A note on closedness of algebraic sum of sets

Hubert Przybycień

Faculty of Mathematics and Computer Science, Adam Mickiewicz University Umultowska 87, 61-614 Poznań, Poland

E-mail: `hubert@amu.edu.pl`

Abstract

In this note we generalize the fact that in topological vector spaces the algebraic sum of closed set A and compact set B is closed. We also prove some conditions that are equivalent to reflexivity of Banach spaces.

2010 Mathematics Subject Classification. **46A22**. 46N10
Keywords. Closed convex sets.

1 Inroduction

Let X be a Hausdorff topological vector space. By $\mathcal{C}(X)$ we denote the family of all closed subsets of X and by $\mathcal{K}(X)$ the family of all compact subset of X. For a nonempty subsets $A, B \subset X$ we define the algebraic sum (*Minkowski sum*) as follow

$$A + B = \{a + b : a \in A, b \in B\}.$$

It is wll known that if A, B are closed sets then $A + B$ need not to be closed, but also it is known that if $A \in \mathcal{C}(X)$ and $B \in \mathcal{K}(X)$ then $A + B \in \mathcal{C}(X)$. In this note we prove the last result with some more abstract point of view.

2 Reflexive spaces

In this section we give some results which show the connections of closedness of algebraic sum of sets with reflexivity of Banach spaces.

Theorem 2.1. Let X be a Banach space and let B be the closed unit ball in X. Then X is a reflexive Banach space if and only if for every closed convex and bounded subset A of X the algebraic sum $A + B$ is closed.

Proof. (Necessity.) Assume that X is a reflexive Banach space, then every closed bounded and convex set is a weakly compact. Since algebraic sum of two weakly compact sets is again weakly compact we conclude that $A + B$ is a weakly compcact set, and therefore closed.
(Sufficiency.) Assume that X is not reflexive Banach space. Then by Theorem of James there exists a continuous linear functional
$f : X \to \mathbb{R}$, such that $||f|| = 1$ and $f(x) < 1$ for every $x \in B$.
Let

$$A = \{x \in X : f(x) \geqslant 1, ||x|| \leqslant 2\}$$

Then A is a closed bounded and convex set but $A + B$ is not closed. Q.E.D.

Tbilisi Mathematical Journal 9(2) (2016), pp. 71–74.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 14 February 2016.
Accepted for publication: 16 September 2016.

Now using the theorem 1 we prove the following theorem

Theorem 2.2. Let X be a Banach space and denote by τ_s the norm topology. Then X is a reflexive Banach space if and only if there exists a linear Hausdorff topology τ on X such that $\tau \subset \tau_s$ and every closed convex and bounded subset of X is compact in the topology τ.

Proof. (Necessity.) If X is a reflexive Banach space then we can take τ to be equal the weak topology on X.
(Sufficiency.) Suppose that every closed convex and bounded subset of X is compact in the topology τ. Then for any closed convex and bounded set A the algebraic sum $A+B$, (where B is a closed unit ball in X) is compact in τ, and hence closed in τ, but since $\tau \subset \tau_s$ it is also closed in τ_s. Thus by theorem 1 the space X is reflexive. Q.E.D.

Definition 2.3. Let X be a topological vector space and $A, B \subset X$. We say that the sets A and B can be strictly separated by hyperplane if there exists a continuous functional $f : X \to \mathbb{R}$, and real numbers u, v such that

$$f(x) < u < v < f(y) \text{ for all } x \in A, y \in B.$$

It is well known that in locally convex topological vector spaces a closed convex sets can be strictly separated from disjoint compact convex sets by a hyperplane.

The next theorem show the connections between a reflexivity and separation.

Theorem 2.4. Let X be a Banach space. Then X is reflexive Banach space if and only if every two disjoint closed bounded and convex sets can be strictly separated by a hyperplane.

Proof. (Necessity.) Let X be a reflexive Banach space and let A, B be a two nonempty disjoint closed convex and bounded sets. Then X with the weak topology τ_w is a locally convex space in which the sets A, B are compact and thus by a separation theorem A, B can be strictly separated by a hyperplane in (X, τ_w). Surely the sets can be also separated by a hyperlane in X since every weak continuous linear functional is also continuous in the norm topology.
(Sufficiency.) Assume that every two disjoint closed bounded and convex sets can be strictly separated by a hyperplane. Let A, B be two disjoint closed bounded and convex sets, we prove that in this case the set $A - B$ is closed. To prove this assume contrary that $A - B$ is not closed. Without loss of generality we may assume that $0 \in \overline{A - B} \setminus (A - B)$. By assumption there exists a continuous linear functional $f : X \to \mathbb{R}$ such that

$$f(x) < u < v < f(y) \text{ for all } x \in A, y \in B,$$

for some $u, v \in \mathbb{R}$.
Therefore

$$f(w) < u - v \text{ for all } w \in A - B,$$

hence

$$f(w) \leqslant u - v < 0 \text{ for all } w \in \overline{A - B},$$

thus $0 \notin \overline{A-B}$ and we get a contradiction.
So we have just proved that for any two disjoint closed bounded and convex sets A, B, the set $A - B$ is closed.
Now let A, B be any two closed convex and bounded sets. Then there exists a vector $x \in X$ such that the sets $x + A$ and B are disjoint, and hence (by what we have just proved above) the set $(x + A) - B$ is closed. But then the set $[(x + A) - B] - x = A - B$ is closed too.
Finally from the equality $A + B = A - (-B)$ we obtain that the sum $A + B$ is closed for any two closed convex and bounded set therefore by theorem 1 the space X is a reflexive Banach space. Q.E.D.

