r; b_1, ..., b_s, q, z, \omega) * f(z)$$

$$= \frac{Bp^n}{(z-\omega)^p} - \sum_{k=p}^{\infty} \Gamma(a, q, k)k^n a_k(z - \omega)^k \tag{9}$$

where

$$\Gamma(a, q, k) = \frac{(a_1, q)_k...(a_r, q)_k}{(q, q)_k...(b_1, q)...(b_s, q)_k}$$

$n = 0, 1, 2.., p \in N = 1, 2, ..$, and * denotes the Hadamard product (or convolution) of $_r\Omega_s$ and f.

Definition A. The function $\rho(z) \in T_\omega(p)$ is said to be a member of the class $T_\omega^n(\alpha, \beta, \gamma, p)$ if it satisfies

$$\left| \frac{(z-\omega)^3(\rho(z))'' + (z-\omega)^2(\rho(z))' - Bp^n}{2(z-\omega)(\rho(z)) - \beta(1+\gamma)Bp^n} \right| < \alpha \tag{10}$$

where $\alpha, \beta, \gamma \in [0, 1)$ and ω is an arbitrary fixed point in U. Let H be a Hilbert space on C and μ be a linear operator on H. Also, let $f(\mu)$ be the operator on H defined by modified Riesz-Dunford integral [4]

$$2\prod if(\mu) = \int_c f(z) \left((z-\omega)I - \mu\right)^{-1} dz \tag{11}$$

where c is a positively oriented simple close rectifiable contour lying in $U = \{z : |z| < 1\}$ and which also contain the spectrum of μ in its interior domain and I is the identity operator on H. see [6], [7].

Definition B. A function $\rho(z)$ given by (9) is in the class $T_\omega^n(\alpha, \beta, \gamma, p, \mu)$ if for all operator μ with $\|\mu\| < 1$ and $\mu \neq 0$, it satisfies the inequality

$$\left\| \mu^3[I_p f(\mu)]'' + \mu^2[I_p f(\mu)]' - Bp^{n+2} \right\| \leq \alpha \left\| -2\mu I_p f(\mu) - \beta(1+\gamma)Bp^n \right\| \tag{12}$$

where $\alpha, \beta, \gamma \in [0, 1)$ Recently, Ghanim and Darus [9], Joshi [10], Xiaopei [16] and Najafzadeh et al [12] considered the operator on Hilbert space and they obtained useful and interesting results.

2 Main result

Here, the author obtain coefficient inequalities and distortion property for a function $\rho(z) \in T_\omega^n(\alpha, \beta, \gamma, p, \mu)$

Theorem 1. A function $\rho(z)$ given by (9) is in the class $T_\omega^n(\alpha, \beta, \gamma, p, \mu)$ for all $\mu = 0$ if and only if

$$\sum_{k=p}^{\infty} \frac{\Gamma(a, q, k)(k^{n+2} + 2\alpha)}{\alpha Bp^n[2 - \beta(1+\gamma)]} a_k \leq 1 \tag{13}$$

The result is sharp for the function $\rho(z)$ given by

$$M(z) = \frac{B}{(z-\omega)^p} - \frac{\alpha B p^n [2 - \beta(1+\gamma)]}{\Gamma(a,q,k)(k^{n+2}+2\alpha)}(z-\omega)^k \tag{14}$$

for $k \geq 1$ and $p \in N = 1, 2, 3, ..$

Proof. Supposing that (13) holds true, we have

$$\left\| \mu^3 f''(\mu) + \mu^2 f'(\mu) - Bp^{n+2} \right\| - \alpha \left\| 2\mu f(\mu) - \beta(1+\gamma)Bp^n \right\| \tag{15}$$

$$= \left\| -\sum_{k=p}^{\infty} \Gamma(a,q,k)k^2 a_k \mu^{n+p} \right\| - \alpha \left\| Bp^n[2-\beta(1+\gamma)] - \sum_{k=p}^{\infty} 2\Gamma(a,q,k)k^n a_k \mu^{n+p} \right\|$$

$$\leq \sum_{k=p}^{\infty} \Gamma(a,q,k)(k^2 + 2\alpha) - \alpha Bp^n[2-\beta(1+\gamma)] \leq 0 \tag{16}$$

