Advances on the coefficient bounds for m-fold symmetric bi-close-to-convex functions†

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Abstract

In 1955, Waadeland considered the class of m-fold symmetric starlike functions of the form $f_m(z) = z + \sum_{n=1}^{\infty} a_{mn+1} z^{mn+1}$; $m \geq 1$; $|z| < 1$ and obtained the sharp coefficient bounds $|a_{mn+1}| \leq [(2/m + n - 1)!] / [(n!)(2/m - 1)!]$. Pommerenke in 1962, proved the same coefficient bounds for m-fold symmetric close-to-convex functions. Nine years later, Keogh and Miller confirmed the same bounds for the class of m-fold symmetric Bazilevic functions. Here we will show that these bounds can be improved even further for the m-fold symmetric bi-close-to-convex functions. Moreover, our results improve those corresponding coefficient bounds given by Srivastava et al that appeared in 7(2) (2014) issue of this journal. A function is said to be bi-close-to-convex in a simply connected domain if both the function and its inverse map are close-to-convex there.

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1 Introduction

Let $\mathcal{K}$ be the class of all functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ that satisfy $f'(z) \neq 0$ and

$$\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial \theta} \arg \left[e^{i\theta} f'(re^{i\theta}) \right] d\theta > -\pi; \, \theta_1 < \theta_2, \, 0 \leq r < 1. \tag{1.1}$$

The class $\mathcal{K}$ is the class of close-to-convex functions. It was proved by Kaplan [7] that a function f of the form (1.1) belongs to $\mathcal{K}$ if and only if there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ starlike in $\mathbb{D}$ (that is, $Re\,[zg'(z)/g(z)] > 0$ in $\mathbb{D}$) such that $Re\,(zf'/g) > 0$ in $\mathbb{D}$. In 1955, Waadeland [14] considered the class of m-fold symmetric starlike functions of the form

$$g_m(z) = z + \sum_{n=1}^{\infty} b_{mn+1} z^{mn+1}; \, m \geq 1$$

and obtained the sharp coefficient bounds

$$|b_{mn+1}| \leq \binom{2/m + n - 1}{n} \sim \frac{1}{\Gamma(2/m)} n^{2/m-1}.$$

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Pommerenke [10] in 1962, proved the same coefficient bounds for m-fold symmetric close-to-convex functions $f_m(z) = z + \sum_{n=1}^{\infty} a_{mn+1} z^{mn+1}$; $m \geq 1$. Nine years later, Keogh and Miller [8] confirmed the same bounds for the class of m-fold symmetric Bazilevic functions. Here we will show that these bounds can be improved even further for the m-fold symmetric bi-close-to-convex functions. Moreover, the coefficient bounds presented in this paper for $|a_{m+1}|$, $|a_{2m+1}|$ and $m > 1$ also improve those corresponding coefficient bounds given by Srivastava et al [12]. A function is said to be bi-close-to-convex in a simply connected domain if both the function and its inverse map are close-to-convex there. The class of bi-univalent functions was first introduced and studied by Lewin [9] and has gained momentum in recent years mainly due to the pioneer work of Srivastava et al [11]. Because the bi-univalency requirement makes the behavior of the coefficients of bi-univalent functions unpredictable, no general coefficient bounds for subclasses of bi-univalent functions was known up until the publication of article [6] by Jahangiri and Hamidi. The unpredictability of m-fold symmetric bi-starlike functions was first studied by the authors in [4] followed by the publication of the articles [12] and [13] by Srivastava et al. Here we further improve the bounds given in [4] to include the larger class of m-fold symmetric bi-close-to-convex functions. We begin with the statement of the following

Theorem 1.1. For $m \geq 2$ if $f_m(z) = z + \sum_{n=1}^{\infty} a_{mn+1} z^{mn+1}$ is m-fold symmetric bi-close-to-convex in $\mathbb{D}$, then

(i). $|a_{m+1}| \leq \dfrac{1}{m}\sqrt{\dfrac{2}{m+1}}$,

(ii). $|a_{2m+1}| \leq \dfrac{1}{m^2}$,

(iii). $|a_{mn+1}| \leq \dfrac{2}{m^2 n}$, if $\quad a_{mk+1} = 0$; $(2 \leq k < n)$.

