On algebraic solitons for geometric evolution equations on three-dimensional Lie groups

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Abstract

The relationship between algebraic soliton metrics and self-similar solutions of geometric evolution equations on Lie groups is investigated. After discussing the general relationship between algebraic soliton metrics and self-similar solutions to geometric evolution equations, we investigate the cross curvature flow and the second order renormalization group flow on simply-connected, three-dimensional, unimodular Lie groups, providing a complete classification of left invariant algebraic solitons that give rise to self-similar solutions of the corresponding flows on such spaces.

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1 Introduction

A major theme in modern differential geometry and topology is the use of geometric evolution equations to improve a certain geometric structure or quantity on a smooth manifold $\mathcal{M}$. Perhaps the most celebrated example of this is the Ricci flow as introduced by Richard Hamilton in [18], where one views the Ricci flow equation

$$\frac{\partial \mathbf{g}}{\partial t} = -2\mathbf{Rc}\left[\mathbf{g}\right]; \qquad \mathbf{g}_0 = \mathbf{g}(0) \tag{1.1}$$

as something akin to a heat equation for the evolution of the metric tensor $\mathbf{g}_0$ on the underlying manifold structure. Of particular interest in this situation is the evolution of geometric quantities associated with solutions to (1.1) and the singularities that form (in either finite or infinite time) for the flow. Using a rescaling of the flow, such singularities are typically modeled with self-similar solutions and are referred to as *Ricci soliton* metrics. Ricci soliton metrics are thusly solutions to (1.1) of the form $\mathbf{g}(t) = c(t)\varphi_t^*\mathbf{g}_0$, where φ_t is one-parameter family of diffeomorphsims of $\mathcal{M}$ and c is a positive scalar function, and they have been studied extensively since Hamilton's introduction of the Ricci flow. We refer the reader to [5], [7], [8], [19], and references therein for a further discussion of the role of Ricci solitons in the study of the Ricci flow.

In addition to playing a role in the study of the Ricci flow on closed manifolds, Ricci solitons have also played an important role in the study of preferred and/or distinguished metrics on Lie groups. Namely, in [25], Lauret introduced Ricci solitons as a natural generalization of Einstein metrics for nilpotent Lie groups. More to the point, in [28] Milnor

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establishes that a nilpotent Lie group cannot carry a left invariant (Riemannian) Einstein metric (i.e., a metric $\mathbf{g}$ satisfying $\mathbf{Rc}\,[\mathbf{g}] = \lambda\mathbf{g}$ for some scalar λ). Whereas an Einstein metric $\mathbf{g}$ is a solution of the Ricci flow (1.1) that evolves only by scaling, a Ricci soliton is suitably interpreted as a geometric fixed point for the flow as a Ricci soliton is a metric that evolves by scaling and diffeomorphism. Furthermore, a Ricci soliton $\mathbf{g}$ must satisfy the Ricci soliton equation

$$\beta\mathbf{g} + \mathcal{L}_{\mathbf{g}}\mathbf{X} = -2\mathbf{Rc}\,[\mathbf{g}]\,, \tag{1.2}$$

where β is a scalar, $\mathbf{X}$ is a vector field on $\mathcal{M}$, and $\mathcal{L}_{\mathbf{g}}\mathbf{X}$ is the Lie derivative of the metric $\mathbf{g}$ in the direction of the vector field $\mathbf{X}$. When $\mathbf{g}$ is an Einstein metric, one can take the vector field $\mathbf{X}$ to be a Killing field of $\mathbf{g}$, while when $\mathbf{g}$ is a non-trivial Ricci soliton (i.e., a Ricci soliton that is not an Einstein metric), the vector field $\mathbf{X}$ appearing in (1.2) will be unique up to a Killing field of $\mathbf{g}$. See [5] for full details.

With this generalization in mind, Lauret [25] establishes that Ricci soliton metrics on nilpotent Lie groups can be found via algebraic methods alone. Namely, denoting the (1,1)-Ricci operator by $\widehat{\mathbf{Rc}}\,[\mathbf{g}]$, Lauret notes that any left invariant metric $\mathbf{g}$ (with a corresponding scalar β) such that $\widehat{\mathbf{Rc}}\,[\mathbf{g}] - \beta\mathrm{Id}$ is a derivation of the Lie algebra gives rise to a Ricci soliton metric. Lauret refers to such metrics as *algebraic Ricci solitons*. Further, Lauret shows that on simply-connected nilpotent Lie groups a Ricci soliton metric is also necessarily algebraic and that Ricci soliton metrics on simply-connected nilpotent Lie groups are unique up to a constant scalar multiple and an automorphism of the corresponding Lie algebra. As such, one can argue that algebraic Ricci solitons are candidates for a “best metric” on a class of Lie groups. See [25] for details.

After Lauret introduced algebraic Ricci solitons, they have been studied extensively on Lie groups and homogeneous spaces by numerous authors. It is interesting to note that to date, the only known examples of non-trivial algebraic Ricci solitons on non-compact Lie groups occur on solvable Lie groups. This situation is identical to that of the case of Einstein metrics. See [21], [22], [23], and [24] for further discussion.

In the current paper, our aim is twofold. We begin by showing that Lauret’s ideas pertaining to algebraic solitons apply equally well to arbitrary (but subject to the appropriate conditions) geometric evolution equations for left invariant Riemannian metrics on simply-connected Lie groups. This builds off of the work of Glickenstein in [15] and Lauret in [25]. We then apply these ideas to the cross curvature flow and the second order renormalized group flow on simply-connected, three-dimensional, unimodular Lie groups, where we classify those algebraic solitons that give rise to self-similar solutions of the respective flow. These results can be compared to the results in [15], where Glickenstein investigates the Ricci flow and cross curvature flow and corresponding soliton metrics on three-dimensional homogeneous geometries using Riemannian groupoids, and [17], where Glickenstein and Wu investigate the second order renormalized group flow on three-dimensional unimodular Lie groups using the bracket flow and the evolution of the structure constants of three-dimensional Lie algebras. For more on the use of the bracket flow and the evolution of structure constants of the Lie algebra, see [16] and [24]. In particular, in [16] the authors investigate the Ricci flow on the non-Abelian, three-dimensional, unimodular metric Lie algebras by studying the evolution of the structure constants.

The outline of the paper is as follows. In Section 2 we present a brief review of the cross curvature flow on locally homogeneous three-dimensional manifolds (Section 2.1.1) and the second order renormalization group flow (Section 2.1.2) on smooth Riemannian manifolds. In Section 2.1.3, we introduce the appropriate conditions that must be satisfied by a geometric evolution equation in order to tie algebraic solitons together with self-similar solutions of the corresponding flow. We then follow this up with a brief discussion of some of the necessary algebraic conditions that a simply-connected Lie group and its corresponding Lie algebra must satisfy in order to be able to support an algebraic soliton for a given geometric evolution equation.

We begin Section 3 by reviewing Milnor's construction of the so-called *Milnor frames* on simply-connected, three-dimensional, unimodular Lie groups equipped with a left invariant Riemannian metric. This is followed by a review of the geometric expressions and tensors associated with the cross curvature flow and the second order renormalized group flow as they appear in a Milnor frame. We conclude with the classification of algebraic solitons that give rise to self-similar solutions of the cross curvature flow and second order renormalized group flow on simply-connected, three-dimensional, unimodular Lie groups in Section 3.2 - Section 3.7.

2 Geometric preliminaries and notation

2.1 Geometric evolution equations on 3-manifolds

In this section we recall the definitions of the cross curvature flow (XCF) and the second order renormalization group (RG-2) flow on a three-dimensional Riemannian manifold $(\mathcal{M}, \mathbf{g})$. For general Riemannian manifolds, neither flow is well-posed and one is not able to guarantee short-time existence of solutions. However, if one restricts their attention to left invariant metrics on Lie groups (or more generally to homogenous spaces), then short-time existence and uniqueness follows from the standard existence and uniqueness results of ordinary differential equations. On a Lie group, for example, the selection of a frame at the identity is tantamount to selecting global coordinate functions on the fiber of the bundle of positive-definite symmetric (0,2)-tensors and the study of the geometric evolution equation in question reduces to the evolution of the coefficient functions of the metric tensor with respect to the selected frame. On a three-dimensional Lie group this results in a system of six coupled ordinary differential equations, but we will see that like the situation for the Ricci flow on three-dimensional homogeneous geometries, this can be reduced to a system of three ordinary differential equations for both the XCF and RG-2 flow.

For a more detailed discussion of the cross curvature flow, we refer the readers to [6], where the flow was introduced by Chow and Hamilton on three-dimensional manifolds with strictly positive or strictly negative sectional curvature, and to [3] and [4], where the authors study the XCF on three-dimensional homogeneous geometries. Additionally, in [1], Buckland establishes the existence of solutions to the cross curvature flow on 3-manifolds equipped with an initial Riemannian metric that has everywhere positive or everywhere negative sectional curvature. For examples of the XCF on a square torus bundle over $\mathbf{S}^1$ and on $\mathbf{S}^2$-bundles over $\mathbf{S}^1$, see [27], and for a program trying to utilize the XCF for the purpose of trying to to show that the moduli space of negatively curved metrics on a closed hyperbolic 3-manifold is path-connected, see [9].

For a more detailed discussion of the RG-2 flow we refer the reader to [12], where the authors focus on a geometric introduction to the RG-2 flow, and [13], where the authors study the RG-2 flow on three-dimensional homogeneous spaces in a spirit similar to the analysis of the Ricci flow on three-dimensional homogeneous spaces carried out by Isenberg and Jackson in [20].

2.1.1 XCF

Before we introduce the positive and negative cross curvature flow (±XCF) on a locally homogeneous three-dimensional manifold as defined in [3], a few remarks are in order. In [6], Chow and Hamilton considered the XCF on three-dimensional Riemannian manifolds $(\mathcal{M}, \mathbf{g})$ where the sectional curvatures of the initial metric $\mathbf{g}$ are strictly positive or strictly negative, and the sign of the XCF was chosen depending on the sign of the sectional curvatures ($\pm$ XCF when the sign of the sectional curvatures of $\mathbf{g}$ are $\mp$). One does not expect to obtain general existence results for the XCF, but as noted above, by restricting one's attention to Lie groups or (locally) homogeneous spaces, short-time existence of solutions is easily obtained from standard ODE results (regardless of the signs of the sectional curvatures). Following [3] and [6] we will now define the cross curvature tensor and the positive and negative XCF on locally homogeneous manifolds.

