Gabor frames on local fields of positive characteristic

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Abstract

Gabor frames have gained considerable popularity during the past decade, primarily due to their substantiated applications in diverse and widespread fields of engineering and science. Finding general and verifiable conditions which imply that the Gabor systems are Gabor frames is among the core problems in time-frequency analysis. In this paper, we give some simple and sufficient conditions that ensure a Gabor system $\{M_{u(m)b}T_{u(n)a}g =: \chi_m(bx)g(x-u(n)a)\}_{m,n\in\mathbb{N}_0}$ to be a frame for $L^2(K)$. The conditions proposed are stated in terms of the Fourier transforms of the Gabor system's generating functions.

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1 Introduction

The notion of frame was first introduced by Duffin and Schaeffer [1] in connection with some deep problems in non-harmonic Fourier series. Frames are basis-like systems that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, or obtain other desirable features unavailable with orthonormal bases. The idea of Duffin and Schaeffer did not generate much interest outside non-harmonic Fourier series until the seminal work by Daubechies, Grossmann, and Meyer [2]. They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (affine) frames for $L^2(\mathbb{R})$. After their work, the theory of frames began to be studied widely and deeply. Today, the theory of frames has become an interesting and fruitful field of mathematics with abundant applications in signal processing, image processing, harmonic analysis, Banach space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, and medicine and so on. An introduction to the frame theory and its applications can be found in [3,4].

Gabor frames form a special kind of frames for $L^2(\mathbb{R})$ whose elements are generated by time-frequency shifts of a single window-function or atom. More specifically, let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}^+$, we use $\mathcal{G}(g,a,b)$ to denote the *Gabor family* or *system* $\{M_{mb}T_{na}g : m,n \in \mathbb{Z}\}$ generated by g where $T_{na}f(x) = f(x-na)$ is the translation unitary operator and $M_{mb}f(x) = e^{2\pi imbx}f(x)$ is the modulation unitary operator. The composition $M_{mb}T_{na}$ is called the time-frequency shift operator. The system $\{M_{mb}T_{na}g : m,n \in \mathbb{Z}\}$ is called a *Gabor frame* if there exist constants $A, B > 0$ such that

$$A\big\|f\big\|_2^2 \le \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \big|\big\langle f, M_{mb}T_{na}g\big\rangle\big|^2 \le B\big\|f\big\|_2^2, \quad \text{for all } f \in L^2(\mathbb{R}). \tag{1.1}$$

Gabor systems that form frames for $L^2(\mathbb{R})$ have a wide variety of applications. In practice, once the window function has been chosen, the first question to investigate for Gabor analysis is to find

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the values of the time-frequency parameters a, b such that $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame. A useful tool in this context is the Ron and Shen [5] criterion. By using this criterion, Gröchenig et al.[6] have proved that the system $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ cannot be a frame for $a > 0$ and b integer greater than 1. Many results in this area, including necessary conditions and sufficient conditions have been established during the last two decades [7–10]. We refer the reader to the books [11,12] for a comprehensive treatment of Gabor frames.

A field K equipped with a topology is called a local field if both the additive and multiplicative groups of K are locally compact Abelian groups. For example, any field endowed with the discrete topology is a local field. For this reason we consider only non-discrete fields. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Local fields have attracted the attention of several mathematicians, and have found innumerable applications not only in the number theory, but also in the representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as a part of the standard repertoire of contemporary mathematics. For more details we refer to [13].

The local field K is a natural model for the structure of Gabor frame systems, as well as a domain upon which one can construct Gabor basis functions. There is a substantial body of work that has been concerned with the construction of Gabor frames on K, or more generally, on local fields of positive characteristic. Jiang et al.[14] constructed Gabor frames on local fields of positive characteristic using basic concepts of operator theory and have established a necessary and sufficient conditions for the system $\big\{M_{u(m)b}T_{u(n)a}g =: \chi_m(bx)g\big(x-u(n)a\big)\big\}_{m,n\in\mathbb{N}_0}$ to be a frame for $L^2(K)$. Recently, Li and Jun [15] established a complete characterization of Gabor frames on local fields by virtue of two basic equations in the Fourier domain and show how to construct an orthonormal Gabor basis for $L^2(K)$. Recent results related to wavelet and Gabor frames on local fields of prime characteristic can be found in [16-19] and the references therein.

