Chromatic number of Harary graphs

Adel P. Kazemi* and Parvin Jalilolghadr

College of Mathematical Sciences, University of Mohaghegh Ardabili, P.O.Box 5619911367, Ardabil, Iran
*Corresponding author

E-mail: adelpkazemi@yahoo.com, a.p.kazemi@uma.ac.ir, p_jalilolghadr@yahoo.com

Abstract

A proper coloring of a graph G is a function from the vertices of the graph to a set of colors such that any two adjacent vertices have different colors, and the chromatic number of G is the minimum number of colors needed in a proper coloring of a graph. In this paper, we will find the chromatic number of the Harary graphs, which are the circulant graphs in some cases.

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1 Introduction

All graphs considered here are finite, undirected and simple. For standard graph theory terminology not given here we refer to [3]. Let $G = (V, E)$ be a graph with the *vertex set* V of *order* n and the *edge set* E of *size* m.

A *proper coloring* of a graph G is a function from the vertices of the graph to a set of colors such that any two adjacent vertices have different colors, and the *chromatic number* $\chi(G)$ of G is the minimum number of colors needed in a proper coloring of a graph [3]. In a proper coloring of a graph a *color class* is the independent set of all same colored vertices of the graph. If f is a proper coloring of G with the color classes V_1, V_2, ..., V_l such that every vertex in V_i has color i, we simply write $f = (V_1, V_2, ..., V_l)$.

As you see in many references such as [3], the Harary graphs are defined as follows: given $2m < n$, place n vertices around a circle, equally spaced. Form $H_{2m,n}$ by making each vertex adjacent to the nearest m vertices in each direction around the circle. If n is even, form $H_{2m+1,n}$ by making each vertex adjacent to the nearest m vertices in each direction and to the diametrically opposite vertex. Both kinds are regular. When n is odd, index the vertices by the integers modulo n. Construct $H_{2m+1,n}$ from $H_{2m,n}$ by adding the edges $i \leftrightarrow i + \frac{n-1}{2}$ for $1 \leq i \leq \frac{n+1}{2}$. Obviously, $H_{2,n} = C_n$ and $H_{n-1,n} = K_n$ which C_n and K_n denote the cycle and complete graph of order n, respectively. These cases are not the subject of our study.

For $[n] = \{1, 2, ..., n\}$ and a subset D of it, the *circulant* graph $G(n, D)$ is a graph with the vertex set $[n]$, and ij is an edge if and only if $i - j$ (to modulo n) belongs to $D \cup (-D)$. The study of chromatic number of circulant graphs of small degree has been widely considered in many papers such as [1, 2, 4].

Every one can see that the Harary graph $H_{2m,n}$ is the circulant graph $G(n, D)$ with $D = \{1, 2, ..., m\}$, and the Harary graph $H_{2m+1,n}$ is the circulant graph $G(n, D)$ with $D = \{1, 2, ..., m\} \cup \{n/2\}$ when n is even, while for odd n the Harary graph $H_{2m+1,n}$ is not a circulant graph. Here, we will find the chromatic number of Harary graphs.

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Recall that for a graph G of order n, $\alpha(G)$ or simply α denotes the *independence number* of G, which is the maximum cardinality of an independent set S in G. It can be easily verify that for any Harary graph H,

$$\alpha(H) = \begin{cases} \lfloor \frac{n}{m+1} \rfloor - 1 & \text{if } H = H_{2m+1,n},\ n \text{ is even and } n \equiv 0 \pmod{2m+2}, \\ \lfloor \frac{n}{m+1} \rfloor & \text{otherwise.} \end{cases}$$

By considering this fact that each color class is an independent set, and every independent set has cardinality at most $\alpha(G)$, we conclude

$$\chi(G) \geq \lceil \frac{n}{\alpha(G)} \rceil. \tag{1.1}$$

Since the Harary graph $H_{m,n}$ is a complete graph of order n if and only if $n = m+1$, and so its chromatic number is n, so in this paper, we always assume that $n > m+1$, and prove

$$\chi(H_{m,n}) = \begin{cases} \lceil \frac{n}{\alpha} \rceil + 1 & \text{if } n = m+3 \geq 10, \text{ and } m \equiv 3 \pmod{4}, \\ \lceil \frac{n}{\alpha} \rceil & \text{otherwise.} \end{cases}$$

Through this paper, we consider

$$n - t \equiv s \pmod{t-1}, \tag{1.2}$$

where $t = \lceil \frac{n}{\alpha} \rceil$, and α denotes the independence number of a Harary graph.