The following construction/definition of the cross curvature tensor is taken from [6]. Let $(\mathcal{M}, \mathbf{g})$ be a three-dimensional Riemannian manifold and let $\mathbf{e}_i$ denote a local frame field on $\mathcal{M}$ with corresponding dual co-frame field ω^i, $1 \leq i \leq 3$. Let the Ricci tensor of the metric $\mathbf{g} = g_{ij}\omega^i \otimes \omega^j$ be denoted by $\mathbf{Rc}[\mathbf{g}] = \mathbf{Rc} = R_{ij}\omega^i \otimes \omega^j$, with the corresponding scalar curvature denoted by $\mathrm{S}[\mathbf{g}] = \mathrm{S}$. We let the Einstein tensor of the metric $\mathbf{g}$ be denoted by $\mathbf{E}[\mathbf{g}] = \mathbf{E} = \mathbf{Rc} - \frac{\mathrm{S}}{2}\mathbf{g}$ and define a $(2,0)$-tensor $\mathbf{P}[\mathbf{g}] = \mathbf{P}$ by raising the indices on the Einstein tensor $\mathbf{E}$. The component functions of $\mathbf{P}$ with respect to a local frame are thusly

$$P^{ij} = g^{ik}g^{jl}R_{kl-}\frac{\mathrm{S}}{2}g^{ij},$$

where g^{ij} denote the component functions of $\mathbf{g}^{-1}$. Provided that $\left(P^{ij}\right)$ is invertible, we denote $\left(P^{ij}\right)^{-1}$ by (V_{ij}) and then define the cross curvature tensor $\mathbf{H}[\mathbf{g}] = \mathbf{H} = H_{ij}\omega^i \otimes \omega^j$ by the component functions

$$H_{ij} = \left(\frac{\det P^{ij}}{\det g^{ij}}\right) V_{ij}. \tag{2.1}$$

As observed in [3] and [27], the cross curvature tensor $\mathbf{H}$ takes a particularly simple form in a local orthonormal frame $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ where the Ricci tensor $\mathbf{Rc}$ is diagonalized. For any permutation (l, m, n) of $(1, 2, 3)$, let $K_l = K(\mathbf{e}_m \wedge \mathbf{e}_n)$ denote the principal sectional curvature of the plane that is orthogonal to $\mathbf{e}_l$. We thus have $R_{ll} = K_m + K_n$, and it follows that the cross curvature tensor $\mathbf{H}$ is diagonalized with respect to the indicated frame and the non-zero component functions of $\mathbf{H}$ are

$$H_{ll} = K_m K_n, \qquad (l, m, n) \text{ a permutation of } (1, 2, 3). \tag{2.2}$$

Note that (2.2) defines the cross curvature tensor when $\mathbf{P}[\mathbf{g}]$ fails to be invertible. Additionally, note that $\mathbf{H}$ obeys the following important homogeneity property for a scaling of the metric tensor $\mathbf{g}$:

$$\mathbf{H}[c\mathbf{g}] = \frac{1}{c}\mathbf{H}[\mathbf{g}], \quad c \in \mathbb{R}_{>0}. \tag{2.3}$$

Following [3], we now define the XCF on locally homogeneous three-dimensional manifolds.

Definition 2.1 (Cross Curvature Flow)**.** Let $(\mathcal{M}, \mathbf{g}_0)$ be a locally homogeneous three-dimensional manifold. The *positive cross curvature flow* (+XCF) is defined by

$$\frac{\partial \mathbf{g}}{\partial t} = 2\mathbf{H}[\mathbf{g}]; \qquad \mathbf{g}(0) = \mathbf{g}_0,$$

and the *negative cross curvature flow* (-XCF) is defined by

$$\frac{\partial \mathbf{g}}{\partial t} = -2\mathbf{H}[\mathbf{g}]; \qquad \mathbf{g}(0) = \mathbf{g}_0.$$

Due to the original use of the cross curvature flow by Chow and Hamilton, there is not a clear choice as to what should be the forward direction for the XCF when the signs of the sectional curvature vary, and both directions of the flow on three-dimensional homogeneous spaces are investigated in [2], [3], and [4] in a spirit similar to the analysis carried out by Isenberg and Jackson for the Ricci flow on homogeneous three-dimensional geometries in their seminal paper [20].

2.1.2 The RG-2 flow

The RG-2 flow is a second order approximation of the renormalization group flow that corresponds to a perturbative analyses of nonlinear sigma model quantum field theories from a world sheet into $(\mathcal{M}, \mathbf{g})$, and unlike the XCF, which has currently only been defined and investigated on three-dimensional Riemannian manifolds, the RG-2 flow has been studied in arbitrary dimensions. For an introduction to the physics of the renormalization group flow, we refer the reader to [10] and [11]. We will mostly follow the geometrical introduction to the renormalization group flow as found in the work of Gimre, Guenther, and Isenberg in [12] and [13].

The evolution of the metric tensor $\mathbf{g} = g_{ij}\omega^i \otimes \omega^j$ under the nonlinear sigma quantum field theories takes the form

$$\frac{\partial \mathbf{g}}{\partial t} = -\alpha \mathbf{Rc}[\mathbf{g}] - \frac{\alpha^2}{2}\mathbf{Rm}^2[\mathbf{g}] + \mathcal{O}\left(\alpha^3\right), \tag{2.4}$$

where α denotes a positive coupling constant and the tensor $\mathbf{Rm}^2[\mathbf{g}] = \check{R}_{ij}\omega^i \otimes \omega^j$ involves quadratic terms stemming from the full Riemannian curvature tensor of the metric tensor $\mathbf{g}$. The component functions of $\mathbf{Rm}^2[\mathbf{g}] = \check{R}_{ij}\omega^i \otimes \omega^j$ are

$$\check{R}_{ij} = Rm_{iklm}Rm_{jpqr}g^{kp}g^{lq}g^{mr},$$

where $\mathbf{Rm}[\mathbf{g}] = Rm_{ijkl}\omega^i \otimes \omega^j \otimes \omega^k \otimes \omega^l$ denotes the Riemannian curvature tensor of $\mathbf{g}$. After an appropriate rescaling of the parameter t, one finds that the second order approximation of the renormalization group flow takes the form

$$\frac{\partial \mathbf{g}}{\partial t} = -2\mathbf{Rc}[\mathbf{g}] - \frac{\alpha}{2}\mathbf{Rm}^2[\mathbf{g}]. \tag{2.5}$$

We will denote the tensor $-2\mathbf{Rc}[\mathbf{g}] - \frac{\alpha}{2}\mathbf{Rm}^2[\mathbf{g}]$ by $\mathbf{RG}[\mathbf{g}]$ and define the second order renormalization group flow (the RG-2 flow) for a Riemannian metric on a smooth manifold $\mathcal{M}$ as follows.

Definition 2.2 (RG-2 Flow)**.** Let $(\mathcal{M}, \mathbf{g}_0)$ be a smooth Riemannian manifold. The *second order renormalization group (*RG-2*) flow* is the geometric evolution equation for the Riemannian metric $\mathbf{g}_0$ defined by

$$\frac{\partial \mathbf{g}}{\partial t} = \mathbf{RG}[\mathbf{g}] = -2\mathbf{Rc}[\mathbf{g}] - \frac{\alpha}{2}\mathbf{Rm}^2[\mathbf{g}]; \qquad \mathbf{g}(0) = \mathbf{g}_0. \tag{2.6}$$

The two terms comprising the tensor $\mathbf{RG}[\mathbf{g}]$ behave in the following manner under a positive scaling of the metric tensor $\mathbf{g}$:

$$\mathbf{Rc}[c\mathbf{g}] = \mathbf{Rc}[\mathbf{g}] \qquad \text{and} \qquad \mathbf{Rm}^2[c\mathbf{g}] = \frac{1}{c}\mathbf{Rm}^2[\mathbf{g}], \qquad c \in \mathbb{R}_{>0}. \tag{2.7}$$

Note that the first order approximation of the renormalization group flow is the Ricci flow. In fact, in [12] and [13], the authors take this viewpoint and investigate the RG-2 flow mathematically as a nonlinear deformation of the Ricci flow. In [13], the authors study the RG-2 flow on three-dimensional homogenous geometries with an emphasis on how the asymptotic behavior of the RG-2 flow depends on the parameter α and how this compares with the asymptotic behavior of the Ricci flow, where they use the results of [20] as their benchmark and guide.

The short-time existence and uniqueness problem for the RG-2 flow is unsettled for a general Riemannian manifold. However, in [14], the authors establish the short-time existence and uniqueness for the second order renormalization group flow initial value problem on closed Riemannian manifolds $(\mathcal{M}, \mathbf{g}_0)$ in general dimensions in the case where the sectional curvatures of the initial metric $\mathbf{g}_0$ satisfy $1 + \alpha K_P > 0$ at all points $p \in \mathcal{M}$ and two-planes $P \subset T_p\mathcal{M}$. As noted above, we remind the reader that the existence and uniqueness of solutions to the RG-2 flow on Lie groups (or more generally homogeneous geometries) is guaranteed as the flow reduces to a system of ordinary differential equations for the inner product at a chosen basepoint.

2.1.3 Solitons

We now make several general observations concerning the XCF and the RG-2 flow and the relationship between self-similar solutions of the flows and so-called solitons. For simplicity, we will assume that $(\mathcal{M}, \mathbf{g})$ is a three-dimensional Riemannian manifold. We let $\mathbf{T} : \mathrm{Sym}^2(T^*\mathcal{M}) \to \mathrm{Sym}^2(T^*\mathcal{M})$ be a bundle map of the bundle of symmetric (0,2)-tensors on $\mathcal{M}$ and consider a geometric evolution equation on $\mathcal{M}$ of the form

$$\frac{\partial \mathbf{g}}{\partial t} = \mathbf{T}[\mathbf{g}]; \qquad \mathbf{g}(0) = \mathbf{g}_0. \tag{2.8}$$

Observe that the Ricci flow, the RG-2 flow and the XCF are all of the indicated form. Following [15], we take note of the following important properties that can be satisfied by the bundle map $\mathbf{T}$.

Definition 2.3 (Natural and Homogeneous)**.** We say that $\mathbf{T} : \mathrm{Sym}^2\left(T^*\mathcal{M}\right) \to \mathrm{Sym}^2\left(T^*\mathcal{M}\right)$ is

1. *natural* if for all diffeomorphisms $\varphi : \mathcal{M} \to \mathcal{M}$ and all Riemannian metrics $\mathbf{g}$ on $\mathcal{M}$, $\mathbf{T}$ satisfies $\mathbf{T}\left[\varphi^*\mathbf{g}\right] = \varphi^*\mathbf{T}\left[\mathbf{g}\right]$, and
2. *homogeneous of degree q* if for any positive scaling $c\mathbf{g}$ of a Riemannian metric $\mathbf{g}$, $\mathbf{T}$ satisfies $\mathbf{T}\left[c\mathbf{g}\right] = c^q\mathbf{T}\left[\mathbf{g}\right]$.

Remark 2.4. Note that $\mathbf{Rc}$, $\mathbf{H}$ and $\mathbf{RG}$ are all natural, while $\mathbf{Rc}$ is homogeneous of degree zero and $\mathbf{H}$ is homogeneous of degree $q = -1$. $\mathbf{RG}$ fails to be homogeneous as it is comprised of two homogeneous terms of different degrees (2.7). Further, note that the definition of natural amounts to saying that the symmetry group of the evolution equation (2.8) contains the full diffeomorphism group of $\mathcal{M}$. In the case where $(\mathcal{M}, \mathbf{g})$ is a homogeneous space it follows that $\mathbf{T}\left[\mathbf{g}\right]$ is invariant under the group acting on $\mathcal{M}$. In particular, if $\mathbf{T}$ is natural and $\mathbf{g}$ is a left invariant metric on a Lie group, then $\mathbf{T}\left[\mathbf{g}\right]$ will be left invariant as well.

Given a geometric evolution equation on $\mathcal{M}$ of the form (2.8), a solution $\mathbf{g}(t)$ that evolves by scaling and diffeomorphism is said to be a *self-similar solution*. Thus a self-similar solution is of the form $\mathbf{g}(t) = c(t)\varphi_t^*\mathbf{g}_0$, where φ_t is a one-parameter family of diffeomorphisms of $\mathcal{M}$ with $\varphi_0 = \mathrm{Id}$ and $c(t)$ is a real-valued function satisfying $c(0) = 1$. Such a solution should be regarded as a geometric fixed point for the flow (2.8).

Related to self-similar solutions of the flow will be the so-called $\mathbf{T}$-*soliton structures*. The quadruple $(\mathcal{M}, \mathbf{g}, \mathbf{X}, \beta)$, where $\mathbf{X}$ is a vector field on $\mathcal{M}$ and $\beta \in \mathbb{R}$, is said to be a $\mathbf{T}$-*soliton structure* for the geometric evolution equation (2.8) if

$$\beta\mathbf{g} + \mathcal{L}_{\mathbf{X}}\mathbf{g} = \mathbf{T}\left[\mathbf{g}\right], \tag{2.9}$$

where $\mathcal{L}_{\mathbf{X}}\mathbf{g}$ denotes the Lie derivative of the metric $\mathbf{g}$ in the direction of the vector field $\mathbf{X}$. $(\mathcal{M}, \mathbf{g}, \mathbf{X}, \beta)$ is said to be *expanding* if β is positive, *steady* if β is zero, and *shrinking* if β is negative. The reason for the terminology expanding, steady, and shrinking will be evident from Proposition 2.5. We will refer to the metric $\mathbf{g}$ from the quadruple $(\mathcal{M}, \mathbf{g}, \mathbf{X}, \beta)$ as a $\mathbf{T}$-soliton, emphasizing $\mathbf{X}$ and β only when needed. As per usual, the vector field $\mathbf{X}$ in the soliton structure $(\mathcal{M}, \mathbf{g}, \mathbf{X}, \beta)$ is unique up to the addition of a Killing field of the metric tensor $\mathbf{g}$.