Hence, ρ is in the class $T_\omega^n(\alpha, \beta, \gamma, p, \mu)$

Conversely, supposing that

$$\left\| \mu^3 f''(\mu) + \mu^2 f'(\mu) - Bp^{n+2} \right\| < \alpha \left\| 2\mu f(\mu) - \beta(1+\gamma)Bp^n \right\| \tag{17}$$

such that

$$\left\| -\sum_{k=p}^{\infty} \Gamma(a,q,k)k^{n+2} a_k \mu^{n+p} \right\| < \alpha \left\| Bp^n[2-\beta(1+\gamma)] - \sum_{k=p}^{\infty} 2\Gamma(a,q,k)k^n a_k \mu^{n+p} \right\| \tag{18}$$

On setting $\mu = \mu I (0 < \mu < 1)$ in (18), we obtain

$$\frac{\sum_{k=p}^{\infty} \Gamma(a,q,k)k^{n+2} a_k \mu^{n+p}}{Bp^n[2-\beta(1+\gamma)] - \sum_{k=p}^{\infty} 2\Gamma(a,q,k)k^n a_k \mu^{n+p}} < \alpha \tag{19}$$

By letting $\mu \to 1$ then (19) becomes

$$\sum_{k=p}^{\infty} \Gamma(a,q,k)k^{n+2} a_k < \alpha Bp^n[2-\beta(1+\gamma)] - \sum_{k=p}^{\infty} 2\alpha\Gamma(a,q,k)k^n a_k \tag{20}$$

or simply written as

$$\sum_{k=p}^{\infty} \Gamma(a,q,k)(k^{n+2} + 2\alpha)a_k \leq Bp^n[2-\beta(1+\gamma)] \tag{21}$$

and this complete the proof.

Corollary. If $\rho(z)$ given by (9) be in the class $T_\omega^n(\alpha, \beta, \gamma, p, \mu)$, then

$$a_k \leq \frac{Bp^n[2-\beta(1+\gamma)]}{\Gamma(a,q,k)(k^{n+2}+2\alpha)}, k, p \in N \tag{22}$$

Theorem 2. If $\rho(z)$ of the form (9) is in the class $T_\omega^n(\alpha, \beta, \gamma, p, \mu)$, $\|\mu\| \leq 1$ and $\mu \neq 0$, then

$$\left\| \frac{B}{\mu^p} \right\| - \frac{\alpha B p^n [2 - \beta(1 + \gamma)]}{\Gamma(a, q, k)(k^{n+2} + 2\alpha)} \|\mu\|^k \leq \|f(\mu)\|$$

$$\leq \left\| \frac{B}{\mu^p} \right\| + \frac{\alpha B p^n [2 - \beta(1 + \gamma)]}{\Gamma(a, q, k)(k^{n+2} + 2\alpha)} \|\mu\|^k$$

The result is sharp for the function $M(z)$ given by (14)

Proof. By theorem 1, we have

$$\sum_{k=p}^{\infty} a_k \leq \frac{\alpha B p^n [2 - \beta(1 + \gamma)]}{\Gamma(a, q, k)(k^{n+2} + 2\alpha)} \tag{23}$$

and

$$\|f(\mu)\| \geq \left\| \frac{B}{\mu^P} \right\| - \|\mu\|^k \sum_{k=p}^{\infty} a_k$$

hence, we obtain

$$\geq \left\| \frac{B}{\mu^p} \right\| - \frac{\alpha B p^n [2 - \beta(1 + \gamma)]}{\Gamma(a, q, k)(k^{n+2} + 2\alpha)} \|\mu^k\| \tag{24}$$

and

$$\|f(\mu)\| \leq \left\| \frac{B}{\mu^P} \right\| + \|\mu\|^k \sum_{k=p}^{\infty} a_k$$

$$\leq \left\| \frac{B}{\mu^p} \right\| + \frac{\alpha B p^n [2 - \beta(1 + \gamma)]}{\Gamma(a, q, k)(k^{n+2} + 2\alpha)} \|\mu^k\| \tag{25}$$

and the proof is complete

Next, the author discuss about extreme point of $T_\omega^n(\alpha, \beta, \gamma, p, \mu)$ and effect of operator on functions in this class