2 The chromatic number of $H_{2m,n}$

In the following theorem we prove $\chi(H_{2m,n}) = \lceil \frac{n}{\alpha} \rceil$.

Theorem 2.1. For any integers $m > 1$ and $n \geq 2m+2$, $\chi(H_{2m,n}) = \lceil \frac{n}{\alpha} \rceil$.

Proof. Since $\alpha = \lfloor \frac{n}{m+1} \rfloor$, we may assume that $n = \alpha(m+1) + r$, for some $0 \leq r \leq m$. Let $r = 0$. Since for each $1 \leq i \leq m+1$, the set $V_i = \{i + (m+1)j \mid 0 \leq j \leq \alpha - 1\}$ is independent, we conclude $\chi(H_{2m,n}) = m+1 = \lceil \frac{n}{\alpha} \rceil$, by (1.1). So, we may assume $r \neq 0$. Clearly, $t > m+1$, and we continue our proof in the following two cases.

Case 1: $n - m > (\alpha - 1)t$. Let $f = (V_1, V_2, ..., V_t)$ be a coloring function of $H_{2m,n}$ in which

$$V_i = \{i + kt \mid 0 \leq k \leq \alpha - 1\}, \quad \text{for } 1 \leq i \leq n - (\alpha - 1)t,$$

and

$$V_i = \{i + kt \mid 0 \leq k \leq \alpha - 2\}, \quad \text{for } n - (\alpha - 1)t + 1 \leq i \leq t.$$

The condition $n-m > (\alpha-1)t$ guarantees that each of the sets V_i is independent, and so $\chi(H_{2m,n}) = m + 1 = \lceil \frac{n}{\alpha} \rceil$, by (1.1).

Case 2: $n - m \leq (\alpha - 1)t$. Let $f = (V_1, V_2, ..., V_t)$ be a coloring function of $H_{2m,n}$ in which

$$V_i = \{i + kt \mid 0 \leq k \leq s - 1\} \cup \{i + st + k(t-1) \mid 0 \leq k \leq \alpha - s - 1\},$$

for $1 \le i \le t-1$, and $V_t = \{n\} \cup \mathcal{B}$, where

$$\mathcal{B} = \begin{cases} \varnothing & \text{if} \quad s = 0 \\ \{kt \mid 1 \le k \le s\} & \text{if} \quad s \ne 0. \end{cases}$$

Since $|V_t| = s+1$ and $|V_i| = \alpha$ for each $1 \le i \le t-1$, and $n = \alpha(t-1)+s+1$, we conclude that the distance between every two vertices in each V_i is at least $t-1$, ans so $t-1 \ge m+1$ implies that each of the sets V_i is independent. Hence $\chi(H_{2m,n}) = m+1 = \lceil \frac{n}{\alpha} \rceil$, by (1.1). Q.E.D.

3 The chromatic number of $H_{2m+1,n}$ with even n

By considering this fact that

$$\alpha(H_{2m+1,n}) = \begin{cases} \lfloor \frac{n}{m+1} \rfloor - 1 & \text{if } n \text{ is even and } n \equiv 0 \pmod{2m+2}, \\ \lfloor \frac{n}{m+1} \rfloor & \text{otherwise,} \end{cases}$$

we have $n = \alpha(m+1)+r$ or $n = (\alpha+1)(m+1)+r$, for some $0 \le r \le m$. Now, let $r = 0$. In the first case, $\chi(H_{2m+1,n}) = m+1 = t$ and in the second case, $t > m+1$. If $r \ne 0$, then $2m+2 \nmid n$ and so $n = \alpha(m+1)+r$ for some $1 \le r \le m$ which easily implies that $t > m+1$. So in this section, Without loss of generality, we may assume that $t > m+1$. Also, without loss of generality, we may assume that $s \ne 0$ if $n - m \le (\alpha-1)t$. Because $s = 0$ implies $2t-2 \nmid n$, and so the set $\{1 + k(t-1) | 0 \le k \le \alpha - s - 1\}$ is independent for each $1 \le i \le t$. Therefore $\chi(H_{2m+1,n}) = t$.