It is tempting to use the $\mathbf{T}$-soliton equation to define self-similar solutions to (2.8), but as the following proposition shows, this requires that $\mathbf{T}$ be natural and homogeneous. The key link between self-similar solutions of the evolution equation (2.8) and soliton metrics was established in [15] and is a straight forward generalization of the relationship between Ricci solitons and self-similar solutions of the Ricci flow.

Proposition 2.5. Let $\mathbf{T} : \mathrm{Sym}^2\left(T^*\mathcal{M}\right) \to \mathrm{Sym}^2\left(T^*\mathcal{M}\right)$ be a bundle map and assume that $\mathbf{T}$ is natural.

1. If $\mathbf{g}(t) = c(t)\varphi_t^*\mathbf{g}_0$ is a self-similar solution to the geometric evolution equation (2.8), then $\mathbf{g}_0$ is a $\mathbf{T}$-soliton.

2. If $\mathbf{g}_0$ is a steady $\mathbf{T}$-soliton, then there is a self-similar solution to (2.8) of the form $\mathbf{g}(t) = \varphi_t^*\mathbf{g}_0$ (i.e., $\mathbf{g}_0$ evolves by diffeomorphisms).

3. If $\mathbf{g}_0$ is a $\mathbf{T}$-soliton for (2.8) and $\mathbf{T}$ is homogeneous of degree q, then there is a self-similar solution to (2.8) of the form $\mathbf{g}(t) = c(t)\varphi_t^*\mathbf{g}_0$.

Proof. *1*. Assume that $\mathbf{g}(t) = c(t)\varphi_t^*\mathbf{g}_0$ is a self-similar solution to (2.8) and let $\mathbf{X}$ be the vector field on $\mathcal{M}$ defined by $\mathbf{X}(p) = \frac{d}{dt}\left(\varphi_t(p)\right)$, $p \in \mathcal{M}$. Differentiating $\mathbf{g}(t) = c(t)\varphi_t^*\mathbf{g}_0$ with respect to t and using properties of pullbacks we obtain

$$\frac{\partial \mathbf{g}(t)}{\partial t} = c'(t)\varphi_t^*\mathbf{g}_0 + c(t)\varphi_t^*\left(\mathcal{L}_{\mathbf{X}}\mathbf{g}_0\right) = \varphi_t^*\left(c'(t)\mathbf{g}_0 + c(t)\mathcal{L}_{\mathbf{X}}\mathbf{g}_0\right). \tag{2.10}$$

Using the assumptions that $\mathbf{T}$ is natural and that $\mathbf{g}(t) = c(t)\varphi_t^*\mathbf{g}_0$ is a self-similar solution to (2.8) we have also that

$$\frac{\partial \mathbf{g}(t)}{\partial t} = \mathbf{T}\left[c(t)\varphi_t^*\mathbf{g}_0\right] = \varphi_t^*\mathbf{T}\left[c(t)\mathbf{g}_0\right] \tag{2.11}$$

Evaluating (2.10) and (2.11) at time $t = 0$, we find $c'(0)\mathbf{g}_0 + \mathcal{L}_{\mathbf{X}}\mathbf{g}_0 = \mathbf{T}\left[\mathbf{g}_0\right]$ and we conclude that $\mathbf{g}_0$ is a $\mathbf{T}$-soliton for the geometric flow (2.8).

2. Let $\mathbf{X}$ be a vector field such that $\mathcal{L}_{\mathbf{X}}\mathbf{g}_0 = \mathbf{T}\left[\mathbf{g}_0\right]$ and let φ_t be the one-parameter family of diffeomorphisms generated by $\mathbf{X}$. It follows that

$$\frac{\partial \varphi_t^*\mathbf{g}_0}{\partial t} = \varphi_t^*\left(\mathcal{L}_{\mathbf{X}}\mathbf{g}_0\right) = \varphi_t^*\mathbf{T}\left[\mathbf{g}_0\right] = \mathbf{T}\left[\varphi_t^*\mathbf{g}_0\right],$$

and we conclude that $\mathbf{g}(t) = \varphi_t^*\mathbf{g}_0$ is a solution of (2.8).

3. Now, we assume that $(\mathcal{M}, \mathbf{g}_0, \mathbf{X}, \beta)$ is a $\mathbf{T}$-soliton structure for (2.8) and additionally that $\mathbf{T}$ is homogeneous of degree q. Define $c(t)$ to be the solution of the differential equation $\frac{dc}{dt} = \beta c^q; c(0) = 1$ and define the time-dependent vector field $\mathbf{Y}_t$ by $\mathbf{Y}_t = c(t)^{q-1}\mathbf{X}$. Denote the corresponding flow of $\mathbf{Y}_t$ by φ_t and observe that if $\mathbf{g}(t) = c(t)\varphi_t^*\mathbf{g}_0$, then it follows from properties of Lie derivatives that

$$\begin{aligned}
\frac{\partial \mathbf{g}(t)}{\partial t} &= c'(t)\varphi_t^*\mathbf{g}_0 + c(t)\varphi_t^*\left(\mathcal{L}_{\mathbf{Y}}\mathbf{g}_0\right) \\
&= c'(t)\varphi_t^*\mathbf{g}_0 + c(t)\varphi_t^*\left(\mathcal{L}_{c(t)^{q-1}\mathbf{X}}\mathbf{g}_0\right) && \text{(Defn. of } \mathbf{Y}) \\
&= \beta c(t)^q\varphi_t^*\mathbf{g}_0 + c(t)^q\varphi_t^*\left(\mathcal{L}_{\mathbf{X}}\mathbf{g}_0\right) && \text{(Assumptions on } c(t)) \\
&= c(t)^q\varphi_t^*\left(\beta\mathbf{g}_0 + \mathcal{L}_{\mathbf{X}}\mathbf{g}_0\right) && \text{(Properties of pull-backs)} \\
&= c(t)^q\varphi_t^*\left(\mathbf{T}\left[\mathbf{g}_0\right]\right) && ((\mathcal{M}, \mathbf{g}_0, \mathbf{X}, \beta) \text{ is a } \mathbf{T}\text{-soliton}) \\
&= \mathbf{T}\left[c(t)\varphi_t^*\mathbf{g}_0\right], && \mathbf{T} \text{ is natural and homogeneous.}
\end{aligned}$$

We conclude that $\mathbf{T}$-solitons give rise to self-similar solutions of the flow (2.8) whenever $\mathbf{T}$ is natural and homogeneous.

Q.E.D.

Following the work of Lauret on the Ricci flow on nilpotent Lie groups in [25], we will now define algebraic $\mathbf{T}$-solitons for the geometric evolution equation (2.8) and establish the relationship between algebraic $\mathbf{T}$-solitons and $\mathbf{T}$-solitons. Similar considerations apply in all dimensions, but for ease of exposition we restrict ourselves to dimension three and assume that $(\mathcal{M}, \mathbf{g}) = (\mathcal{H}, \mathbf{g})$ is a simply-connected, three-dimensional Lie group with Lie algebra $\mathfrak{h}$ and left invariant Riemannian metric $\mathbf{g}$.

Remark 2.6. For a symmetric $(0,2)$-tensor $\mathbf{A}$ on $(\mathcal{M}, \mathbf{g})$, we refer to the $(1,1)$-tensor $\widehat{\mathbf{A}}$ obtained by using $\mathbf{g}$ to raise an index as the $\mathbf{A}$-*operator*. Note that $\widehat{\mathbf{A}}$ is defined implicitly by requiring that $\mathbf{g}\left(\widehat{\mathbf{A}}\left(\mathbf{X}\right), \mathbf{Y}\right) = \mathbf{A}\left(\mathbf{X}, \mathbf{Y}\right)$ for all vector fields $\mathbf{X}, \mathbf{Y}$ on $\mathcal{M}$, and that with respect to a given frame, the components $\widehat{A}^i_j$ of $\widehat{\mathbf{A}}$ are related to the components A_{ij} of $\mathbf{A}$ by $\widehat{A}^i_j = g^{ik} A_{kj}$. While $\widehat{\mathbf{A}}$ depends on both $\mathbf{A}$ and the $\mathbf{g}$, the metric $\mathbf{g}$ being used to define $\widehat{\mathbf{A}}$ will always be clear from the context.

Definition 2.7 (Algebraic $\mathbf{T}$-soliton)**.** Let $\mathcal{H}$ be a simply-connected Lie group with Lie algebra $\mathfrak{h}$ and left invariant metric $\mathbf{g}$. The triple $(\mathcal{H}, \mathbf{g}, \beta)$, where $\beta \in \mathbb{R}$, is said to be an *algebraic* $\mathbf{T}$-*soliton* for the geometric evolution equation (2.8) if the operator $\mathbf{D} : \mathfrak{h} \to \mathfrak{h}$ defined by

$$\mathbf{D} = \widehat{\mathbf{T}}\left[\mathbf{g}\right] - \beta \mathrm{Id} \tag{2.12}$$

is a derivation of $\mathfrak{h}$. When there is no potential for confusion, we we will simply refer to the left invariant metric $\mathbf{g}$ as an algebraic $\mathbf{T}$-soliton.

The following proposition is a straightforward adaptation from [25] and establishes the relationship between algebraic $\mathbf{T}$-solitons and $\mathbf{T}$-solitons. The proof provided is essentially identical to the proof provided by Onda in [29], where the author investigates algebraic Ricci solitons in the case of pseudo-Riemannian metrics.

Proposition 2.8. If $\mathcal{H}$ is a simply-connected Lie group and the left invariant metric $\mathbf{g}$ is an algebraic $\mathbf{T}$-soliton for the geometric evolution equation (2.8), then $\mathbf{g}$ is a $\mathbf{T}$-soliton.

Proof. Assume that $\mathbf{g}$ is an algebraic $\mathbf{T}$-soliton with soliton constant of β and that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is an orthonormal basis for $\mathbf{g}$. Set $\mathbf{D} = \widehat{\mathbf{T}}\left[\mathbf{g}\right] - \beta\mathrm{Id}$ and define $\varphi_t : \mathcal{H} \to \mathcal{H}$ by declaring its differential at the identity element of $\mathcal{H}$ to be $d\varphi_t = \exp\left(\frac{t\mathbf{D}}{2}\right)$. Now define a vector field $\mathbf{X}$ on $\mathcal{H}$ by setting $\mathbf{X}(p) = \left.\frac{d\varphi_t(p)}{dt}\right|_{t=0}$, $p \in \mathcal{H}$. By properties of Lie derivatives, it follows that

$$\mathcal{L}_{\mathbf{X}}\mathbf{g}\left(\mathbf{e}_i, \mathbf{e}_j\right) = \frac{d}{dt}\varphi_t^*\mathbf{g}\left(\mathbf{e}_i, \mathbf{e}_j\right) = \frac{1}{2}\left(\mathbf{g}\left(\mathbf{D}\left(\mathbf{e}_i\right), \mathbf{e}_j\right) + \mathbf{g}\left(\mathbf{e}_i, \mathbf{D}\left(\mathbf{e}_j\right)\right)\right),$$

and from the defining characteristics of $\mathbf{D}$, we find that

$$\frac{1}{2}\left(\mathbf{g}\left(\mathbf{D}\left(\mathbf{e}_i\right), \mathbf{e}_j\right) + \mathbf{g}\left(\mathbf{e}_i, \mathbf{D}\left(\mathbf{e}_j\right)\right)\right) = \frac{1}{2}\left(\mathbf{g}\left(\widehat{\mathbf{T}}\left[\mathbf{g}\right]\left(\mathbf{e}_i\right), \mathbf{e}_j\right) + \mathbf{g}\left(\widehat{\mathbf{T}}\left[\mathbf{g}\right]\left(\mathbf{e}_j\right), \mathbf{e}_i\right)\right) - \beta\mathbf{g}\left(\mathbf{e}_i, \mathbf{e}_j\right).$$