The following example justifies the existence of functions satisfying the bounds given in Theorem 1.1.

Example 1.2. Let $f(z) = z + \frac{2}{m^2 n} z^{mn+1}$; $m \geq 2,\ n \geq 2,\ z \in \mathbb{D}$. Then for the starlike function $g(z) = z - \frac{2}{m^2 n} z^{mn+1}$; $m \geq 2,\ n \geq 2,\ z \in \mathbb{D}$ we have

$$\frac{zf'(z)}{g(z)} = \frac{1 + \frac{2(mn+1)}{m^2 n} z^{mn}}{1 - \frac{2}{m^2 n} z^{mn}} = 1 + \sum_{k=1}^{\infty} \frac{2(mn+2)}{(m^2 n)^k} z^{mk} = 1 + \sum_{k=1}^{\infty} A_k z^{mk}.$$

We note that A_k is a convex null sequence since $\lim\limits_{k\to\infty} A_k = 0$, $1 - A_1 \geq 0$ and $A_k - A_{k+1} \geq 0$. Therefore, $Re(zf'/g) > 0$.

On the other hand, for $F(w) = f^{-1}(w) = w - \frac{2}{m^2 n} w^{mn+1}$; $m \geq 2,\ n \geq 2,\ w \in \mathbb{D}$ consider the starlike function $G(w) = w + \frac{2}{m^2 n} w^{mn+1}$; $m \geq 2,\ n \geq 2,\ w \in \mathbb{D}$. Then we have

$$\frac{wF'(w)}{G(w)} = \frac{1 - \frac{2(mn+1)}{m^2 n} w^{mn}}{1 + \frac{2}{m^2 n} w^{mn}} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2(mn+2)}{(m^2 n)^k} w^{mk} = 1 + \sum_{k=1}^{\infty} (-1)^k A_k w^{mk}.$$

Once again, since A_k is a convex null sequence, $Re(wF'/G) > 0$.

2 Proofs

In order to prove our theorem we shall need the following well-known lemma.

Lemma 2.1. (See Duren [3] or Jahangiri [5])

For the positive real part functions $P_1(z) = 1 + \Sigma_{n=1}^{\infty} p_n z^n$ and $P_m(z) = \sqrt[m]{P_1(z^m)}$ where $P_m(z) = 1 + \Sigma_{n=1}^{\infty} p_{mn} z^{mn}$, $z \in \mathbb{D}$, $m \in \mathbb{N}$ we have

(i). $|p_n| \leq 2$,

(ii). $\left|p_2 + \lambda p_1^2\right| \leq 2 + \lambda \left|p_1\right|^2$ if $\lambda \geq -1/2$,

(iii). $p_m = \dfrac{1}{m} p_1$,

(iv). $p_{2m} = \dfrac{1}{m}\left[p_2 - \dfrac{m-1}{2m} p_1^2\right]$,

(v). $p_{mn} = \dfrac{1}{m} p_n;\ if\ \ p_{mk} = 0; (2 \leq k < n)$.

Proof of Theorem 1.1. If $F = f^{-1}$ is the inverse of a function f univalent in $\mathbb{D}$, then F has a Maclaurin series expansion in some disk about the origin (e.g. see [3] or [9]). According to Airault [1] or Airault and Ren [2, p. 349], the function $F = f^{-1}$, the inverse map of the univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ has the Faber polynomial expansion

$$F(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^n;\ w \in \mathbb{D}$$

where K_{n-1}^{-n} is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$.