To find the chromatic number of the Harary graph $H_{2m+1,n}$, with even n, we need the following two lemmas.

Lemma 3.1. Let $n - m \le (\alpha-1)t$. If $2t \mid n$, then $\frac{n}{2t} > s$.

Proof. Set $\frac{n}{2t} := q$. First, let $\frac{n}{2} - st = j(t-1) - it$, for some $0 \le i \le s-1$ and some $1 \le j \le \alpha-s-1$. Then $(i+q-s)t = j(t-1)$. Since t and $t-1$ are coprime, j is a multiple of t and $i+q-s$ is a multiple of $t-1$. Specially, $i+q-s \ge t-1$. Moreover, $i-s \le -1$ and so $t-1 \le i+q-s \le q-1$ which implies that $t \le q$. Since $q \le s$ implies $t \le s$, we have $q > s$.

Now, let $\frac{n}{2} - st \ne j(t-1) - it$, for each $0 \le i \le s-1$ and each $1 \le j \le \alpha - s - 1$. By knowing $n = \alpha(t-1)+s+1$, since $q \le s$ implies $\frac{n}{2} - st = j(t-1) - it$ for $i = q-1$ and $j = \alpha - s - 1$, we obtain $q > s$. Q.E.D.

Let $V_i' = \{i + kt \mid 0 \le k \le s-1\}$ and $V_i'' = \{i + st + k(t-1) \mid 0 \le k \le \alpha - s - 1\}$, for $1 \le i \le t$, be subsets of the vertex set of the Harary graph $H_{2m+1,n}$. We note that $V_i' \cap V_i'' = \varnothing$, and V_i' is independent, by Lemma 3.1, because V_i' is independent if and only if either $2t \nmid n$ or $2t \mid n$ and $\frac{n}{2t} > s$. Also V_i'' is independent if and only if either $2t-2 \nmid n$ or $2t-2 \mid n$ and $\frac{n}{2t-2} > \alpha - s - 1$. So for each $0 \le i \le s-1$, the set $V_i = V_i' \cup V_i''$ is independent if and only if the set V_i'' is independent and $\frac{n}{2} - st \ne j(t-1) - it$, for each $1 \le j \le \alpha - s - 1$. Next lemma states that if $\frac{n}{2} - st = j(t-1) - it$ for some $0 \le i \le s-1$ and some $1 \le j \le \alpha - s - 1$, then the set V_i'' is again independent.

Lemma 3.2. Let $n - m \le (\alpha-1)t$. If $\frac{n}{2} - st = k(t-1) - pt$ for some $0 \le p \le s-1$ and some $1 \le k \le \alpha - s - 1$, then the set $V_i'' = \{i + st + k(t-1) \mid 0 \le k \le \alpha - s - 1\}$ is independent, where $1 \le i \le t$.

Proof. It is sufficient to prove $2t-2 \nmid n$. Let $2t-2 \mid n$ and $\frac{n}{2}-st=k(t-1)-pt$, for some $0 \le p \le s-1$ and some $1 \le k \le \alpha - s - 1$. Then $(s-p)t = (\frac{n}{2t-2}-k)(t-1)$. Since t and $t-1$ are coprime, it follows that $\frac{n}{2t-2}-k$ is a multiple of t and $s-p$ is a multiple of $t-1$, specially $t-1 \le s-p$. On the other hand, we have $s+1 \le t-1$, which implies $p \le -1$, a contradiction. Hence $2t-2 \nmid n$, and so the set V_i'' is independent, where $1 \le i \le t$. Q.E.D.