By definition of $\widehat{\mathbf{T}}[\mathbf{g}]$ and the symmetry of the metric tensor $\mathbf{g}$, we have also that

$$\begin{aligned}\mathbf{T}[\mathbf{g}](\mathbf{e}_i,\mathbf{e}_j)-\beta\mathbf{g}(\mathbf{e}_i,\mathbf{e}_j) &= \frac{1}{2}\left(\mathbf{T}[\mathbf{g}](\mathbf{e}_i,\mathbf{e}_j)+\mathbf{T}[\mathbf{g}](\mathbf{e}_j,\mathbf{e}_i)\right)-\beta\mathbf{g}(\mathbf{e}_i,\mathbf{e}_j)\\ &= \frac{1}{2}\left(\mathbf{g}\left(\widehat{\mathbf{T}}[\mathbf{g}](\mathbf{e}_i),\mathbf{e}_j\right)+\mathbf{g}\left(\widehat{\mathbf{T}}[\mathbf{g}](\mathbf{e}_j),\mathbf{e}_i\right)\right)-\beta\mathbf{g}(\mathbf{e}_i,\mathbf{e}_j).\end{aligned}$$

Thus $\mathcal{L}_{\mathbf{X}}\mathbf{g}(\mathbf{e}_i,\mathbf{e}_j)=\mathbf{T}[\mathbf{g}](\mathbf{e}_i,\mathbf{e}_j)-\beta\mathbf{g}(\mathbf{e}_i,\mathbf{e}_j)$ holds for all basis vectors $\mathbf{e}_i,\mathbf{e}_j$, and we conclude that $\mathcal{L}_{\mathbf{X}}\mathbf{g}=\mathbf{T}[\mathbf{g}]-\beta\mathbf{g}$ and that $\mathbf{g}$ is a $\mathbf{T}$-soliton for (2.8). Q.E.D.

Remark 2.9. It follows from Proposition 2.5 that if $\mathbf{T}$ is natural and homogeneous, then every algebraic $\mathbf{T}$-soliton gives rise to a self-similar solution of the geometric evolution equation $\frac{\partial\mathbf{g}}{\partial t}=\mathbf{T}[\mathbf{g}]$. In light of this observation, we say that the algebraic $\mathbf{T}$-soliton $(\mathcal{H},\mathbf{g},\beta)$ is expanding, steady, or shrinking, depending on whether or not β is positive, zero, or negative, respectively. Note that if $\mathbf{T}$ is natural but not homogeneous, then a steady algebraic $\mathbf{T}$-soliton gives rise to self-similar solution of (2.8) that evolves by diffeomorphism only.

Remark 2.10. Note that $\mathbf{g}$ is a shrinking (res. expanding) soliton for the $+$XCF if and only if $\mathbf{g}$ is an expanding (res. shrinking) soliton for the $-$XCF. Specifically, $(\mathcal{M},\mathbf{g},\mathbf{X},\beta)$ is a $+$XCF soliton structure on $\mathcal{M}$ if and only if $(\mathcal{M},\mathbf{g},-\mathbf{X},-\beta)$ is a $-$XCF soliton structure on $\mathcal{M}$. This follows directly from the fact that $\beta\mathbf{g}+\mathcal{L}_{\mathbf{X}}\mathbf{g}=2\mathbf{H}[\mathbf{g}]$ if and only if $-\beta\mathbf{g}+\mathcal{L}_{-\mathbf{X}}\mathbf{g}=-2\mathbf{H}[\mathbf{g}]$. Further note that when looking for soliton structures for the evolution equation $\frac{\partial\mathbf{g}}{\partial t}=\mathbf{T}[\mathbf{g}]$, one can replace $\mathbf{T}$ with any positive scalar multiple of $\mathbf{T}$ without changing the qualitative nature of the soliton structure. In our classification of algebraic soliton structures for the XCF on three-dimensional unimodular Lie groups, we will use $\mathbf{T}=\pm\mathbf{H}$ as opposed to $\mathbf{T}=\pm2\mathbf{H}$ which is found in Definition 2.1 and is used in [3], [4], and [15].

Whereas the search for $\mathbf{T}$-solitons often involves a complicated system of partial differential equations, algebraic $\mathbf{T}$-solitons can be found via algebraic methods alone. Furthermore, there are algebraic conditions that must be satisfied in order for a Lie group $\mathcal{H}$ to be able to support an algebraic $\mathbf{T}$-soliton. The following proposition shows that for a simply-connected Lie group to admit an algebraic $\mathbf{T}$-soliton the corresponding Lie algebra must have a derivation that can be diagonalized.

Proposition 2.11. Let $\mathcal{H}$ be a simply-connected Lie group with Lie algebra $\mathfrak{h}$. If $\mathcal{H}$ admits an algebraic $\mathbf{T}$-soliton structure for the geometric evolution equation (2.8) and $\mathbf{T}$ is natural, then $\mathfrak{h}$ admits a derivation that is diagonalizable.

Proof. Suppose that $(\mathcal{H},\mathbf{g},\beta)$ is an algebraic $\mathbf{T}$-soliton structure on $\mathcal{H}$ and note that the assumption of $\mathbf{T}$ being natural implies that $\mathbf{T}[\mathbf{g}]$ is left invariant. Identifying both $\mathbf{g}$ and $\mathbf{T}[\mathbf{g}]$ with their values on $\mathfrak{h}\simeq T_e\mathcal{H}$, then since $\mathbf{g}$ is positive definite, there exists a basis $\mathcal{B}=\{\mathbf{e}_i\}$ for $\mathfrak{h}$ that diagonalizes both $\mathbf{g}$ and $\mathbf{T}[\mathbf{g}]$. It follows that $\widehat{\mathbf{T}}[\mathbf{g}]$ is also diagonalized with respect to the indicated basis. The assumption that $\mathbf{g}$ is an algebraic $\mathbf{T}$-soliton implies that $\mathbf{D}=\widehat{\mathbf{T}}[\mathbf{g}]-\beta\mathrm{Id}$ is a diagonal derivation of the Lie algebra $\mathfrak{h}$ and that the basis $\mathcal{B}=\{\mathbf{e}_i\}$ serves as a basis of eigenvectors for $\mathbf{D}$. Q.E.D.

Remark 2.12. Note that if $\mathbf{g}$ is an algebraic $\mathbf{T}$-soliton and $\mathbf{D} = \widehat{\mathbf{T}}[\mathbf{g}] - \beta \mathrm{Id} = \mathbf{0}$, then $\mathbf{T}[\mathbf{g}]$ is a scalar multiple of the metric and we have $\mathbf{T}[\mathbf{g}] = \beta\mathbf{g}$. Such an algebraic $\mathbf{T}$-soliton should be regarded as a trivial algebraic $\mathbf{T}$-soliton. With this in mind, the above proposition can be restated so as to avoid the zero derivation by noting that if $\mathcal{H}$ admits a non-trivial algebraic $\mathbf{T}$-soliton, then $\mathfrak{h}$ must admit a non-zero derivation that is diagonalizable.

The following simple lemma concerning diagonal derivations will be used extensively in the classification of XCF and RG-2 algebraic solitons that give rise to self-similar solutions of their respective flows on three-dimensional unimodular Lie groups. We will state the lemma in arbitrary dimensions.

Lemma 2.13. Let $\mathfrak{h}$ be a Lie algebra and suppose that the linear operator $\mathbf{D} : \mathfrak{h} \to \mathfrak{h}$ is diagonalizable with an ordered basis of eigenvectors $\mathcal{B} = \{\mathbf{e}_i\}$ and corresponding eigenvalues d_i, $1 \leq i \leq n$. Then $\mathbf{D}$ is a derivation of $\mathfrak{h}$ if and only if for all basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$, we have that $[\mathbf{e}_i, \mathbf{e}_j] = 0$ or $[\mathbf{e}_i, \mathbf{e}_j]$ is an eigenvector with eigenvalue $d_i + d_j$.

As it pertains to algebraic $\mathbf{T}$-solitons, there are two extreme cases concerning the eigenvalues of a diagonal derivation on a non-Abelian Lie algebra $\mathfrak{h}$ that merit mention. The first is the case where $\mathbf{D} = \widehat{\mathbf{T}[\mathbf{g}]} - \beta \mathrm{Id}$ has only one eigenvalue (i.e., a repeated eigenvalue of multiplicity $n = \dim \mathfrak{h}$). In light of Lemma 2.13, then in this case we must have that $\mathbf{D} = \widehat{\mathbf{T}}[\mathbf{g}] - \beta \mathrm{Id} = \mathbf{0}$ and $\mathbf{T}[\mathbf{g}]$ is a scalar multiple of the metric $\mathbf{g}$. Note that a self-similar solution of the indicated form evolves by scaling only and there is no diffeomorphism action. Such a metric would play the role for $\mathbf{T}$ that an Einstein metric does for $\mathbf{Rc}$.

The second case is when all eigenvalues of the diagonal derivation $\mathbf{D} = \widehat{\mathbf{T}[\mathbf{g}]} - \beta \mathrm{Id}$ have multiplicity one. In this case, the Lie algebra $\mathfrak{h}$ must admit a *nice basis*. Following [26], we say that a basis $\mathcal{B} = \{\mathbf{e}_i\}$ for a Lie algebra $\mathfrak{h}$ is *nice* if the structure constants defined by $[\mathbf{e}_i, \mathbf{e}_j] = c_{ij}^k \mathbf{e}_k$ satisfy

- for all i, j, there exists at most one k such that $c_{ij}^k \neq 0$, and
- for all i, k, there exists at most one j such that $c_{ij}^k \neq 0$.

The condition on a basis of $\mathfrak{h}$ being nice can thusly be interpreted as requiring the Lie bracket of any two basis basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ be zero or belong to the span of a third basis vector $\mathbf{e}_k$, and two non-zero brackets $[\mathbf{e}_i, \mathbf{e}_j]$ and $[\mathbf{e}_p, \mathbf{e}_q]$ are non-zero scalar multiples of each other if and only if $\{i, j\} = \{p, q\}$ or $\{i, j\}$ and $\{p, q\}$ are disjoint. It follows from Lemma 2.13 that if $\mathbf{D} = \widehat{\mathbf{T}[\mathbf{g}]} - \beta \mathrm{Id}$ is a diagonal derivation with distinct eigenvalues, then the Lie algebra $\mathfrak{h}$ must admit a nice basis. Note that all three-dimensional unimodular Lie groups admit a nice basis for their Lie algebras (see Section 3.1). See [26] for further discussion of Lie algebras admitting a nice basis and the relationship between a nice basis and stably Ricci diagonal flows as introduced in [30] by Payne.

3 Algebraic solitons on three-dimensional unimodular Lie groups

3.1 Milnor frames

Let $\mathcal{H}$ be a three-dimensional unimodular Lie group with Lie algebra $\mathfrak{h}$ and left invariant metric $\mathbf{g}$. Throughout what follows, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ will be a left invariant frame with dual co-frame denoted by ω^1, ω^2, and ω^3.

