A generalization of $\lambda-$slant Toeplitz operators

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Abstract

We compute and study the behavior of the solutions of the equation $\lambda M_z X = X M_{z^k}$, which are referred as generalized $\lambda-$slant Toeplitz operators, for general complex number λ and $k \geq 2$.

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1 Introduction

Toeplitz operators on the Hardy space H^2, are characterized by the operator equation $U^*XU = X$, where U is the forward unilateral shift operator on the Hardy space H^2. However, Toeplitz operators on the space L^2 are nothing but the operators in the commutant of the multiplication operator M_z and thus can be written as solutions to the operator equation $M_z X = X M_z$. The set $\{e_n : n \in \mathbb{Z}\}$, where $e_n(z) = z^n$, is the standard orthonormal basis of the Hilbert space L^2. Multiple papers have been published in the 1960s, 1970s, 1980s, 1990s, and the 2000s that extend and generalize the study made in the paper [4] of Brown and Halmos. For an integer $k \geq 2$, the $k^{th}-$order slant Toeplitz operators are defined as $U_\varphi = W_k M_\varphi$, where M_φ is the Laurent operator on L^2 induced by φ and W_k is an operator on L^2 such that $W_k(e_i) = e_{i/k}$, if i is divisible by k, otherwise 0. In [1], $k^{th}-$order slant Toeplitz operators are characterized as the solutions of the operator equation $M_z X = X M_{z^k}, k \geq 2$.

Question imposed by *Barría* and Halmos [3] led to the introduction of classes such as class of $\lambda-$Toeplitz operators, $\lambda-$slant Toeplitz operators [5, 6, 8-10]. Motivated by the work of *Avendaño* [2] and *Barría* and Halmos [3], we are inspired to solve the operator equation $\lambda M_z X = X M_{z^k}$, for $\lambda \in \mathbb{C}$ and $k \geq 2$. We call the solutions of the operator equation $\lambda M_z X = X M_{z^k}$, for $\lambda \in \mathbb{C}$ and $k \geq 2$ to be "Generalized $\lambda-$slant Toeplitz operators".

In this paper, we find an explicit formula for the generalized $\lambda-$slant Toeplitz operators and also give a matrix characterization to the generalized $\lambda-$slant Toeplitz operators. We obtain some spectral properties of the generalized $\lambda-$slant Toeplitz operators, which have always been a topic of interest of many mathematicians. An attempt is also made to discuss the properties of the compression of generalized $\lambda-$ slant Toeplitz operators to the Hardy space H^2.

2 Generalized $\lambda-$slant Toeplitz operators

$\lambda-$slant Toeplitz operators are characterized as the operators satisfying the operator equation $\lambda M_z X = X M_{z^2}$ and are discussed in [6]. Now we ask about the solutions of the equation $\lambda M_z X = X M_{z^k}$, for general complex number λ and integer $k \geq 2$. Throughout this paper, we assume k is an integer satisfying $k \geq 2$. We begin with the following definition.

Definition 2.1. For $k \geq 2$ and a fixed complex number λ, an operator X on L^2 is said to be generalized $\lambda-$slant Toeplitz operator if it is a solution of the equation $\lambda M_z X = X M_{z^k}$.

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It is very interesting to obtain the following.

Theorem 2.2. If X is a solution of $\lambda M_z X = XM_{z^k}$, $|\lambda| \neq 1$ then $X = 0$.

Proof. Suppose X is a solution of the equation $\lambda M_z X = XM_{z^k}$, $|\lambda| \neq 1$. We first consider the case $|\lambda| < 1$. In this case, define an operator τ on $\mathfrak{B}(L^2)$ such that $\tau(X) = \lambda M_z X M_{\overline{z}^k}$. Then $\|\tau\| < 1$, which implies that $(I - \tau)$ is invertible. X being solution of the equation $\lambda M_z X = XM_{z^k}$, $(I - \tau)X = 0$. This gives that X is zero operator.

Now consider the case $|\lambda| > 1$. In this case, we define τ as $\tau(X) = M_{\overline{z}} X M_{z^k}$. Now we find the invertibility of $(\lambda I - \tau)$, which provides that X is zero operator. Q.E.D.

We now consider the case for $|\lambda| = 1$ and claim the following.

Theorem 2.3. For $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the operator equation $\lambda M_z X = XM_{z^k}$ admits of non-zero solutions and each non-zero solution is of the form $X = D_{\overline{\lambda}} S$, where S is a $k^{th}-$order slant Toeplitz operator and $D_{\overline{\lambda}}$ is the composition operator on L^2 induced by $z \mapsto \overline{\lambda} z$, i.e, $D_{\overline{\lambda}} f(z) = f(\lambda z)$ for all $f \in L^2$.