Theorem 3. Let $\rho_o = \frac{B}{(z-\omega)^p}$ and $\rho_n(z) = \frac{B}{(z-\omega)^p} - \frac{\alpha B p^n [2-\beta(1+\gamma)]}{\Gamma(a,q,k)(k^{n+2}+2\alpha)}(z-\omega)^k$, then $\rho(z) \in T_\omega^n(\alpha, \beta, \gamma, p, \mu)$ if and only if it can be expressed as $\rho(z) = \sum_{k=0}^{\infty} \lambda_k \rho_k(z)$ where $\lambda \geq 0$ and $\sum_{k=0}^{\infty} \lambda_k = 1$

Proof. Let $\rho(z) = \sum_{k=0}^{\infty} \lambda_k \rho_k(z)$

$$= \frac{B}{(z-\omega)^p} - \sum_{k=p}^{\infty} \lambda_k \frac{\alpha B p^n [2 - \beta(1 + \gamma)]}{\Gamma(a, q, k)(k^{n+2} + 2\alpha)}(z-\omega)^k \tag{26}$$

since

$$\sum_{k=p}^{\infty} \frac{\Gamma(a, q, k)(k^{n+2} + 2\alpha)}{\alpha B p^n [2 - \beta(1 + \gamma)]} X \frac{\alpha B p^n [2 - \beta(1 + \gamma)]}{\Gamma(a, q, k)(k^{n+2} + 2\alpha)} \lambda_k = \sum_{k=p}^{\infty} \lambda_k = 1 - \lambda_o \leq 1 \tag{27}$$

So, by theorem 1, we find that $\rho(z) \in T_\omega^n(\alpha, \beta, \gamma, p, \mu)$.
Conversely, suppose that $\rho(z) \in T_\omega^n(\alpha, \beta, \gamma, p, \mu)$, then by (22) we have

$$a_k \le \frac{\alpha B p^n [2 - \beta(1 + \gamma)]}{\Gamma(a, q, k)(k^{n+2} + 2\alpha)} \tag{28}$$

Setting

$$\lambda_k = \frac{\Gamma(a, q, k)(k^{n+2} + 2\alpha)}{\alpha B p^n [2 - \beta(1 + \gamma)]} a_k \tag{29}$$

and $\lambda_o = 1 - \sum_{k=p}^{\infty} \lambda_k$ and the proof completes

Theorem 4. If $\rho(z) \in T_\omega^n(\alpha, \beta, \gamma, p, \mu)$, then the function $F_{c,p}(z)$ defined by $F_{c,p} = c \int_0^1 [s^c \rho(zs + \omega(1 - s))]ds$, $c \ge 1$,is also in this class.

Proof. Since $\rho(z) \in T_\omega^n(\alpha, \beta, \gamma, p, \mu)$ is of the form (9), so

$$F_{c,p} = c \int_d^1 s^c \left[\frac{B}{s(z - \omega)^p} - \sum_{k=p}^{\infty} \Gamma(a, q, k) k^n a_k (s(z - \omega))^k \right] ds, d < 1 \tag{30}$$

$$= \frac{B}{(z - \omega)^p} - \sum_{k=p}^{\infty} \Gamma(a, q, k) k^n \frac{c}{c + k + 1} a_k (z - \omega)^k \tag{31}$$

$\frac{c}{c+k+1} < 1$, hence by theorem 1, we conclude that

$$F_{c,p}(z) \in T_\omega^n(\alpha, \beta, \gamma, p, \mu) \tag{32}$$

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