The first few terms of the coefficients K_{n-1}^{-n} are $K_1^{-2} = -2a_2$, $K_2^{-3} = +3(2a_2^2 - a_3)$, $K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4)$ and $K_4^{-5} = +5(14a_2^2 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5)$.

In general, for $n \geq 1$ and for real values of κ, these coefficients are calculated according to

$$K_{n-1}^{\kappa} = \kappa a_n + \frac{\kappa(\kappa-1)}{2} D_{n-1}^2 + \frac{\kappa!}{(\kappa-3)!3!} D_{n-1}^3 + \cdots + \frac{\kappa!}{(\kappa-n+1)!(n-1)!} D_{n-1}^{n-1},$$

where $D_{n-1}^{\kappa} = D_{n-1}^{\kappa}(a_2, a_3, \cdots, a_n)$ are homogeneous polynomials explicated in

$$D_{n-1}^{\kappa}\ (a_2, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{\kappa!(a_2)^{\mu_1} \ldots (a_n)^{\mu_{n-1}}}{\mu_1! \ \ldots \ \mu_{n-1}!} \qquad \text{for} \qquad \kappa \leq n-1,$$

and the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_{n-1}$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_{n-1} = \kappa, \\ \mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} = n-1. \end{cases}$$

Evidently $D_n^n(a_2, a_3, \ldots, a_n) = a_2^n$.

Therefore, the m-fold symmetric function $f_m(z) = \sqrt[m]{f_1(z^m)}$ has the Faber polynomial expansion

$$\begin{aligned} f_m(z) &= \sqrt[m]{f(z^m)} = z + \sum_{n=1}^{\infty} K_n^{\frac{1}{m}}(a_2, a_3, \cdots, a_{n+1}) z^{mn+1} \\ &= z + a_{m+1} z^{m+1} + a_{2m+1} z^{2m+1} + \dots . \end{aligned} \tag{2.1}$$

According to Kaplan ([7], Theorem 2), for the m-fold symmetric close-to-convex function f_m, the corresponding starlike function is also m-fold symmetric. So there exists an m-fold symmetric starlike function $g_m(z) = z + b_{m+1} z^{m+1} + b_{2m+1} z^{2m+1} + \dots$ so that

$$Re\left(\frac{z f_m'(z)}{g_m(z)}\right) > 0;\ z \in \mathbb{D}.$$

By the same token, there exists a positive real part function $\varphi_m(z) = 1 + \sum_{n=1}^{\infty} \varphi_{mn} z^{mn}$ so that

$$\frac{z f_m'(z)}{g_m(z)} = \varphi_m(z);\ z \in \mathbb{D}. \tag{2.2}$$

On the other hand, the Faber polynomial expansion for $z f_m'/g_m$ would be

$$\frac{z f_m'(z)}{g_m(z)} = 1 + \sum_{n=1}^{\infty} \{((nm+1)a_{nm+1} - b_{nm+1}) \tag{2.3}$$

$$+ \sum_{\ell=1}^{n-1} K_\ell^{-1}(b_{m+1}, b_{2m+1}, \cdots, b_{\ell m+1}) \left[((n-\ell)m+1)a_{(n-\ell)m+1} - b_{(n-\ell)m+1}\right]\} z^{mn}.$$

For the inverse map $F_m = f_m^{-1}$, the Faber polynomial expansion is

$$\begin{aligned} F_m(w) &= w + \sum_{n=1}^{\infty} A_{mn+1} w^{mn+1} \\ &= w + \sum_{n=1}^{\infty} \frac{1}{mn+1} K_n^{-(mn+1)}(a_{m+1}, a_{2m+1}, \dots, a_{mn+1}) w^{mn+1}. \end{aligned} \tag{2.4}$$