In [28], Milnor establishes the existence of a left invariant $\mathbf{g}$-orthonormal frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for $\mathfrak{h}$ such that the algebraic structure of $\mathfrak{h}$ is determined by the non-zero bracket relations

$$[\mathbf{e}_2, \mathbf{e}_3] = \lambda^1 \mathbf{e}_1 \qquad [\mathbf{e}_3, \mathbf{e}_1] = \lambda^2 \mathbf{e}_2 \qquad [\mathbf{e}_1, \mathbf{e}_2] = \lambda^3 \mathbf{e}_3. \tag{3.1}$$

The Levi-Civita connection of $\mathbf{g}$ is completely determined by the Koszul formula, and with respect to the indicated frame the defining covariant derivatives are

$$\nabla_{\mathbf{e}_1}\mathbf{e}_2 = \mu^1 \mathbf{e}_3 \qquad \nabla_{\mathbf{e}_1}\mathbf{e}_3 = -\mu^1 \mathbf{e}_2 \qquad \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mu^2 \mathbf{e}_1$$

$$\nabla_{\mathbf{e}_3}\mathbf{e}_2 = -\mu^3 \mathbf{e}_1 \qquad \nabla_{\mathbf{e}_3}\mathbf{e}_1 = \mu^3 \mathbf{e}_2 \qquad \nabla_{\mathbf{e}_2}\mathbf{e}_1 = -\mu^2 \mathbf{e}_3,$$

where $\mu^i = \frac{1}{2}\left(\lambda^1 + \lambda^2 + \lambda^3\right) - \lambda^i$. The principal sectional curvatures are thusly given by

$$K_l = K\left(\mathbf{e}_m \wedge \mathbf{e}_n\right) = \lambda^l \mu^l - \mu^m \mu^n, \qquad (l, m, n) \text{ is a permutation of } (1, 2, 3). \tag{3.2}$$

The Ricci tensor $\mathbf{Rc}\left[\mathbf{g}\right] = R_{ij}\omega^i \otimes \omega^j$ is diagonalized with respect to the indicated frame, and the non-zero components are

$$R_{ll} = 2\mu^m \mu^n = K_m + K_n, \qquad (l, m, n) \text{ is a permutation of } (1, 2, 3). \tag{3.3}$$

Following [28], we observe that if the metric $\mathbf{g}$ is altered by declaring the basis

$$\tilde{\mathbf{e}}_1 = BC\mathbf{e}_1, \qquad \tilde{\mathbf{e}}_2 = AC\mathbf{e}_2, \qquad \tilde{\mathbf{e}}_3 = AB\mathbf{e}_3, \qquad A, B, C \in \mathbb{R}_{>0}$$

to be orthonormal, then we find that the resulting structure constants are all scaled by positive constants:

$$[\tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3] = A^2\lambda^1 \tilde{\mathbf{e}}_1, \qquad [\tilde{\mathbf{e}}_3, \tilde{\mathbf{e}}_1] = B^2\lambda^2 \tilde{\mathbf{e}}_2, \qquad [\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2] = C^2\lambda^3 \tilde{\mathbf{e}}_3.$$

As such, we can assume that the left invariant metric $\mathbf{g}$ is expressed relative to a left invariant frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and its dual co-frame $\omega^1, \omega^2, \omega^3$ as $\mathbf{g} = A\,\omega^1 \otimes \omega^1 + B\,\omega^2 \otimes \omega^2 + C\,\omega^3 \otimes \omega^3$, with the structure constants (3.1) defining the Lie algebra satisfying $\lambda^i \in \{1, 0, -1\}$. Further, by assuming an orientation for the Lie algebra $\mathfrak{h}$, we can assume that there are at least as many positive structure constants as negative structure constants, and that by appropriately ordering our basis we have $\lambda^1 \geq \lambda^2 \geq \lambda^3$.

Given a left invariant metric $\mathbf{g}$ on $\mathcal{H}$, we will refer to a left invariant frame $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ that

1. diagonalizes the metric $\mathbf{g}$ (i.e., $\mathbf{g} = A\,\omega^1 \otimes \omega^1 + B\,\omega^2 \otimes \omega^2 + C\,\omega^3 \otimes \omega^3$), and
2. diagonalizes the structure constants of $\mathfrak{h}$ as in (3.1) with $\lambda^i \in \{1, 0, -1\}$ and $\lambda^1 \geq \lambda^2 \geq \lambda^3$,

as a *Milnor frame* for $\mathbf{g}$. In what follows, we will work exclusively with Milnor frames for $\mathbf{g}$.

The six possibilities for the structure constants of an oriented Lie algebra $\mathfrak{h}$ corresponding to a simply-connected, three-dimensional, unimodular Lie group are recorded in the following table:

$\left(\lambda^1, \lambda^2, \lambda^3\right)$	Lie group
$(1,1,1)$	SU(2)
$(1,1,-1)$	$\widetilde{\mathrm{SL}(2,\mathbb{R})}$
$(1,1,0)$	E(2)
$(1,0,-1)$	E(1,1)
$(1,0,0)$	3-dim Heisenberg group
$(0,0,0)$	$\mathbb{R}^3$

Note that if $\mathbf{g}$ is a left invariant metric on a three-dimensional unimodular Lie group $\mathcal{H}$ and $\mathbf{g}$ expressed in a Milnor frame as $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$, then the principal sectional curvatures are given by

$$K_l = K\left(\mathbf{e}_m \wedge \mathbf{e}_n\right) = \tilde{\lambda}^l \tilde{\mu}^l - \tilde{\mu}^m \tilde{\mu}^n, \qquad (l,m,n) \text{ is a permutation of } (1,2,3), \tag{3.4}$$

where $\tilde{\lambda}^1 = \sqrt{\frac{A}{BC}}\lambda^1$, $\tilde{\lambda}^2 = \sqrt{\frac{B}{CA}}\lambda^2$, $\tilde{\lambda}^3 = \sqrt{\frac{A}{BC}}\lambda^3$, and $\tilde{\mu}^i = \frac{1}{2}\left(\tilde{\lambda}^1 + \tilde{\lambda}^2 + \tilde{\lambda}^3\right) - \tilde{\lambda}^i$, $1 \leq i \leq 3$. The non-zero components of the Ricci tensor $\mathbf{Rc}\left[\mathbf{g}\right] = R_{ij}\omega^i \otimes \omega^j$ are thusly

$$R_{11} = A\left(K_2 + K_3\right), \qquad R_{22} = B\left(K_1 + K_3\right), \qquad R_{33} = C\left(K_1 + K_2\right), \tag{3.5}$$

and the tensor $\mathbf{Rm}^2\left[\mathbf{g}\right] = \check{R}_{ij}\omega^i \otimes \omega^j$ is diagonalized with non-zero components given by

$$\begin{cases} \check{R}_{11} &= 2A\left(\left(K_2\right)^2 + \left(K_3\right)^2\right) \\ \check{R}_{22} &= 2B\left(\left(K_3\right)^2 + \left(K_1\right)^2\right) \\ \check{R}_{33} &= 2C\left(\left(K_3\right)^2 + \left(K_1\right)^2\right). \end{cases} \tag{3.6}$$

It also follows that both the cross curvature tensor $\mathbf{H}\left[\mathbf{g}\right] = H_{ij}\omega^i \otimes \omega^j$ and the RG-2 tensor $\mathbf{RG}\left[\mathbf{g}\right] = RG_{ij}\omega^i \otimes \omega^j$ are diagonalized with respect to a Milnor frame. The non-zero components are, respectively,

$$H_{11} = AK_2K_3, \quad H_{22} = BK_1K_3, \quad \text{and} \quad H_{33} = CK_1K_2, \tag{3.7}$$

and

$$\begin{cases} RG_{11} &= -A\left(2\left(K_2 + K_3\right) + \alpha\left(\left(K_2\right)^2 + \left(K_3\right)^2\right)\right) \\ RG_{22} &= -B\left(2\left(K_3 + K_1\right) + \alpha\left(\left(K_3\right)^2 + \left(K_1\right)^2\right)\right) \\ RG_{33} &= -C\left(2\left(K_1 + K_2\right) + \alpha\left(\left(K_1\right)^2 + \left(K_2\right)^2\right)\right). \end{cases} \tag{3.8}$$

Finally, we note that the corresponding operators $\widehat{\mathbf{H}}\left[\mathbf{g}\right]$ and $\widehat{\mathbf{RG}}\left[\mathbf{g}\right]$ are diagonalized with respect to the indicated frame and take the form

$$\widehat{\mathbf{H}}\left[\mathbf{g}\right] = \operatorname{diag}\left(K_2K_3\,,\, K_3K_1\,,\, K_1K_2\right), \tag{3.9}$$
$$\widehat{\mathbf{RG}}\left[\mathbf{g}\right] = \operatorname{diag}\left(\frac{RG_{11}}{A}, \frac{RG_{22}}{B}, \frac{RG_{33}}{C}\right).$$

We will now classify the algebraic solitons on simply-connected, three-dimensional, unimodular Lie groups that give rise to self-similar solutions of the XCF and RG-2 flow. Before proceeding, recall from Proposition 2.5 and Proposition 2.8 that all algebraic XCF-solitons give rise to self-similar solutions of the XCF, whereas only steady algebraic RG-2-solitons give rise to self-similar solutions of the RG-2 flow.

3.2 $\mathbb{R}^3$

We begin with the trivial case of $\mathcal{H} = \mathbb{R}^3$. Since all left invariant metrics on the Abelian group $\mathbb{R}^3$ are flat, then it follows immediately that all left invariant metrics $\mathbf{g}$ are fixed points for the XCF and RG-2 flow. Such metrics can be regarded as trivial solitons.

3.3 Heisenberg group

Let $\mathbf{g}$ be a left invariant metric on the three-dimensional Heisenberg group and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a Milnor frame where $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$ and the Lie algebra structure is determined by the non-zero bracket $[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1$. By using an automorphism of the Lie algebra, one can further assume that $B = C = 1$ and the metric takes the form $\mathbf{g} = A\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$.

From (3.4) we find that the principal sectional curvatures are

$$K_1 = K(\mathbf{e}_2 \wedge \mathbf{e}_3) = -\frac{3}{4}A, \qquad K_2 = K(\mathbf{e}_1 \wedge \mathbf{e}_3) = \frac{1}{4}A, \qquad K_3 = K(\mathbf{e}_1 \wedge \mathbf{e}_2) = \frac{1}{4}A. \tag{3.10}$$

We will now make use of the algebraic structure of the Lie algebra to give a simple criterion for when a left invariant metric $\mathbf{g}$ on the Heisenberg group is an algebraic $\mathbf{T}$-soliton for the evolution equation $\frac{\partial \mathbf{g}}{\partial t} = \mathbf{T}[\mathbf{g}]$. Recall from Remark 2.6 that for a symmetric $(0,2)$-tensor $\mathbf{A}$ we denote the $(1,1)$-tensor obtained by using $\mathbf{g}$ to raise an index on $\mathbf{A}$ by $\widehat{\mathbf{A}}$.

Lemma 3.1. Let $\mathbf{g}$ be a left invariant metric on the three-dimensional Heisenberg group, with $\mathbf{g}$ expressed relative to a Milnor frame as $\mathbf{g} = A\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$. If $\widehat{\mathbf{T}}[\mathbf{g}]$ is diagonalized relative to the Milnor frame, then $\mathbf{g}$ is an algebraic $\mathbf{T}$-soliton for the evolution equation $\frac{\partial \mathbf{g}}{\partial t} = \mathbf{T}[\mathbf{g}]$ with soliton constant $\beta = \widehat{T}^2_2 + \widehat{T}^3_3 - \widehat{T}^1_1$.

Proof. Let $\mathbf{D} = \widehat{\mathbf{T}}[\mathbf{g}] - \beta \mathrm{Id}$ and assume that $\widehat{\mathbf{T}}[\mathbf{g}] : \mathfrak{h} \to \mathfrak{h}$ is diagonalized relative to the Milnor frame. Owing to the Lie bracket structure of the Lie algebra $\mathfrak{h}$, it follows from Lemma 2.13 that the operator $\mathbf{D} = \left(d^i_j\right) = \left(\widehat{T}^i_j - \beta\delta^i_j\right)$ is a derivation of $\mathfrak{h}$ if and only if $[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1$ is an eigenvector for $\mathbf{D}$ with eigenvalue $d^1_1 = d^2_2 + d^3_3$. Accordingly, we must have $\widehat{T}^1_1 - \beta = \left(\widehat{T}^2_2 - \beta\right) + \left(\widehat{T}^3_3 - \beta\right)$, or equivalently, $\beta = \widehat{T}^2_2 + \widehat{T}^3_3 - \widehat{T}^1_1$. This completes the proof. Q.E.D.