The last theorem in this section gives also a condition which is equivalent to reflexivity for Banach spaces.

Theorem 2.5. A Banach space X is reflexive if and only if every closed and convex set has an element with minimal norm.

Proof. (Necessity.) Assume that X is a reflexive Banach space and let A be a nonempty closed and convex set. Without loss of generality we may assume that $0 \notin A$. Let $a = \inf_{x \in A} ||x||$ we'll show that there is an element $u \in A$ such that $||u|| = a$. To prove this let

$$B_n = \{x \in X : ||x|| \leqslant a + n^{-1}\}$$

then the sets $C_n = B_n \cap A$ form a descending sequence of nonempty closed convex and bounded and hence weakly compact and convex sets. Thus

$$C = \bigcap_n C_n \neq \varnothing$$

and it is easy to observe that for $v \in C$ we have $||v|| = a$.
(Sufficiency.) Assume that every closed and convex set has an element with minimal norm. Let $f : X \to \mathbb{R}$ be a continuous linear functional with $||f|| = 1$. Then the set $A = \{x \in X : f(x) \geqslant 1\}$ is closed and convex therefore by assumption it has an element x_0 with minimal norm. It is easy to observe that $||x_0|| = 1$ and thus f attains its maximum on unit ball. Hence by theorem of James X is reflexive Banach space. Q.E.D.

3 Closedness of algebraic sum

In this section we prove a theorem which is a generalization of the fact that in Hausdorff topological vector space the algebraic sum of closed set A and compact set B is closed.

Theorem 3.1. Let X, Y, Z be Hausdorff topological spaces and let $f : X \times Y \to Z$ be a function such that:

(a) f is continuous, and for every $y \in Y$ the function $f(\cdot, y)$ is an injection.

(b) there exists a continuous function $\varphi : Y \times Z \to X$ such that $f(\varphi(y, z), y) = z$ for all $(y, z) \in Y \times Z$.

Assume that $A \subset X \times Y$ is a closed set such that:

(c) the projection $\pi_Y(A) = \{y \in Y : (x, y) \in A\}$ of the set A is compact.

Then the image $f(A) = \{z \in Z : z = f(x, y) \text{ for some } (x, y) \in A\}$ of the set A is closed.

Proof. Let $z_\alpha \in f(A), \alpha \in \Lambda$ be an MS-sequence tending to z_0. Then there exists a MS-sequences $x_\alpha, y_\alpha, \alpha \in \Lambda$ such that $(x_\alpha, y_\alpha) \in A$ and $z_\alpha = f(x_\alpha, y_\alpha)$. Therefore $x_\alpha \in \pi_X(A), y_\alpha \in \pi_Y(A)$, but since $\pi_Y(A)$ is compact there exists a MS-subseqence $y_\beta, \beta \in \Sigma$ of the MS-sequence $y_\alpha, \alpha \in \Lambda$ such that $y_\beta \to y_0 \in \pi_Y(A)$. Moreover by the continuity of φ we get $x_\beta = \varphi(y_\beta, z_\beta) \to \varphi(y_0, z_0) = x_0 \in X$. But A is closed and $(x_\beta, y_\beta) \in A$, $(x_\beta, y_\beta) \to (x_0, y_0)$ thus $(x_0, y_0) \in A$. Therefore $f(x_\beta, y_\beta) \to f(x_0, y_0) \in f(A)$. But $f(x_\beta, y_\beta) \to z_0$ and thus $z_0 = f(x_0, y_0) \in f(A)$ and the proof is complete. Q.E.D.

Corollary 3.2. Let $(G, +)$ be a Hausdorff topological group and let $B, C \subset G$, be subsets such that B is closed and C is compact. Then the algebraic sum $B + C$ is closed.

Proof. To prove this it is enough to take in theorem 5 : $X = Y = Z = G$, $A = B \times C$, $f(x, y) = x + y$, $\varphi(y, z) = z - y$. Q.E.D.

References

[1] A. Alexiewicz, *Functional Analysis* (1969) (in Polish)

[2] J. Dieudonne, *Sur la sparation des ensembles convexes*, Math. Ann. **163** (1966)

[3] Robert C. James, *Reflexivity and the sup of linear functionals*, Israel J. Math **13** (1972), 3-4

[4] W. Rudin, *Functional Analysis*, McGraw-Hill Book Company, New York, 1973