The close-to-convexity of the inverse function F_m implies the existence of an m-fold symmetric function $G_m(w) = w + \sum_{n=1}^{\infty} B_{mn+1} w^{mn+1}$ starlike in $\mathbb{D}$ so that $Re(wF_m'(w)/G_m(w)) > 0$ in $\mathbb{D}$. So, there exists a positive real part function $\Psi_m(w) = 1 + \sum_{n=1}^{\infty} \psi_{mn} w^{mn}$ in $\mathbb{D}$ representing

$$\frac{wF_m'(w)}{G_m(w)} = 1 + \psi_m w^m + \psi_{2m} w^{2m} + \cdots; \quad w \in \mathbb{D}. \tag{2.5}$$

The Faber polynomial expansion for $wF_m'(w)/G(w)$ is given by

$$\frac{wF_m'(w)}{G_m(w)} = 1 + \sum_{n=1}^{\infty} \{((nm+1)A_{nm+1} - B_{nm+1}) \tag{2.6}$$

$$+\sum_{\ell=1}^{n-1} K_{\ell}^{-1}(B_{m+1}, B_{2m+1}, \cdots, B_{\ell m+1}) \left[((n-\ell)m+1)A_{(n-\ell)m+1} - B_{(n-\ell)m+1}\right]\} \ w^{mn}.$$

Comparing the corresponding coefficients of (2.2), (2.3), (2.5) and (2.6), we obtain

$$\varphi_{mn} = ((nm+1)a_{nm+1} - b_{nm+1}) \tag{2.7}$$

$$+\sum_{\ell=1}^{n-1} K_{\ell}^{-1}(b_{m+1}, b_{2m+1}, \cdots, b_{\ell m+1}) \left[((n-\ell)m+1)a_{(n-\ell)m+1} - b_{(n-\ell)m+1}\right],$$

and

$$\psi_{mn} = ((nm+1)A_{nm+1} - B_{nm+1}) \tag{2.8}$$

$$+\sum_{\ell=1}^{n-1} K_{\ell}^{-1}(B_{m+1}, B_{2m+1}, \cdots, B_{\ell m+1}) \left[((n-\ell)m+1)A_{(n-\ell)m+1} - B_{(n-\ell)m+1}\right].$$

Letting $n = 1$ and $n = 2$, the above two equations (2.7) and (2.8) yield

$$A_{m+1}, = -a_{m+1}, \qquad A_{2m+1} = (m+1)a_{m+1}^2 - a_{2m+1},$$

and consequently

$$(m+1)a_{m+1} - b_{m+1} = \varphi_m, \tag{2.9}$$

$$(2m+1)a_{2m+1} - b_{2m+1} - b_{m+1}[(m+1)a_{m+1} - b_{m+1}] = \varphi_{2m} \tag{2.10}$$

$$-(m+1)a_{m+1} - B_{m+1} = \psi_m, \tag{2.11}$$

and

$$(2m+1)[(m+1)a_{m+1}^2 - a_{2m+1}] - B_{2m+1} + B_{m+1}[(m+1)a_{m+1} + B_{m+1}] = \psi_{2m}. \tag{2.12}$$

Substituting (2.9) in (2.10) and (2.11) in (2.12) and then adding them we obtain

$$(m+1)(2m+1)a_{m+1}^2 - b_{2m+1} - B_{2m+1} - (b_{m+1})(\varphi_m) - (B_{m+1})(\psi_m) = \varphi_{2m} + \psi_{2m}. \tag{2.13}$$

On the other hand, for the starlike function $g_m(z) = z + b_{m+1}z^{m+1} + b_{2m+1}z^{2m+1} + \ldots$, we set $\frac{zg'_m(z)}{g_m(z)} = P_m(z) = 1 + \Sigma_{n=1}^{\infty} p_{mn} z^{mn}$ where $ReP_m(z) > 0$ in $\mathbb{D}$. Similarly, for the starlike function $G_m(w) = w + B_{m+1}w^{m+1} + B_{2m+1}w^{2m+1} + \ldots$, we set $\frac{wG'_m(w)}{G_m(w)} = Q_m(w) = 1 + \Sigma_{n=1}^{\infty} q_{mn} w^{mn}$ where $ReQ_m(w) > 0$ in $\mathbb{D}$. Comparing the corresponding coefficients we obtain

$$b_{m+1} = \frac{1}{m}p_m, \quad b_{2m+1} = \frac{1}{2m}\left[p_{2m} + \frac{1}{m}p_m^2\right]$$

and

$$B_{m+1} = \frac{1}{m}q_m, \quad B_{2m+1} = \frac{1}{2m}\left[q_{2m} + \frac{1}{m}q_m^2\right].$$