The classification of algebraic XCF and RG-2 solitons on the Heisenberg group that give rise to self-similar solutions of their respective flows now follows easily.

3.3.1 Algebraic XCF-solitons

From (3.7) and (3.10), we find that the cross curvature tensor of the metric $\mathbf{g}$ is given by $\mathbf{H}[\mathbf{g}] = \frac{A^3}{16}\omega^1 \otimes \omega^1 - \frac{3A^2}{16}\omega^2 \otimes \omega^2 - \frac{3A^2}{16}\omega^3 \otimes \omega^3$, with corresponding cross curvature operator

$\widehat{\mathbf{H}}[\mathbf{g}]$ represented in the given frame by

$$\widehat{\mathbf{H}}[\mathbf{g}] = \operatorname{diag}\left(\frac{A^2}{16}, -\frac{3A^2}{16}, -\frac{3A^2}{16}\right). \tag{3.11}$$

Theorem 3.2. For all choices of A, the left invariant metric $\mathbf{g} = A\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$ is a shrinking algebraic XCF soliton for the +XCF on the Heisenberg group. (And likewise, an expanding algebraic −XCF soliton.)

Remark 3.3. Recall from Remark 2.10 that $\mathbf{g}$ is a shrinking (res. expanding) soliton structure for the +XCF if and only if $\mathbf{g}$ is an expanding (res. shrinking) soliton structure for the −XCF.

Proof. The cross curvature operator $\widehat{\mathbf{H}}[\mathbf{g}]$ is diagonalized in a Milnor frame and is given as in (3.11). Applying Lemma 3.1 we find that $\mathbf{g}$ is an algebraic +XCF soliton with soliton constant $\beta = \widehat{H}_2^2 + \widehat{H}_3^3 - \widehat{H}_1^1 = -\frac{7}{16}A^2$, which completes the proof. Q.E.D.

3.3.2 Algebraic RG-2 Solitons

Combining (3.8) and (3.10), the RG-2 tensor of the metric $\mathbf{g}$ is

$$\mathbf{RG}[\mathbf{g}] = -\left(\frac{1}{8}\alpha A^3 + A^2\right)\omega^1 \otimes \omega^1 + \left(A - \frac{5}{8}\alpha A^2\right)\omega^2 \otimes \omega^2 + \left(A - \frac{5}{8}\alpha A^2\right)\omega^3 \otimes \omega^3, \tag{3.12}$$

and the corresponding RG-2 operator is

$$\widehat{\mathbf{RG}}[\mathbf{g}] = \operatorname{diag}\left(-\frac{\alpha}{8}A^2 - A, A - \frac{5}{8}\alpha A^2, A - \frac{5}{8}\alpha A^2\right). \tag{3.13}$$

Since we are focused on finding self-similar solutions to the RG-2 flow on the Heisenberg group, we will only be looking for steady algebraic RG-2 solitons of the flow. We immediately establish the following.

Theorem 3.4. The left invariant metric $\mathbf{g} = A\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$ is a steady algebraic soliton that gives rise to a self-similar solution of the RG-2 flow $\frac{\partial \mathbf{g}}{\partial t} = -2\mathbf{Rc}[\mathbf{g}] - \frac{4}{3A}\mathbf{RG}[\mathbf{g}]$ (i.e., when $\alpha = \frac{8}{3A}$).

Proof. The RG-2 operator $\widehat{\mathbf{RG}}[\mathbf{g}]$ (3.13) is diagonalized with respect to the Milnor frame. Setting $\beta = 0$ and applying Lemma 3.1, we find that $\mathbf{g}$ is a steady algebraic RG-2 soliton that gives rise to a self-similar solution of the RG-2 flow when $\widehat{RG}_1^1 = \widehat{RG}_2^2 + \widehat{RG}_3^3$. From (3.13) we see that for the equation $\widehat{RG}_1^1 = \widehat{RG}_2^2 + \widehat{RG}_3^3$ to be satisfied we must have $-\frac{\alpha}{8}A^2 - A = 2\left(A - \frac{5}{8}\alpha A^2\right)$, or equivalently, $3A\left(\frac{3}{8}\alpha A - 1\right) = 0$. Since $A = \mathbf{g}(\mathbf{e}_1, \mathbf{e}_1)$, the result follows. Q.E.D.

3.4 E(2)

Let $\mathbf{g}$ be a left invariant metric on E(2), the three-dimensional group of (orientation preserving) isometries of the Euclidean plane, and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a Milnor frame for $\mathbf{g}$ such that $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$, with the structure of the Lie algebra determined by the non-zero brackets

$$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1 \qquad \text{and} \qquad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2.$$

Using an automorphism of the Lie algebra, we can further assume that $A = 1$ and write $\mathbf{g} = \omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$.

According to (3.4), the principal sectional curvatures of $\mathbf{g}$ are

$$K_1 = \frac{(B+3)(B-1)}{4BC}, \qquad K_2 = \frac{(3B+1)(1-B)}{4BC}, \qquad K_3 = \frac{(B-1)^2}{4BC}. \tag{3.14}$$

Note that when $B = 1$, we obtain a family of flat metrics on E(2).

As before, we can make use of the algebraic structure of the Lie algebra to provide a simple criterion for when a left invariant metric is an algebraic $\mathbf{T}$-soliton for a geometric evolution equation of the form $\frac{\partial \mathbf{g}}{\partial t} = \mathbf{T}[\mathbf{g}]$.

Lemma 3.5. Let $\mathbf{g}$ be a left invariant metric on E(2), with $\mathbf{g}$ expressed relative to a Milnor frame as $\mathbf{g} = \omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$. If $\widehat{\mathbf{T}}[\mathbf{g}]$ is diagonalized relative to the Milnor frame, then $\mathbf{g}$ is an algebraic $\mathbf{T}$-soliton for the evolution equation $\frac{\partial \mathbf{g}}{\partial t} = \mathbf{T}[\mathbf{g}]$ if and only if $\widehat{T}^1_1 = \widehat{T}^2_2$. Moreover, the soliton constant is $\beta = \widehat{T}^3_3$.

Proof. The proof proceeds in an identical manner to that of the proof of Lemma 3.1. Let $\mathbf{D} = \widehat{\mathbf{T}}[\mathbf{g}] - \beta \mathrm{Id}$ and assume that $\widehat{\mathbf{T}}[\mathbf{g}] : \mathfrak{h} \to \mathfrak{h}$ is diagonalized relative to the Milnor frame. Owing to the Lie bracket structure of the Lie algebra $\mathfrak{h}$, it follows from Lemma 2.13 that the operator $\mathbf{D} = \left(d^i_j\right) = \left(\widehat{T}^i_j - \beta\delta^i_j\right)$ is a derivation of $\mathfrak{h}$ if and only if $[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1$ and $[\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2$ are eigenvectors for $\mathbf{D}$ with eigenvalues $d^1_1 = d^2_2 + d^3_3$ and $d^2_2 = d^3_3 + d^1_1$, respectively. Accordingly, we must have $d^3_3 = \widehat{T}^3_3 - \beta = 0$ and $\widehat{T}^1_1 = \widehat{T}^2_2$, which completes the proof. Q.E.D.

3.4.1 XCF-solitons

It follows from (3.7) and (3.14) that the cross curvature tensor of $\mathbf{g}$ is

$$\mathbf{H}[\mathbf{g}] = -\frac{(3B+1)(B-1)^3}{16B^2C^2}\omega^1 \otimes \omega^1 + \frac{(B+3)(B-1)^3}{16BC^2}\omega^2 \otimes \omega^2 - \frac{(B+3)(B-1)^2(3B+1)}{16B^2C}\omega^3 \otimes \omega^3. \tag{3.15}$$

The non-zero components of $\widehat{\mathbf{H}}[\mathbf{g}]$ with respect to the indicated frame are $\widehat{H}^1_1 = H_{11}$, $\widehat{H}^2_2 = \frac{H_{22}}{B}$, and $\widehat{H}^3_3 = \frac{H_{33}}{C}$, and the cross curvature operator is

$$\widehat{\mathbf{H}}[\mathbf{g}] = \mathrm{diag}\left(-\frac{(3B+1)(B-1)^3}{16B^2C^2}, \frac{(B+3)(B-1)^3}{16B^2C^2}, \frac{(B+3)(B-1)^2(3B+1)}{16B^2C^2}\right).$$

Theorem 3.6. All flat metrics on E(2) are fixed points for the XCF on E(2) and are thus (trivial) steady algebraic solitons. These are the only algebraic solitons for the XCF on E(2).

Proof. That the flat metrics on E(2) are fixed points for the XCF is clear from the definition of the cross curvature tensor. To see that these are the only algebraic solitons for the XCF on E(2), we will employ Lemma 3.5. By Lemma 3.5, for $\mathbf{g}$ to be an algebraic XCF soliton we must have $\widehat{H}_1^1 = \widehat{H}_2^2$ and $\beta = \widehat{H}_3^3$. Owing to (3.9), we see that $\widehat{H}_1^1 = \widehat{H}_2^2$ if and only if $K_2K_3 = K_1K_3$. Thus we must have $K_3 = 0$ or $K_1 = K_2$. From (3.14), we see that $K_3 = 0$ if and only if $B = 1$ and that $K_1 = K_2$ if and only if $B = 1$ or $B = -1$. Since $B = \mathbf{g}(\mathbf{e}_2, \mathbf{e}_2)$, we conclude that $\mathbf{g} = \omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$ is an algebraic soliton for the XCF if and only if $B = 1$. It follows that such a metric is flat and is thus a fixed point of the flow. Q.E.D.

3.4.2 RG-2 solitons

The RG-2 tensor of $\mathbf{g}$ is $\mathbf{RG}[\mathbf{g}] = RG_{11}\omega^1 \otimes \omega^1 + RG_{22}\omega^2 \otimes \omega^2 + RG_{33}\omega^3 \otimes \omega^3$, where

$$\begin{aligned} RG_{11} &= -\frac{(B-1)\left(5\alpha B^3 - 3\alpha B^2 - 8B^2C - B\alpha - 8BC - \alpha\right)}{8B^2C^2} \\ RG_{22} &= -\frac{(B-1)\left(\alpha B^3 + \alpha B^2 + 8B^2C + 3B\alpha + 8BC - 5\alpha\right)}{8BC^2} \\ RG_{33} &= -\frac{(B-1)^2\left(5\alpha B^2 + 6\alpha B - 8BC + 5\alpha\right)}{8B^2C}, \end{aligned} \tag{3.16}$$

and the corresponding RG-2 operator is $\widehat{\mathbf{RG}}[\mathbf{g}] = \operatorname{diag}\left(\widehat{RG}_1^1, \widehat{RG}_2^2, \widehat{RG}_3^3\right)$, where

$$\widehat{RG}_1^1 = RG_{11}, \qquad \widehat{RG}_2^2 = \frac{RG_{22}}{B}, \qquad \text{and} \qquad \widehat{RG}_3^3 = \frac{RG_{33}}{C}. \tag{3.17}$$

Theorem 3.7. All flat metrics on E(2) are fixed points for the RG-2 flow and are thus trivial steady algebraic solitons. Furthermore, the flat metrics on E(2) are the only left invariant steady algebraic solitons for the RG-2 flow on E(2).

Proof. Owing to the definition of the RG-2 tensor, it is clear that the flat metrics are fixed points (and thus trivial algebraic solitons for the RG-2 flow). To see that the flat metrics are the only left invariant steady algebraic solitons for the RG-2 flow on E(2), note by Lemma 3.5 that in order for $\mathbf{g} = \omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$ to be a steady algebraic RG-2 soliton we must have $\widehat{RG}_3^3 = 0$ and $\widehat{RG}_1^1 = \widehat{RG}_2^2$.