Theorem 3.3. For each even n with $n \ge 2m+3$,

$$\chi(H_{2m+1,n}) = \begin{cases} \lceil \frac{n}{\alpha} \rceil + 1 & \text{if } n = 2m+4 \ge 10, \text{ and } m \equiv 1 \pmod 2, \\ \lceil \frac{n}{\alpha} \rceil & \text{otherwise.} \end{cases}$$

Proof. First let $n = 2m+4 \ge 10$ and $m \equiv 1 \pmod 2$. Let $\chi(H_{2m+1,n}) = t = \lceil \frac{n}{\alpha} \rceil$, and let f be a proper coloring function of $H_{2m+1}n$. Since the subgraph of the graph induced by the vertices 1, 2, ..., $m+1$ is a clique, we may assume that $f(i) = i$ for each $1 \le i \le m+1$. Then for each vertex $m+2 \le i \le 2m+4$,

$$f(m+2) \in \{1, m+2\},\ f(m+3) \in \{2, m+2\},\ f(n) \in \{m+1, m+2\},$$

$$f(n-1) \in \{m, m+2\},\ f(m+i) \in \{i-3, i-1, m+2\},\ 4 \le i \le m+2.$$

Now we discuss on the following two cases.

Case 1. $f(m+2) = m+2$. Then $f(n) = m+1$, and $f(m+2i+1) = 2i$, for $1 \le i \le \frac{m+1}{2}$, which implies $f(n) = f(n-2) = m+1$, a contradiction. Because the distance between the vertices n and $n-2$ is less than m.

Case 2. $f(m+2) = 1$. First, by a proof similar to Case 1, we have $f(n) \ne m+2$. Hence $f(n) = m+1$. We also see that for at most one vertex $m+3 \le i \le n-1$, we may have $f(i) = m+2$. Let $f(m+3) = m+2$. Then $f(n-1) = m$, which implies $f(m+2i) = 2i-1$, for $2 \le i \le \frac{m+1}{2}$, a contradiction. Hence $f(m+3) = 2$. Also, with a similar proof, we will have $f(n-1) \ne m+2$. If for each vertex $m+3 \le i \le n-1$, $f(i) \ne m+2$, then $f(n-2i-1) = m-2i$, for $0 \le i \le \frac{m-1}{2}$, a contradiction (Because $f(m+4) = f(m+2) = 1$). Therefore, for only one vertex $m+3 \le i \le n-2$, $f(i) = m+2$. Then $f(n-1) = m$ and $f(n-3) \in \{m+2, m-2\}$.

Since $f(n-3) = m+2$ implies $f(n-2) = m+1$, which is a contradiction to $f(n) = m+1$, we have $f(n-3) = m-2$. Then $f(n-5) \in \{m+2, m-4\}$. By continuing this method, we will have $f(m+4) = m+2$. Hence $f(n+2i+1) = 2i$, for $2 \le i \le \frac{m+1}{2}$, a contradiction. Therefore $\chi(H_{2m+1,n}) > \lceil \frac{n}{\alpha} \rceil$. Now since the coloring function f with criterion

$$f(i) = i,\ \text{for } 1 \le i \le m+2,\ f(m+3) = 2,\ f(m+4) = 3,\ f(m+5) = m+3,$$

$$f(m+2i) = 2i-1, f(m+2i+1) = 2i-2,\ \text{for } 3 \le i \le \frac{m+3}{2}$$

is a proper coloring of the graph with $\lceil \frac{n}{\alpha} \rceil + 1$ colors, we obtain $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil + 1$, where $n = 2m+4$ and m is odd.

Now, in the second part of our proof, we may assume that if $n = 2m+4 \ge 10$, then $m \equiv 0 \pmod 2$, and we continue our proof in the following four cases. We recall that $t = \lceil \frac{n}{\alpha} \rceil$.