In this article, we continue our investigation on Gabor frames on local fields and will present generalized inequalities for Gabor frames on local fields of positive characteristic via Fourier transform. The inequalities we proposed are stated in terms of the Fourier transforms of the Gabor system's generating functions. Although, we consider Gabor frames generated by a single function here, our results can easily be verified to Gabor frames with multi-generators on local fields of positive characteristic.

The paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and state the main results. Section 3 gives the proofs of the results.

2 Preliminaries on local fields

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. We use the notation of the book by Taibleson [13]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties:

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|xy| = |x||y|$ for all $x, y \in K$;

(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the *ultrametric inequality.* The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. It is the unique maximal compact subring of K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [13], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [13]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [13], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

Definition 2.1. If $f \in L^1(K)$, then the Fourier transform of f is defined by

$$\mathscr{F}\{f(x)\} = \hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\, dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\, \overline{\chi_\xi(x)} dx = \int_K f(x)\chi(-\xi x)\, dx.$$

The properties of Fourier transform on local field K are much similar to those of on the real line. In fact, the Fourier transform have the following properties:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \leq \|f\|_1$.
- If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(K) \cap L^2(K)$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(K)$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\overline{\chi_\xi(x)}\, dx, \tag{2.2}$$

where $f_k = f\,\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\, dx. \tag{2.3}$$

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q, \quad n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \quad 0 \le a_k < p, \quad \text{and } k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}.$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \le b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}.$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \ge 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \ne 1. \end{cases} \tag{2.4}$$

We also denote the test function space on K by Ω, that is, each function f in Ω is a finite linear combination of functions of the form $\mathbf{\Phi}_k(x-h), h \in K, k \in \mathbb{Z}$, where $\mathbf{\Phi}_k$ is the characteristic function of $\mathfrak{B}^k$. This class of functions can also be described in the following way. A function $g \in \Omega$ if and only if there exist integers k, ℓ such that g is constant on cosets of $\mathfrak{B}^k$ and is supported on $\mathfrak{B}^\ell$. It follows that Ω is closed under Fourier transform and is an algebra of continuous functions with compact support, which is dense in $\mathcal{C}_0(K)$ as well as in $L^p(K), 1 \le p < \infty$.

For a given $g \in L^2(K)$, define the Gabor system

$$\mathcal{G}(g, a, b) := \Big\{ g_{m,n}(x) =: \chi_m(bx)g\big(x - u(n)a\big) : n, m \in \mathbb{N}_0 \Big\}. \tag{2.5}$$

We call the Gabor system $\mathcal{G}(g,a,b)$ a *Gabor frame* for $L^2(K)$, if there exist positive numbers $0 < C \le D < \infty$ such that for all $f \in L^2(K)$

$$C\|f\|_2^2 \le \sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} \left|\langle f, M_{u(m)b}T_{u(n)a}g\rangle\right|^2 \le D\|f\|_2^2. \tag{2.6}$$

Before stating our results, we introduce some notations. For any $g \in L^2(K)$ and $a, b > 0$. We set

$$\begin{aligned}
\Delta_k(\xi) &= \sum_{m\in\mathbb{N}_0} \left|\hat{g}\big(\xi - bu(m)\big)\hat{g}\big(\xi - bu(m) + a^{-1}u(k)\big)\right|,\\
\alpha_k &= \operatorname{ess\,sup}_{\xi} \Delta_k(\xi),\ \ k \in \mathbb{N}_0, \quad \beta = \sum_{k\in\mathbb{N}} \alpha_k, \quad \gamma = \operatorname{ess\,inf}_{\xi} \Delta_0(\xi),\\
\Lambda_k(\xi) &= \sum_{k\in\mathbb{N}_0} \hat{g}\big(\xi - bu(m)\big)\hat{g}\big(\xi - bu(m) + a^{-1}u(k)\big),\\
\delta_k &= \operatorname{ess\,sup}_{\xi} \left|\Lambda_k(\xi)\right|,\ \ k \in \mathbb{N}_0, \quad \mu = \sum_{k\in\mathbb{N}} \delta_k\,.
\end{aligned}$$

Theorem 2.1. *Let $a, b > 0$ and $g \in L^2(K)$. If α_0, β and γ satisfy*

$$\beta < \gamma \le \alpha_0 < \infty,$$

then system in (2.5) constitutes a Gabor frame for $L^2(K)$ with bounds $\dfrac{C_1}{a}$ and $\dfrac{D_1}{a}$, where $C_1 = \gamma - \beta$ and $D_1 = \alpha_0 + \beta$.