Combining (3.8) and (3.9), we see that $\widehat{RG}_1^1 = \widehat{RG}_2^2$ if and only if

$$K_2(2 + \alpha K_2) = K_1(2 + \alpha K_1).$$

Substituting (3.14) into the above and making using of the fact that we can assume that $B \neq 1$, this is equivalent to the equation

$$-(3B+1)\left(2 + \alpha\frac{(3B+1)(1-B)}{4BC}\right) = (B+3)\left(2 + \alpha\frac{(B+3)(B-1)}{4BC}\right).$$

The above equation admits solutions when $B=-1$ and when $C=\frac{\alpha(B-1)^2}{4B}$. Since B must be greater than zero, we only concern ourselves with the later. Substituting $C=\frac{\alpha(B-1)^2}{4B}$ into $\widehat{RG}_3^3$ as in in (3.17), we find that for $\widehat{RG}_3^3$ to be zero, we must have $-\frac{1}{2}\frac{(3B+1)(B+3)}{B}=0$. Since B must be greater than zero, we conclude that the only left invariant steady algebraic RG-2 solitons on E(2) are the flat metrics. Q.E.D.

3.5 $\mathrm{E}(1,1)$

Let $\mathbf{g}$ be a left invariant metric for $\mathrm{E}(1,1)$, the solvable three-dimensional group of isometries of the standard Lorentz-Minkowski plane, expressed relative to a Milnor frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as $\mathbf{g}=A\omega^1\otimes\omega^1+B\omega^2\otimes\omega^2+C\omega^3\otimes\omega^3$. The corresponding Lie algebra structure is then generated by the non-zero brackets

$$[\mathbf{e}_2,\mathbf{e}_3]=\mathbf{e}_1 \qquad \text{and} \qquad [\mathbf{e}_1,\mathbf{e}_2]=-\mathbf{e}_3.$$

Using an automorphism of the Lie algebra, we can further require that $C=1$ and that the metric takes the form $\mathbf{g}=A\omega^1\otimes\omega^1+B\omega^2\otimes\omega^2+\omega^3\otimes\omega^3$.

Making the appropriate substitutions into (3.4), we find the principal sectional curvatures of $\mathbf{g}$ are

$$\begin{aligned} K_1 &= K(\mathbf{e}_2\wedge\mathbf{e}_3) = -\frac{(A+1)(3A-1)}{4AB} \\ K_2 &= K(\mathbf{e}_1\wedge\mathbf{e}_3) = \frac{(A+1)^2}{4AB} \\ K_3 &= K(\mathbf{e}_1\wedge\mathbf{e}_2) = \frac{(A+1)(A-3)}{4AB}. \end{aligned} \tag{3.18}$$

We now state a lemma for $\mathrm{E}(1,1)$ that is equivalent to Lemma 3.5 for $\mathrm{E}(2)$.

Lemma 3.8. Let $\mathbf{g}$ be a left invariant metric on $\mathrm{E}(1,1)$ that is expressed relative to a Milnor frame as $\mathbf{g}=A\omega^1\otimes\omega^1+B\omega^2\otimes\omega^2+\omega^3\otimes\omega^3$. If $\widehat{\mathbf{T}}[\mathbf{g}]$ is diagonalized relative to the Milnor frame, then $\mathbf{g}$ is an algebraic $\mathbf{T}$-soliton for the evolution equation $\frac{\partial\mathbf{g}}{\partial t}=\mathbf{T}[\mathbf{g}]$ if and only if $\widehat{T}_1^1=\widehat{T}_3^3$. Moreover, the soliton constant is $\beta=\widehat{T}_2^2$.

Proof. The proof proceeds in an identical fashion to the proof of Lemma 3.5 and the details are left to the reader. Q.E.D.

3.5.1 XCF solitons

Making the appropriate substitutions into (3.7), the non-zero components of the cross curvature tensor $\mathbf{H}[\mathbf{g}]=H_{11}\omega^1\otimes\omega^1+H_{22}\omega^2\otimes\omega^2+H_{33}\omega^3\otimes\omega^3$ are found to be

$$H_{11}=\frac{(A+1)^3(A-3)}{16AB^2}, \quad H_{22}=\frac{(A+1)^2(1-3A)(A-3)}{16A^2B}, \quad H_{33}=\frac{(A+1)^3(1-3A)}{16A^2B^2}, \tag{3.19}$$

and the cross curvature operator $\widehat{\mathbf{H}}[\mathbf{g}]$ is diagonalized with respect to the given frame and represented by $\widehat{\mathbf{H}}[\mathbf{g}]=\operatorname{diag}\left(\widehat{H}_1^1,\widehat{H}_2^2,\widehat{H}_3^3\right)$, where $\widehat{H}_1^1=\frac{H_{11}}{A}$, $\widehat{H}_2^2=\frac{H_{22}}{B}$, and $\widehat{H}_3^3=H_{33}$.

Theorem 3.9. The left invariant metric $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$ is an expanding (res. shrinking) algebraic soliton for the +XCF (res. −XCF) on E(1, 1) if and only if $A = 1$ and the soliton constant is $\beta = \frac{1}{B^2}$ (res. $\beta = -\frac{1}{B^2}$).

Proof. Since $\widehat{\mathbf{H}}[\mathbf{g}]$ is diagonalized relative to the Milnor frame, it follows from Lemma 3.8 that $\mathbf{g}$ is an algebraic soliton for the +XCF with $\mathbf{D} = \widehat{\mathbf{H}}[\mathbf{g}] - \beta\mathrm{Id}$ a derivation of $\mathfrak{h}$ if and only if $\widehat{H}^1_1 = \widehat{H}^3_3$ and $\beta = \widehat{H}^2_2$. Owing to (3.9) and (3.19), it follows that we must have

$$\beta = \widehat{H}^2_2 = \frac{(A+1)^2(1-3A)(A-3)}{16A^2B^2},$$

and

$$\widehat{H}^1_1 = \widehat{H}^3_3 \iff K_2K_3 = K_1K_2.$$

Thus, we must have $K_2 = 0$ (which only happens if $A = -1$) or $K_1 = K_3$. Since $A > 0$, then according to (3.18), we see that $K_1 = K_3$ if and only if $1 - 3A = A - 3$. We thus find that we have an algebraic soliton which gives rise to a self-similar solution when $A = 1$. Substituting $A = 1$ into $\beta = \widehat{H}^2_2$ we find $\beta = \frac{1}{B^2}$. Q.E.D.

3.5.2 RG-2 solitons

From (3.8) and (3.9), respectively, we find that the non-zero components of the RG-2 tensor $\mathbf{RG}[\mathbf{g}] = RG_{ij}\omega^i \otimes \omega^j$ are

$$\begin{aligned}
RG_{11} &= -\frac{(A+1)\left(\alpha A^3 + 8A^2B - \alpha A^2 + 3\alpha A - 8AB + 5\alpha\right)}{8AB^2} \\
RG_{22} &= -\frac{(A+1)^2\left(5\alpha A^2 - 6\alpha A - 8AB + 5\alpha\right)}{8A^2B} \\
RG_{33} &= -\frac{(A+1)\left(5\alpha A^3 + 3\alpha A^2 - 8A^2B - \alpha A + 8AB + \alpha\right)}{8A^2B^2},
\end{aligned} \tag{3.20}$$

and the corresponding RG-2 operator is $\widehat{\mathbf{RG}}[\mathbf{g}] = \mathrm{diag}\left(\widehat{RG}^1_1, \widehat{RG}^2_2, \widehat{RG}^3_3\right)$, where

$$\widehat{RG}^1_1 = \frac{RG_{11}}{A}, \qquad \widehat{RG}^2_2 = \frac{RG_{22}}{B}, \qquad \widehat{RG}^3_3 = RG_{33}. \tag{3.21}$$

Theorem 3.10. The left invariant metric $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$ is a steady algebraic soliton that gives rise to a self-similar solution of the RG-2 flow on E(1, 1) if and only if

1. $A = 1$ and $\alpha = 2B$, or
2. $A = \frac{1}{3}$ or $A = 3$ and $\alpha = \frac{3}{4}B$.

Proof. Noting that $\widehat{\mathbf{RG}}[\mathbf{g}]$ is diagonalized relative to the Milnor frame, then in accordance with Lemma 3.8, we see that for $\mathbf{g}$ to be steady algebraic soliton we must have $\widehat{RG}^2_2 = 0$ and $\widehat{RG}^1_1 = \widehat{RG}^3_3$.

From (3.20) and (3.21), we see that $\widehat{RG}_1^1 = \widehat{RG}_3^3$ if and only if

$$\frac{1}{2}\frac{(A+1)(A-1)\left(\alpha A^2 + 2\alpha A - 4AB + \alpha\right)}{A^2B^2} = 0.$$

If $A = 1$, then from (3.20) and (3.21) we see that for $\widehat{RG}_2^2$ to be equal to zero we must have$-\frac{1}{2}\frac{4\alpha-8B}{B^2} = 0$,which establishes *1.* If $\alpha A^2 + 2\alpha A - 4AB + \alpha = 0$, then solving for $\alpha = \frac{4AB}{(A+1)^2}$ and substituting into (3.20) and (3.21) for $\widehat{RG}_2^2$, we find that $\widehat{RG}_2^2 = 0$ if and only if

$$-\frac{1}{2}\frac{(3A-1)(A-3)}{AB} = 0,$$

which establishes *2.* Q.E.D.

Remark 3.11. The left invariant metrics $\mathbf{g} = 3\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$ and $\mathbf{g} = \frac{1}{3}\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$ are isometric under an automorphism of the Lie algebra of E(1,1). With respect to the Milnor Frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, it is readily established that any invertible linear operator that is represented in matrix form relative to the chosen frame as $F = \begin{pmatrix} f_1^1 & f_2^1 & f_3^1 \\ 0 & \pm 1 & 0 \\ f_3^1 & f_2^3 & f_1^1 \end{pmatrix}$ is an automorphism of the Lie algebra. Taking $F = \begin{pmatrix} 0 & 0 & \sqrt{3} \\ 0 & -1 & 0 \\ \sqrt{3} & 0 & 0 \end{pmatrix}$ establishes the isometry between the two metrics in question. Similarly, one is able to show that every left invariant metric on E(1,1) is equivalent to one of the form $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$ with $A \geq 1$.

3.6 $\widetilde{\mathrm{SL}(2)}$

Let $\mathcal{H} = \widetilde{\mathrm{SL}(2)}$, the universal cover of $\mathrm{SL}(2,\mathbb{R})$, and let $\mathbf{g}$ be a left invariant metric on $\mathcal{H}$ with Milnor frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The metric then takes the form $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$ and the Lie algebra structure is determined by the non-zero brackets

$$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, \qquad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2, \qquad \text{and} \qquad [\mathbf{e}_1, \mathbf{e}_2] = -\mathbf{e}_3.$$

We will now show that $\widetilde{\mathrm{SL}(2)}$ does not support either a left invariant algebraic XCF soliton or a left invariant algebraic RG-2 soliton that gives rise to a self-similar solution of the corresponding flow. The proof relies on a lemma appropriately adapted from Lemma 2.13 and on the possible signatures for the Ricci tensor of a left invariant metric on $\widetilde{\mathrm{SL}(2)}$. Specifically, in [28] (Corollary 4.7), Milnor establishes that the signature of the Ricci tensor of any left invariant metric on $\widetilde{\mathrm{SL}(2)}$ must be either $(+,-,-)$ or $(0,0,-)$.

Lemma 3.12. Let $\mathbf{g}$ be a left invariant metric on $\widetilde{\mathrm{SL}(2)}$ expressed relative to a Milnor frame as $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$. If $\widehat{\mathbf{T}}[\mathbf{g}]$ is diagonalized relative the Milnor frame, then $\mathbf{g}$ is an algebraic $\mathbf{T}$-soliton for the evolution equation $\frac{\partial \mathbf{g}}{\partial t} = \mathbf{T}[\mathbf{g}]$ if and only if $\widehat{T}_1^1 = \widehat{T}_2^2 = \widehat{T}_3^3$, and in this instance $\mathbf{T}[\mathbf{g}]$ must be a scalar multiple of $\mathbf{g}$.