Case 1. $n - m > (\alpha - 1)t$ and $2t \nmid n$. Let $f = (V_1, V_2, ..., V_t)$ be a coloring function of $H_{2m+1,n}$ in which

$$V_i = \{i + kt \mid 0 \le k \le \alpha - 1\}, \quad \text{for } 1 \le i \le n - (\alpha - 1)t,$$

and

$$V_i = \{i + kt \mid 0 \le k \le \alpha - 2\}, \quad \text{for } n - (\alpha - 1)t + 1 \le i \le t.$$

Since the given coloring function $f = (V_1, V_2, ..., V_t)$ is a proper coloring of $H_{2m+1,n}$, we obtain $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil$.

Case 2. $n - m > (\alpha - 1)t$ and $2t \mid n$. For even t, let

$$V_{2i-1} = \{2i + kt \mid 0 \le k \le n/2t - 1\} \cup \{2i - 1 + kt \mid n/2t \le k \le \alpha - 1\},$$

$$V_{2i} = \{2i - 1 + kt \mid 0 \le k \le n/2t - 1\} \cup \{2i + kt \mid n/2t \le k \le \alpha - 1\},$$

where $1 \le i \le t/2$, and for odd t, let

$$V_{2i-1} = \{2i + kt \mid 0 \le k \le n/2t - 1\} \cup \{2i - 1 + kt \mid n/2t \le k \le \alpha - 1\},$$

$$V_{2i} = \{2i - 1 + kt \mid 0 \le k \le n/2t - 1\} \cup \{2i + kt \mid n/2t \le k \le \alpha - 1\},$$

where $1 \le i \le (t - 3)/2$, and

$$V_{t-2} = \{kt - 1 \mid 1 \le k \le n/2t\} \cup \{kt - 2 \mid n/2t + 1 \le k \le \alpha - 1\} \cup \{n\},$$

$$V_{t-1} = \{kt \mid 1 \le k \le n/2t\} \cup \{kt - 1 \mid n/2t + 1 \le k \le \alpha - 1\} \cup \{n - 2\},$$

$$V_t = \{kt - 2 \mid 1 \le k \le n/2t\} \cup \{kt \mid n/2t + 1 \le k \le \alpha - 1\} \cup \{n - 1\}.$$

In each case, the given coloring function $f = (V_1, V_2, ..., V_t)$ is a proper coloring of $H_{2m+1,n}$, and so $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil$.

Case 3. $n - m \le (\alpha - 1)t$ and $\frac{n}{2} - st \ne j(t-1) - it$, for each $0 \le i \le s-1$ and each $1 \le j \le \alpha - s - 1$. Let $V_i' = \{i + kt \mid 0 \le k \le s - 1\}$ and $V_i'' = \{i + st + k(t - 1) \mid 0 \le k \le \alpha - s - 1\}$ be subsets of the vertex set of the Harary graph $H_{2m+1,n}$, where $1 \le i \le t$. We note that $V_i' \cap V_i'' = \varnothing$, and V_i' is independent, by Lemma 3.2.

First let either $2t - 2 \nmid n$ or $2t - 2 \mid n$ and $\frac{n}{2t-2} > \alpha - s - 1$. Then the set V_i'' is independent, and this condition that $\frac{n}{2} - st \ne j(t-1) - it$ for each $0 \le i \le s-1$ and each $1 \le j \le \alpha - s - 1$, implies that each of the sets $V_i = V_i' \cup V_i''$ is independent. Therefore the coloring function $f = (V_1, V_2, ..., V_t)$ is a proper coloring of $H_{2m+1,n}$, where

$$V_i = \{i + kt \mid 0 \le k \le s - 1\} \cup \{i + st + k(t - 1) \mid 0 \le k \le \alpha - s - 1\},$$

for $1 \le i \le t - 1$, and $V_t = \{kt | 1 \le k \le s\} \cup \{n\}$. Hence $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil$.

Now, let $2t - 2 \mid n$ and $\frac{n}{2t-2} \le \alpha - s - 1$. Then the given coloring function $f = (V_1, V_2, ..., V_t)$ is a proper coloring of $H_{2m+1,n}$, where

$$\begin{aligned} V_i &= \{i + kt \mid 0 \le k \le s\} \\ &\cup \{i + st + k(t - 1) \mid 1 \le k \le n/(2t - 2) - 1\} \\ &\cup \{i + 1 + st + k(t - 1) \mid n/(2t - 2) \le k \le \alpha - s - 1\}, \end{aligned}$$

for each $1 \le i \le t-1$, and $V_t = \{kt | 1 \le k \le s\} \cup \{1 + st + \frac{n}{2}\}$. Hence $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil$.