Remark 1. The analogous time domain version of Theorem 2.1 was established by Li and Jiang[14] (Theorem 5.2, pp. 173), while the above frequency version can be verified by using the machinery of shift-invariant spaces in the same way as used by Li et al. in [20].

Drawing inspiration from the general results of Gabor frames on local fields of positive characteristic obtained by Li and Jiang [14], we shall present two sufficient conditions in frequency domain for such frames on local fields. The conditions obtained are better than that of one in Theorem 2.1.

The first result of the paper is stated as follows.

Theorem 2.2. *Let $a, b > 0$ and $g \in L^2(K)$. If α_0, γ and μ satisfy*

$$\mu < \gamma \le \alpha_0 < \infty, \tag{2.7}$$

then the Gabor system $\mathcal{G}(g,a,b)$ as defined in (2.5) is a frame for $L^2(K)$ with bounds $\dfrac{C_2}{a}$ and $\dfrac{D_2}{a}$, where $C_2 = \gamma - \mu$ and $D_2 = \alpha_0 + \mu$.

Remark 2. Obviously, $\mu \le \beta$, so the frame bounds in Theorem 2.2 are better than ones in Theorem 2.1.

Next, we prove a more general result which includes not only the results of Theorem 2.1 and 2.2 as special cases, but also leads to a standard development of interesting generalizations of some well-known results. To do so, we set

$$\sigma = \operatorname*{ess\,sup}_{\xi} \sum_{k\in\mathbb{N}} \big|\Lambda_k\big|.$$

Theorem 2.3. *Let $a, b > 0$ and $g \in L^2(K)$. If α_0, γ and σ satisfy*

$$\sigma < \gamma \leq \alpha_0 < \infty, \tag{2.8}$$

then the Gabor system $\mathcal{G}(g,a,b)$ given by (2.5) constitutes a frame for $L^2(K)$ with bounds $\dfrac{C_3}{a}$ and $\dfrac{D_3}{a}$, where $C_3 = \gamma - \sigma$ and $D_3 = \alpha_0 + \sigma$.

Remark 3. Since $\sigma \leq \mu$, the frame bounds in Theorem 2.3 are better than ones in Theorem 2.2.

3 Proof of the main results

In order to prove Theorems 2.2 and 2.3, we need the following lemma whose proof can be found in Christensen [3] (Lemma 5.1.7, pp. 92).

Lemma 3.1. *Suppose that $\{f_k\}_{k=1}^{\infty}$ is a family of elements in a Hilbert space $\mathbb{H}$ such that there exist constants $0 < A \leq B < \infty$ satisfying*

$$A\big\|f\big\|_2^2 \leq \sum_{k=1}^{\infty} \big|\langle f, f_k\rangle\big|^2 \leq B\big\|f\big\|_2^2,$$

for all f belonging to a dense subset $\mathcal{D}$ of $\mathbb{H}$. Then, the same inequalities are true for all $f \in \mathbb{H}$; that is, $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathbb{H}$.

In view of Lemma 3.1, we will consider the following set of functions:

$$\Omega^0 = \Big\{ f \in \Omega : \operatorname{supp}\hat{f} \subset K\backslash\{0\} \text{ and } \big\|\hat{f}\big\|_\infty < \infty \Big\}.$$

Since Ω is dense in $L^2(K)$ and closed under the Fourier transforms, the set Ω^0 is also dense in $L^2(\mathbb{K})$. Therefore, it is enough to verify that the system $\mathcal{G}(g,a,b)$ given by (2.5) is a frame for $L^2(K)$ if the results of Theorems 2.2 and 2.3 hold for all $f \in \Omega^0$.