An application of Lemma 2.1 yields

$$b_{m+1} = \frac{1}{m^2} p_1, \quad b_{2m+1} = \frac{1}{2m^2}\left[p_2 - \frac{m^2-m-2}{2m^2} p_1^2\right]$$

and

$$B_{m+1} = \frac{1}{m^2} q_1, \quad B_{2m+1} = \frac{1}{2m^2}\left[q_2 - \frac{m^2-m-2}{2m^2} q_1^2\right].$$

Now we are ready to prove the bound for $|a_{m+1|}$. Solving (2.13) for $(m+1)(2m+1)a_{m+1}^2$ and taking the absolute values in conjunction with an application of the inequalities given in the above Lemma 2.1 we obtain

$$\begin{aligned}
(m+1)(2m+1)\left|a_{m+1}\right|^2 &\leq |b_{2m+1}| + |B_{2m+1}| + |b_{m+1}| \cdot |\varphi_m| + |B_{m+1}| \cdot |\psi_m| + |\varphi_{2m}| + |\psi_{2m}| \\
&\leq \frac{1}{2m^2}\left|p_2 - \frac{m^2-m-2}{2m^2} p_1^2\right| + \frac{1}{m^3}\,|p_1|\,|\varphi_1| + \frac{1}{m}\left|\varphi_2 - \frac{m-1}{2m}\varphi_1^2\right| \\
&\quad + \frac{1}{2m^2}\left|q_2 - \frac{m^2-m-2}{2m^2} q_1^2\right| + \frac{1}{m^3}\,|q_1|\,|\psi_1| + \frac{1}{m}\left|\psi_2 - \frac{m-1}{2m}\psi_1^2\right| \\
&\leq \frac{1}{2m^2}\left(2 - \frac{m^2-m-2}{2m^2}|p_1|^2\right) + \frac{1}{m^3}\,|p_1|\,|\varphi_1| + \frac{1}{m}\left(2 - \frac{m-1}{2m}|\varphi_1|^2\right) \\
&\quad + \frac{1}{2m^2}\left(2 - \frac{m^2-m-2}{2m^2}|q_1|^2\right) + \frac{1}{m^3}\,|q_1|\,|\psi_1| + \frac{1}{m}\left(2 - \frac{m-1}{2m}|\psi_1|^2\right) \\
&\leq \frac{2(2m+1)}{m^2} - \frac{m^2-m-2}{4m^4}|p_1|^2 + \frac{1}{m^3}\,|p_1|\,|\varphi_1| - \frac{m-1}{2m^2}|\varphi_1|^2 \\
&\quad - \frac{m^2-m-2}{4m^4}|q_1|^2 + \frac{1}{m^3}\,|q_1|\,|\psi_1| - \frac{m-1}{2m^2}|\psi_1|^2 \\
&= \frac{2(2m+1)}{m^2} - \frac{m-1}{2m^2}\left(|\varphi_1| - \frac{1}{m(m-1)}|p_1|\right)^2 - \frac{m^2-m-2}{2m^3(m-1)}|p_1|^2 \\
&\quad - \frac{m-1}{2m^2}\left(|\psi_1| - \frac{1}{m(m-1)}|q_1|\right)^2 - \frac{m^2-m-2}{2m^3(m-1)}|q_1|^2.
\end{aligned}$$

For $m \geq 2$ it follws that $(m+1)(2m+1)\left|a_{m+1}\right|^2 \leq \frac{2(2m+1)}{m^2}$ or $|a_{m+1}| \leq \frac{1}{m}\sqrt{\frac{2}{m+1}}$.