Case 4. $n - m \le (\alpha - 1)t$ and $\frac{n}{2} - st = q(t-1) - pt$, for some $0 \le p \le s-1$ and some $1 \le q \le \alpha - s - 1$. Then each of the sets $V_i'' = \{i + st + k(t-1) \mid 0 \le k \le \alpha - s - 1\}$ is independent, by Lemma 3.2. Now we define for odd t,

$$\begin{array}{rcl} V_{2l-1} & = & \{2l - 1 + kt \mid 0 \le k \le p-1\} \\ & \cup & \{2l + pt\} \\ & \cup & \{2l - 1 + kt \mid p+1 \le k \le s-1\} \\ & \cup & V''_{2l-1}\ , \end{array}$$

$$\begin{array}{rcl} V_{2l} & = & \{2l + kt \mid 0 \le k \le p-1\} \\ & \cup & \{2l - 1 + pt\} \\ & \cup & \{2l + kt \mid p+1 \le k \le s-1\} \\ & \cup & V''_{2l}\ , \end{array}$$

$$V_t = \{kt \mid 1 \le k \le s\} \cup \{n\},$$

while for even t,

$$\begin{array}{rcl} V_{2l-1} & = & \{2l - 1 + kt \mid 0 \le k \le p-1\} \\ & \cup & \{2l + pt\} \\ & \cup & \{2l - 1 + kt \mid p+1 \le k \le s-1\} \\ & \cup & V''_{2l-1}\ , \end{array}$$

$$\begin{array}{rcl} V_{2l} & = & \{2l + kt \mid 0 \le k \le p-1\} \\ & \cup & \{2l - 1 + pt\} \\ & \cup & \{2l + kt \mid p+1 \le k \le s-1\} \\ & \cup & V''_{2l} \end{array}$$

where $1 \le l \le \lfloor \frac{t}{2} \rfloor$. Then the coloring function $f = (V_1, V_2, ..., V_t)$ is a proper coloring of $H_{2m+1,n}$, and so $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil$. Q.E.D.

4 The chromatic number of $H_{2m+1,n}$ with odd n

Since $\alpha(H_{2m+1,n}) = \lfloor \frac{n}{m+1} \rfloor$ for odd n, we have $n = \alpha(m+1) + r$ for some $0 \le r \le m$. Without loss of generality, we may assume that $t > m+1$. A simple calculation shows that if $s = 0$, then $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil$ (we recall that $n - t \equiv s \pmod{t-1}$, and $t = \lceil \frac{n}{\alpha} \rceil$). So we assume $s \ne 0$.

Theorem 4.1. For each odd n with $n \ge 2m+3$, $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil$.

Proof. We presend our proof in the following two cases.

Case 1. $n - m > (\alpha - 1)t$. We first prove that $2t \nmid n-1$. Assume on the contrary $2t \mid n-1$. Then $n = 1 + s + \alpha(t-1)$ implies that $\alpha - s = (\alpha - \frac{n-1}{t})t$. The condition $\alpha \ne s$ implies that $\alpha - s \ge t$ and so $t < \alpha$. On the other hand, since $n = \alpha(m+1) + r$ for some $0 \le r \le m$, we obtain

$$\begin{array}{rcl} n - m & \le & \alpha(m+1) \\ & \le & \alpha(t-1) \\ & < & (\alpha - 1)t, \end{array}$$

which is a contradiction. Therefore $2t \nmid n-1$. Now let $f = (V_1, V_2, ..., V_t)$ be a coloring function of $H_{2m+1,n}$, in which

$$V_i = \{i + kt \mid 0 \leq k \leq \alpha - 1\}, \quad \text{where } 1 \leq i \leq n - (\alpha - 1)t,$$

and

$$V_i = \{i + kt \mid 0 \leq k \leq \alpha - 2\}, \quad \text{where } n - (\alpha - 1)t + 1 \leq i \leq t.$$

Since the condition $n - m > (\alpha - 1)t$ guarantees that each of the sets V_i is independent, we obtain $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil$.