Proof. Assume that $\widehat{\mathbf{T}}[\mathbf{g}]$ is diagonalized with respect to the selected frame. Since $[\mathbf{e}_l, \mathbf{e}_m] \neq 0$ for all pairs (l, m) where l and m are distinct, then according to Lemma 2.13 the operator $\mathbf{D} = \widehat{\mathbf{T}}[\mathbf{g}] - \beta\mathrm{Id} : \mathfrak{h} \to \mathfrak{h}$ is a derivation of the Lie algebra if and only if $[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1$, $[\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2$, and $[\mathbf{e}_1, \mathbf{e}_2] = -\mathbf{e}_3$ are eigenvectors for $\mathbf{D} = \left(d^i_j\right) = \left(\hat{T}^i_j - \beta\delta^i_j\right)$ with eigenvalues $d^1_1 = d^2_2 + d^3_3$, $d^2_2 = d^3_3 + d^1_1$, and $d^3_3 = d^1_1 + d^2_2$, respectively. The corresponding system of equations admits a solution if and only if $d^1_1 = d^2_2 = d^3_3 = 0$, or equivalently, $\widehat{T}^1_1 = \widehat{T}^2_2 = \widehat{T}^3_3 = \beta$. Q.E.D.

This allows us to establish the following.

Theorem 3.13. The Lie group $\mathcal{H} = \widetilde{\mathrm{SL}(2)}$ does not support either a left invariant XCF algebraic soliton or a left invariant RG-2 algebraic soliton that gives rise to a self-similar solution to the RG-2 flow.

Proof. We will first establish that $\widetilde{\mathrm{SL}(2)}$ does not support a left invariant algebraic XCF soliton. By Lemma 3.12, $\mathbf{g}$ is an algebraic soliton for the XCF if and only if $\beta = \widehat{H}^1_1 = \widehat{H}^2_2 = \widehat{H}^3_3$. According to (3.7), the cross curvature operator is $\widehat{\mathbf{H}}[\mathbf{g}] = \mathrm{diag}\left(K_2K_3, K_1K_3, K_1K_2\right)$ and it follows that we must have $\beta = 0$ (and the algebraic soliton is actually a fixed point) or the sectional curvature of the metric $\mathbf{g}$ must be constant. The restrictions on the possible signatures of the Ricci tensor ([28], Corollary 4.7) show that $\widetilde{\mathrm{SL}(2)}$ does not support a metric of constant sectional curvature. Furthermore, $\beta = 0$ requires that at least two of three principal sectional curvatures must be zero. From (3.5), we find that in this case the signature of the Ricci tensor would be $(+, +, 0)$, $(0, -, -)$, or $(0, 0, 0)$, none of which are possible for a left invariant metric on $\widetilde{\mathrm{SL}(2)}$. We conclude that $\widetilde{\mathrm{SL}(2)}$ does not support an algebraic XCF-soliton.

The proof that $\widetilde{\mathrm{SL}(2)}$ does not support an algebraic soliton that gives rise to self similar solution for the RG-2 flow follows similarly. According to Proposition 2.5 and Proposition 2.8 we only need to concern ourselves with steady algebraic solitons (i.e., $\beta = 0$). From Lemma 3.12, we find that $\mathbf{D} = \widehat{\mathbf{RG}}[\mathbf{g}]$ is a derivation of the Lie algebra if and only $\widehat{RG}^1_1 = \widehat{RG}^2_2 = \widehat{RG}^3_3 = 0$. From (3.8) it follows that for $\mathbf{D} = \widehat{\mathbf{RG}}[\mathbf{g}]$ to be a derivation of the Lie algebra the system of equations

$$\begin{aligned}
\alpha\left((K_2)^2 + (K_3)^2\right) &= -2\left(K_2 + K_3\right) = -2Rc_{11} \\
\alpha\left((K_3)^2 + (K_1)^2\right) &= -2\left(K_3 + K_1\right) = -2Rc_{22} \\
\alpha\left((K_1)^2 + (K_2)^2\right) &= -2\left(K_1 + K_2\right) = -2Rc_{33}
\end{aligned} \tag{3.22}$$

must admit a solution. Again, relying on the permissible signatures of the Ricci tensor of a left invariant Riemannian metric on $\widetilde{\mathrm{SL}(2)}$ ($(+, -, -)$ or $(0, 0, -)$), we conclude that (3.22) does not admit any solutions. Thus, $\widetilde{\mathrm{SL}(2)}$ does not support a steady algebraic RG-2 soliton structure that gives rise to a self-similar solution of the RG-2 flow. Q.E.D.

3.7 SU(2)

Let $\mathcal{H} = \mathrm{SU}(2)$ and let $\mathbf{g}$ be a left invariant metric on $\mathcal{H}$. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a Milnor frame for $\mathbf{g}$, with the corresponding Lie algebra structure determined by the non-zero brackets

$$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, \qquad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2, \qquad [\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3,$$

and $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$.

Making the appropriate substitutions into (3.4) and (3.5), respectively, we find that the principal sectional curvatures of $\mathbf{g}$ are

$$\begin{aligned} K_1 = K(\mathbf{e}_2 \wedge \mathbf{e}_3) &= \frac{-3A^2 + B^2 + C^2 + 2AB + 2AC - 2BC}{4ABC} \\ K_2 = K(\mathbf{e}_1 \wedge \mathbf{e}_3) &= \frac{-3B^2 + C^2 + A^2 + 2BC + 2BA - 2CA}{4ABC} \\ K_3 = K(\mathbf{e}_1 \wedge \mathbf{e}_2) &= \frac{-3C^2 + A^2 + B^2 + 2CA + 2CB - 2BA}{4ABC}. \end{aligned} \tag{3.23}$$

In finding the left invariant metrics on SU(2) that are algebraic solitons giving rise to self similar solutions of the XCF or the RG-2 flow, we will again rely on the permissible signatures of the corresponding Ricci tensor, which are stated in [28]. Namely, in [28] (Corollary 4.5), Milnor establishes that the signature of the Ricci curvature tensor of a left invariant metric on SU(2) must be $(+,+,+)$, $(+,0,0)$, or $(+,-,-)$.

3.7.1 XCF

The cross curvature tensor of $\mathbf{g}$ is $\mathbf{H}[\mathbf{g}] = H_{11}\omega^1 \otimes \omega^1 + H_{22}\omega^2 \otimes \omega^2 + H_{33}\omega^3 \otimes \omega^3$, where

$$H_{11} = AK_2K_3, \qquad H_{22} = BK_1K_3, \qquad H_{33} = CK_1K_2,$$

and the corresponding cross curvature operator $\widehat{\mathbf{H}}[\mathbf{g}] = \left(\widehat{H}^i_j\right)$ is represented in the given basis by

$$\widehat{\mathbf{H}}[\mathbf{g}] = \mathrm{diag}\left(K_2K_3, K_1K_3, K_1K_2\right), \tag{3.24}$$

with K_1, K_2, and K_3 as in (3.23).

Proposition 3.14. A left invariant metric $\mathbf{g} = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$ is a steady algebraic soliton for the XCF on SU(2) if and only if $\mathbf{g}$ has constant sectional curvature (i.e., $A = B = C$). Moreover, if $\mathbf{g}$ has constant sectional curvature, then $\mathbf{g}$ is an expanding algebraic soliton for the +XCF and a shrinking algebraic soliton for the −XCF.

Remark 3.15. Note that Lemma 3.12 applies equally well to SU(2) and that the proof carries through without any modifications.

Proof. From Lemma 3.12, it follows that for $\mathbf{g}$ to be an algebraic +XCF-soliton with derivation $\mathbf{D} = \widehat{\mathbf{H}}[\mathbf{g}] - \beta\mathrm{Id}$ we must have $\beta = \widehat{H}^1_1 = \widehat{H}^2_2 = \widehat{H}^3_3$. Since $\widehat{H}^1_1 = K_2K_3$, $\widehat{H}^2_2 = K_1K_3$ and $\widehat{H}^3_3 = K_1K_2$, $\mathbf{g}$ being an algebraic soliton requires that $\beta = 0$ or that $\mathbf{g}$ have constant sectional curvature (which only occurs when $A = B = C$). That $\mathbf{g}$ is an algebraic soliton when $A=B=C$ is clear. To verify that there are no steady algebraic solitons for the XCF, observe that $\beta=0$ forces at least two of the three principal sectional curvatures to be equal to zero. Since $\mathbf{Rc}[\mathbf{g}] = A(K_2+K_3)\,\omega^1\otimes\omega^1 + B(K_1+K_3)\,\omega^2\otimes\omega^2 + C(K_1+K_2)\,\omega^3\otimes\omega^3$, then having two of the three principal sectional curvatures equal to zero would force the signature of the Ricci tensor to be $(+,+,0)$, $(-,-,0)$, or $(0,0,0)$, none of which are possible. Thus, we find that there are no left invariant metrics on SU(2) where at least two of the three principal sectional curvatures are zero and we conclude that the only algebraic solitons are the metrics of constant sectional curvature. Q.E.D.

3.7.2 Algebraic RG-2 solitons

Proposition 3.16. A left invariant metric $\mathbf{g} = A\omega^1\otimes\omega^1 + B\omega^2\otimes\omega^2 + C\omega^3\otimes\omega^3$ is an algebraic soliton that gives rise to a self-similar solution for the RG-2 flow on SU(2) if and only if

1. $\mathbf{g}$ has constant sectional curvature ($A=B=C$) and $\alpha=-8A$,
2. $A=\frac{4}{3}B=\frac{4}{3}C$ and $\alpha=-\frac{9}{2}A$,
3. $A=B=\frac{3}{4}C$ and $\alpha=-6A$, or
4. $A=C=\frac{3}{4}B$ and $\alpha=-6A$.

Remark 3.17. As previously noted, the RG-2 flow is not of physical interest when α is negative and all of the algebraic solitons occur when α is negative.

Proof. Applying Lemma 3.12 we find that for a left invariant metric on SU(2) of the form $\mathbf{g} = A\omega^1\otimes\omega^1 + B\omega^2\otimes\omega^2 + C\omega^3\otimes\omega^3$, $\mathbf{D}=\widehat{\mathbf{RG}}[\mathbf{g}]$ is a derivation of the Lie algebra $\mathfrak{h}$ if and only if $\mathbf{D}=\widehat{\mathbf{RG}}[\mathbf{g}]=0$. Combined with (3.8), it follows that for $\mathbf{D}=\widehat{\mathbf{RG}}[\mathbf{g}]$ to be a derivation of the Lie algebra then the system of equations

$$\begin{cases} \alpha\left((K_2)^2+(K_3)^2\right)+2(K_2+K_3)=0 \\ \alpha\left((K_3)^2+(K_1)^2\right)+2(K_3+K_1)=0 \\ \alpha\left((K_1)^2+(K_2)^2\right)+2(K_1+K_2)=0 \end{cases} \tag{3.25}$$

must admit a solution. The system of equations above is equivalent to

$$\begin{cases} K_1(\alpha K_1+2) &= 0 \\ K_2(\alpha K_2+2) &= 0 \\ K_3(\alpha K_3+2) &= 0. \end{cases} \tag{3.26}$$

As noted in the proof of Proposition 3.14, SU(2) does not support a left invariant metric where two or more of the principal sectional curvatures are equal to zero. Thus, the system of equations (3.26) is satisfied when the principal sectional curvatures are all equal to $-\frac{2}{\alpha}$, or when one of the principal sectional curvatures $K_l=0$ and the remaining principal sectional curvatures K_m and K_n are equal to $-\frac{2}{\alpha}$. In this case, the conditions on the permissible Ricci tensors show that all algebraic solitons will have a Ricci tensor with signature $(+,+,+)$ and α will necessarily be negative. The exact solutions can then be obtained by using (3.23) to find the appropriate values of A, B and C. Q.E.D.

Remark 3.18. Note that steady algebraic RG-2 solitons are actually fixed points for the RG-2 flow on SU(2).

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