For the second part of the theorem, we substitute equation (2.9) in (2.10) to obtain

$$\begin{aligned}
(2m+1)\left|a_{2m+1}\right| &\leq |b_{2m+1}| + |b_{m+1}| \cdot |\varphi_m| + |\varphi_{2m}| \\
&\leq \frac{1}{2m^2}\left(2 - \frac{m^2-m-2}{2m^2}|p_1|^2\right) + \frac{1}{m^3}\,|p_1|\,|\varphi_1| + \frac{1}{m}\left(2 - \frac{m-1}{2m}|\varphi_1|^2\right).
\end{aligned}$$

If $m \geq 2$ then

$$\begin{aligned}
(2m+1)\left|a_{2m+1}\right| &\leq \frac{2m+1}{m^2} - \frac{m^2-m-2}{2m^2}|p_1|^2 + \frac{1}{m^3}\,|p_1|\,|\varphi_1| - \frac{m-1}{2m^2}|\varphi_1|^2 \\
&= \frac{2m+1}{m^2} - \frac{m-1}{2m^2}\left(|\varphi_1| - \frac{1}{m(m-1)}|p_1|\right)^2 - \frac{m^2-m-2}{2m^3(m-1)}|p_1|^2.
\end{aligned}$$

Therefore $|a_{2m+1}| \leq \frac{1}{m^2}$.

For the last part of the theorem, set $a_{mk+1} = 0$ for $2 \leq k < n$. Therefore, the equation (2.7) reduces to

$$\varphi_{nm} = (nm+1)a_{nm+1} - b_{nm+1}. \tag{2.14}$$

Once again, under the assumption $a_{mk+1} = 0$; $(2 \leq k < n)$ we note that the early coefficients in equations (2.2) and (2.3) vanish and we are left with $b_{nm+1} = \frac{1}{mn}\varphi_{mn}$. Therefore

$$\begin{aligned} |a_{nm+1}| &\leq \frac{1}{mn+1}\left(|b_{mn+1}| + |\varphi_{mn}|\right) \\ &= \frac{1}{mn+1}\left(\frac{1}{mn}|\varphi_{mn}| + |\varphi_{mn}|\right) \\ &= \frac{1}{mn+1}\left[\frac{1}{mn}\left(\frac{1}{m}|\varphi_n|\right) + \frac{1}{m}|\varphi_n|\right] \\ &\leq \frac{1}{mn+1}\left(\frac{2}{m^2 n} + \frac{2}{m}\right) = \frac{2}{m^2 n}. \end{aligned}$$

Remark 2.2. For odd $(m = 2)$ bi-close-to-convex functions, the above Theorem 1.1 yields $|a_3| \leq \sqrt{1/6}$ and $|a_5| \leq 1/4$ which are far better bounds than $|a_3| \leq 1$ and $|a_5| \leq 1$ obtained by Pommerenke [10] for odd close-to-convex functions. This is also the case for the coefficients of m-fold symmteric $(m \geq 3)$ bi-close-to-convex functions given in Theorem 1.1 versus those obtained by Pommerenke [10] for the m-fold symmetric close-to-convex functions. Our Theorem 1.1 also advances the bounds obtained by the authors in ([4], Theorem 2.2).

Remark 2.3. Srivastava et al [12, Theorem 3] considered the class of m-fold symmetric bi-univalent functions $\sqrt[m]{f(z^m)}$ for which $Re\{f'(z)\} > \beta; 0 \leq \beta < 1$. This is a subclass of m-fold symmetric bi-close-to-convex functions. For $m \geq 2$ the bounds presented by our Theorem 1.1 for $|a_{m+1}|$ and $|a_{2m+1}|$ are far better than those given in [12, Theorem 3].

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