Case 2. $n - m \leq (\alpha - 1)t$. For $1 \leq i \leq t$, let $V'_i = \{i + kt \mid 0 \leq k \leq s - 1\}$ and $V''_i = \{i + st + k(t-1) \mid 0 \leq k \leq \alpha - s - 1\}$ be subsets of the vertex set of the Harary graph $H_{2m+1,n}$. Since V'_i is independent if and only if either $2t \nmid n-1$ or $2t \mid n-1$ and $\frac{n-1}{2t} > s$, so to prove that V'_i is independent, it is sufficient to show that if $2t \mid n-1$, then $\frac{n-1}{2t} > s$. Since $n = 1 + st + (\alpha - s)(t-1)$, we obtain $(\frac{n-1}{t} - s)t = (\alpha - s)(t-1)$. Then $\frac{n-1}{t} - s$ is a multiple of $t-1$ and $\alpha - s$ is a multiple of t. In particular, $\frac{n-1}{t} - s \geq t - 1$. Since by the definition of s, $s < t - 1$, we obtain $\frac{n-1}{2t} > s$.

Since also V''_i is independent if and only if either $2t-2 \nmid n-1$ or $2t-2 \mid n-1$ and $\frac{n-1}{2t-2} > \alpha - s - 1$, to prove that V''_i is independent, it is sufficient to show that $2t - 2 \nmid n - 1$. For this aim, let $n - t = k(t-1) + s$ for some integers k and $0 < s \leq t - 2$. Then $n - 1 = (k+1)(t-1) + s$ implies $t - 1 \nmid n - 1$, and so $2t - 2 \nmid n - 1$.

Therefore $V_i = V'_i \cup V''_i$ is independent if and only if $\frac{n-1}{2} - st \neq j(t-1) - it$, for every $0 \leq i \leq s-1$ and every $1 \leq j \leq \alpha - s - 1$. So, without loss of generality, we may assume that $\frac{n-1}{2} - st = q(t-1) - pt$ for some integers $0 \leq p \leq s-1$ and $1 \leq q \leq \alpha - s - 1$. Now we define for odd t,

$$\begin{aligned} V_{2l-1} &= \{2l - 1 + kt \mid 0 \leq k \leq p - 1\} \\ &\cup \{2l + pt\} \\ &\cup \{2l - 1 + kt \mid p + 1 \leq k \leq s - 1\} \\ &\cup V''_{2l-1}, \end{aligned}$$

$$\begin{aligned} V_{2l} &= \{2l + kt \mid 0 \leq k \leq p - 1\} \\ &\cup \{2l - 1 + pt\} \\ &\cup \{2l + kt \mid p + 1 \leq k \leq s - 1\} \\ &\cup V''_{2l}, \end{aligned}$$

$$V_t = \{kt | 1 \leq k \leq s\} \cup \{n\},$$

where $1 \leq l \leq \lfloor \frac{t}{2} \rfloor$, while for even t,

$$\begin{aligned} V_{2l-1} &= \{2l - 1 + kt \mid 0 \leq k \leq p - 1\} \\ &\cup \{2l + pt\} \\ &\cup \{2l - 1 + kt \mid p + 1 \leq k \leq s - 1\} \\ &\cup V''_{2l-1}, \end{aligned}$$

$$\begin{aligned} V_{2l} &= \{2l + kt \mid 0 \leq k \leq p - 1\} \\ &\cup \{2l - 1 + pt\} \\ &\cup \{2l + kt \mid p + 1 \leq k \leq s - 1\} \\ &\cup V''_{2l} \end{aligned}$$

where $1 \le l \le \lfloor \frac{t}{2} \rfloor$. Then the coloring function $f = (V_1, V_2, ..., V_t)$ is a proper coloring of $H_{2m+1,n}$, and so $\chi(H_{2m+1,n}) = \lceil \frac{n}{\alpha} \rceil$. Q.E.D.

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