Assume that $f \in L^2(K)$ and $h \in \Omega^0$, then by periodization, we have

$$a\int_K h(\xi) f(\xi)\, \chi_k\big(a(\xi-\omega)\big)\, d\xi = a\int_{G_{a^{-1}}} \sum_{m\in\mathbb{N}_0} h\big(\xi + a^{-1}u(m)\big) f\big(\xi + a^{-1}u(m)\big)\, \chi_k\big(a(\xi-\omega)\big)\, d\xi$$

where $G_a = \{x \in K : |x| \leq |a|\}$. Since h lies in Ω^0, so it is bounded and compactly supported, as a consequence, the number of m-terms in the above sum is finite. Thus, we can say the series

$$\sum_{k\in\mathbb{N}_0} \int_K h(\xi) f(\xi)\, \chi_k\big(a(\xi-\omega)\big)\, d\xi \tag{3.1}$$

is convergent to a periodic function $H(\xi) \in L^2(G_{a^{-1}})$, where

$$H(\xi) = \frac{1}{a} \sum_{m\in\mathbb{N}_0} h\big(\xi + a^{-1}u(m)\big) f\big(\xi + a^{-1}u(m)\big). \tag{3.2}$$

Proof of Theorem 2.2.

For any $f \in L^2(K)$, there exists a function sequence $\{f_j\}_{j=1}^{\infty} \subset \Omega^0$, such that

$$\left\| \hat{f}_j - \hat{f} \right\|_2 \to 0 \quad \text{as } j \to \infty, \quad \text{and supp } \hat{f}_j \subset \mathfrak{B}^j$$

since Ω^0 is dense in $L^2(K)$. Let $g_{m,n}(x)$ be the family of functions given by (2.5), then for fixed $m \in \mathbb{N}_0$, we define the functional

$$P_m(h) = \sum_{n\in\mathbb{N}_0} \left|\langle h, g_{m,n}\rangle\right|^2 = \sum_{n\in\mathbb{N}_0} \left|\left\langle \hat{h}, \hat{g}_{m,n}\right\rangle\right|^2, \quad \text{for all } h \in L^2(K). \tag{3.3}$$

Since the Fourier transform of $g_{m,n}$ is

$$\hat{g}_{m,n}(\xi) = \overline{\chi_n\Big(a\big(\xi - bu(m)\big)\Big)}\hat{g}\big(\xi - bu(m)\big),$$

therefore, with the aid of (3.1), we are able to express the relation (3.3) as

$$\begin{aligned}
P_m(f_j) &= \sum_{n\in\mathbb{N}_0} \left|\left\langle \hat{f}_j, \hat{g}_{m,n}\right\rangle\right|^2 \\
&= \sum_{n\in\mathbb{N}_0} \left\langle \hat{f}_j, \hat{g}_{m,n}\right\rangle \overline{\left\langle \hat{f}_j, \hat{g}_{m,n}\right\rangle} \\
&= \sum_{n\in\mathbb{N}_0} \int_K \hat{f}_j(\xi)\, \overline{\hat{g}\big(\xi - bu(m)\big)} \chi_n\Big(a\big(\xi - bu(m)\big)\Big)\, d\xi \\
&\qquad\qquad \times \int_K \overline{\hat{f}_j(\omega)}\, \hat{g}\big(\omega - bu(m)\big) \overline{\chi_n\Big(a\big(\omega - bu(m)\big)\Big)}\, d\omega \\
&= \frac{1}{a} \sum_{k\in\mathbb{N}_0} \int_K \hat{f}_j\big(\xi + a^{-1}u(k)\big) \overline{\hat{g}\big(\xi - bu(m) + a^{-1}u(k)\big)}\, \overline{\hat{f}_j(\xi)}\, \hat{g}\big(\omega - bu(m)\big)\, d\xi.
\end{aligned}$$

Let

$$P(f) = \sum_{m\in\mathbb{N}_0} \sum_{n\in\mathbb{N}_0} \left|\langle f, g_{m,n}\rangle\right|^2 = \sum_{m\in\mathbb{N}_0} P_m(f), \tag{3.4}$$

then

$$P(f_j) = \frac{1}{a} \sum_{k\in\mathbb{N}_0} \sum_{m\in\mathbb{N}_0} \int_K \hat{f}_j\big(\xi + a^{-1}u(k)\big)\overline{\hat{g}\big(\xi - bu(m) + a^{-1}u(k)\big)}\,\overline{\hat{f}_j(\xi)}\,\hat{g}\big(\omega - bu(m)\big)\,d\xi$$

$$= Q_1(f_j) + Q_2(f_j), \tag{3.5}$$

where

$$Q_1(f_j) = \frac{1}{a} \sum_{m\in\mathbb{N}_0} \int_K \left|\hat{f}_j(\xi)\,\hat{g}\big(\omega - bu(m)\big)\right|^2 d\xi \tag{3.6}$$

$$Q_2(f_j) = \frac{1}{a} \sum_{k\in\mathbb{N}} \sum_{m\in\mathbb{N}_0} \int_K \hat{f}_j\big(\xi + a^{-1}u(k)\big)\overline{\hat{g}\big(\xi - bu(m) + a^{-1}u(k)\big)}\,\overline{\hat{f}_j(\xi)}\,\hat{g}\big(\xi - bu(m)\big)\,d\xi. \tag{3.7}$$

Since $\alpha_0 < \infty$, the series $Q_1(f_j)$ is convergent and

$$\frac{\gamma}{a}\left\|\hat{f}_j\right\|_2^2 \le Q_1(f_j) \le \frac{\alpha_0}{a}\left\|\hat{f}_j\right\|_2^2,$$

or equivalently

$$\frac{\gamma}{a}\left\|f_j\right\|_2^2 \le Q_1(f_j) \le \frac{\alpha_0}{a}\left\|f_j\right\|_2^2. \tag{3.8}$$

Next, we claim that $Q_2(f_j)$ is absolutely convergent. To prove this, we set

$$Q_2^*(f_j) = \frac{1}{a} \sum_{k\in\mathbb{N}} \sum_{m\in\mathbb{N}_0} \left|\int_K \hat{f}_j\big(\xi + a^{-1}u(k)\big)\overline{\hat{g}\big(\xi - bu(m) + a^{-1}u(k)\big)}\,\overline{\hat{f}_j(\xi)}\,\hat{g}\big(\xi - bu(m)\big)\,d\xi\right|.$$

Note that

$$\left|\hat{g}\big(\xi - bu(m) + a^{-1}u(k)\big)\hat{g}\big(\xi - bu(m)\big)\right| \le \frac{1}{2}\left(\left|\hat{g}\big(\xi - bu(m) + a^{-1}u(k)\big)\right|^2 + \left|\hat{g}\big(\xi - bu(m)\big)\right|^2\right),$$

hence we have

$$Q_2^*(f_j) \le \frac{1}{a} \sum_{k\in\mathbb{N}} \sum_{m\in\mathbb{N}_0} \int_K \left|\hat{f}_j\big(\xi + bu(m) + a^{-1}u(k)\big)\overline{\hat{f}_j\big(\xi + a^{-1}u(k)\big)}\right| \left|\hat{g}(\xi)\right|^2 d\xi.$$

Since each f_j is bounded and compactly supported on $\mathfrak{B}^j$, and in fact they belongs to Ω^0, hence there exist a constant $M > 0$ such that

$$Q_2^*(f_j) \le M\left\|\hat{f}_j\right\|_\infty^2 \left\|g\right\|^2 < \infty,$$

which proves our claim that $Q_2(f_j)$ is absolutely convergent.

Using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\left|Q_2(f_j)\right| &= \left|\frac{1}{a}\sum_{k\in\mathbb{N}}\int_K \hat{f}_j\big(\xi+a^{-1}u(k)\big)\,\overline{\hat{f}_j(\xi)}\,\Lambda_k(\xi)\,d\xi\right| \\ &\le \frac{1}{a}\sum_{k\in\mathbb{N}}\int_K \left\{\left|\hat{f}_j\big(\xi+a^{-1}u(k)\big)\right|\left|\Lambda_k(\xi)\right|^{1/2}\right\}\left\{\left|\hat{f}_j(\xi)\right|\left|\Lambda_k(\xi)\right|^{1/2}\right\}d\xi \\ &\le \frac{1}{a}\sum_{k\in\mathbb{N}}\left\{\int_K \left|\hat{f}_j\big(\xi+a^{-1}u(k)\big)\right|^2\left|\Lambda_k(\xi)\right|d\xi\right\}^{1/2}\left\{\int_K \left|\hat{f}_j(\xi)\right|^2\left|\Lambda_k(\xi)\right|d\xi\right\}^{1/2}. \end{aligned} \tag{3.9}$$

It is easy to verify that

$$\hat{f}_j\big(\xi+a^{-1}u(k)\big)=\hat{f}_j(\xi), \quad \text{and} \quad \Lambda_k\big(\xi-a^{-1}u(k)\big)=\Lambda_k(\xi),\ \forall\ k\in\mathbb{N}.$$

Thus, we have

$$\left|Q_2(f_j)\right| \le \frac{1}{a}\int_K \left|\hat{f}_j(\xi)\right|^2 d\xi \sum_{k\in\mathbb{N}}\delta_k = \frac{\mu}{a}\left\|f_j\right\|_2^2,$$

or equivalently,

$$-\frac{\mu}{a}\left\|f_j\right\|_2^2 \le Q_2(f_j) \le \frac{\mu}{a}\left\|f_j\right\|_2^2. \tag{3.10}$$

It follows from (3.8) and (3.10) that

$$\frac{\gamma-\mu}{a}\left\|f_j\right\|_2^2 \le P(f_j) \le \frac{\alpha_0+\mu}{a}\left\|f_j\right\|_2^2.$$

Letting $j\to\infty$ in above inequality, we obtain

$$\frac{\gamma-\mu}{a}\left\|f\right\|_2^2 \le P(f) \le \frac{\alpha_0+\mu}{a}\left\|f\right\|_2^2,$$

or

$$\frac{C_2}{a}\left\|f\right\|_2^2 \le \sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0}\left|\langle f, g_{m,n}\rangle\right|^2 \le \frac{D_2}{a}\left\|f\right\|_2^2,$$

where $C_2=\gamma-\mu$ and $D_2=\alpha_0+\mu$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3.

Similar to the proof of Theorem 2.2, (3.6)–(3.9) hold. It follows from (3.9), the Cauchy-Schwarz inequality that

$$\begin{aligned}\left|Q_2(f_j)\right| &\le \frac{1}{a}\left\{\sum_{k\in\mathbb{N}}\int_K \left|\hat{f}_j\big(\xi+a^{-1}u(k)\big)\right|^2\left|\Lambda_k(\xi)\right|d\xi\right\}^{1/2}\left\{\sum_{k\in\mathbb{N}}\int_K \left|\hat{f}_j(\xi)\right|^2\left|\Lambda_k(\xi)\right|d\xi\right\}^{1/2} \\ &= \frac{1}{a}\left\{\sum_{k\in\mathbb{N}}\int_K \left|\hat{f}_j(\xi)\right|^2\left|\Lambda_k\big(\xi-a^{-1}u(k)\big)\right|d\xi\right\}^{1/2}\left\{\sum_{k\in\mathbb{N}}\int_K \left|\hat{f}_j(\xi)\right|^2\left|\Lambda_k(\xi)\right|d\xi\right\}^{1/2}\end{aligned}$$

$$= \frac{1}{a}\left\{\int_K \left|\hat{f}_j(\xi)\right|^2 \sum_{k\in\mathbb{N}} \left|\Lambda_k(\xi)\right| d\xi\right\}^{1/2} \left\{\int_K \left|\hat{f}_j(\xi)\right|^2 \sum_{k\in\mathbb{N}} \left|\Lambda_k(\xi)\right| d\xi\right\}^{1/2}$$

$$\leq \frac{\sigma}{a}\left\|f_j\right\|_2^2, \tag{3.11}$$

which implies that

$$-\frac{\sigma}{a}\left\|f_j\right\|_2^2 \leq Q_2(f_j) \leq \frac{\sigma}{a}\left\|f_j\right\|_2^2. \tag{3.12}$$

Combining (3.8) with (3.12), we obtain

$$\frac{\gamma-\sigma}{a}\left\|f_j\right\|_2^2 \leq P(f_j) \leq \frac{\alpha_0+\sigma}{a}\left\|f_j\right\|_2^2.$$

By taking $j \to \infty$ in above relation, we get

$$\frac{\gamma-\sigma}{a}\left\|f\right\|_2^2 \leq P(f) \leq \frac{\alpha_0+\sigma}{a}\left\|f\right\|_2^2,$$

or

$$\frac{C_3}{a}\left\|f\right\|_2^2 \leq \sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0}\left|\left\langle f, g_{m,n}\right\rangle\right|^2 \leq \frac{D_3}{a}\left\|f\right\|_2^2,$$

where $C_3 = \gamma - \sigma$ and $D_3 = \alpha_0 + \sigma$. This completes the proof of Theorem 2.3. □

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