Proof. Suppose X is an operator of the form $D_{\overline{\lambda}} S$ for some $k^{th}-$order slant Toeplitz operator S. Since $M_z D_{\overline{\lambda}} = \overline{\lambda} D_{\overline{\lambda}} M_z$ and $M_z S = SM_{z^k}$, it is easy to verify that X satisfies $\lambda M_z X = XM_{z^k}$.

Conversely, suppose that X is an operator satisfying $\lambda M_z X = XM_{z^k}$. Then $M_z D_\lambda X = D_\lambda X M_{z^k}$, which implies that $D_\lambda X$ is a $k^{th}-$order slant Toeplitz operator. Therefore, $X = D_{\overline{\lambda}} S$ for some $k^{th}-$order slant Toeplitz operator S. Q.E.D.

Since $k^{th}-$order slant Toeplitz operators are always of the form $U_\varphi (= W_k M_\varphi)$, $\varphi \in L^\infty$ [1], hence in view of Theorem 2.3, we see that generalized $\lambda-$slant Toeplitz operator are always of the form $U_{\varphi,\lambda} = D_{\overline{\lambda}} U_\varphi$. If $\varphi = \sum_{n\in\mathbb{Z}} a_n e_n$ in L^∞, $U_{\varphi,\lambda}$ is given by

$$U_{\varphi,\lambda} e_i = \sum_{m\in\mathbb{Z}} \lambda^m a_{km-i} e_m$$

for each $i \in \mathbb{Z}$.

Since for $|\lambda| \neq 1$, the only generalized $\lambda-$slant Toeplitz operator is the zero operator so now onward the generalized $\lambda-$slant Toeplitz operator $U_{\varphi,\lambda}$, $\varphi \in L^\infty$, is used in reference to the solution of the equation $\lambda M_z X = XM_{z^k}$, where $|\lambda| = 1$. It is clear that $\|U_{\varphi,\lambda}\| \leq \|\varphi\|_\infty$. For $\varphi = \sum_{n\in\mathbb{Z}} a_n e_n \in L^\infty$, $\lambda \in \mathbb{C}$, the adjoint of $U_{\varphi,\lambda}$ satisfies $U^*_{\varphi,\lambda} = (D_{\overline{\lambda}} U_\varphi)^* = U^*_\varphi D_\lambda$ and for each $i, j \in \mathbb{Z}$, $\langle U^*_{\varphi,\lambda} e_j, e_i \rangle = \langle e_j, U_{\varphi,\lambda} e_i \rangle = \langle e_j, \sum_{m\in\mathbb{Z}} \lambda^m a_{km-i} e_m \rangle = \overline{\lambda}^j \overline{a}_{kj-i}$. This helps us to prove the following.

Theorem 2.4. Adjoint of a non-zero generalized $\lambda-$slant Toeplitz operator is not a generalized $\lambda-$slant Toeplitz operator.

Proof. Let, if possible, $U^*_{\varphi,\lambda}$ be a non-zero generalized $\lambda-$slant Toeplitz operator. Then for each $i, j \in \mathbb{Z}$,

$$\begin{aligned}
\lambda \langle U^*_{\varphi,\lambda} e_j, e_i \rangle &= \langle U^*_{\varphi,\lambda} e_{j+k}, e_{i+1} \rangle \\
\Rightarrow \lambda \overline{\lambda}^j \overline{a}_{kj-i} &= \overline{\lambda}^{(j+k)} \overline{a}_{k(j+k)-(i+1)} \\
\Rightarrow \overline{a}_{kj-i} &= \overline{\lambda}^{(k+1)} \overline{a}_{k(j+k)-(i+1)}
\end{aligned}$$

This on substituting $j=0$ provides that $\overline{a}_t = \overline{\lambda}^{n(k+1)}\overline{a}_{n(k^2-1)+t}$ for each $t \in \mathbb{Z}$. Since $a_n \to 0$ as $n \to \infty$, we get that $\overline{a}_t = 0$ for each $t \in \mathbb{Z}$. As a consequence $\varphi = 0$, which contradicts that $U^*_{\varphi,\lambda}$ is non-zero. This completes the proof. Q.E.D.

Next we move on to calculate the norm of the generalized $\lambda-$slant Toeplitz operator $U_{\varphi,\lambda}$. For this, we first prove the following.

Lemma 2.5. Product of a generalized $\lambda-$slant Toeplitz operator and its adjoint is a Laurent operator.

Proof. For $\varphi \in L^\infty$, the k^{th}-order slant Toeplitz operator $U_\varphi = W_k M_\varphi$ satisfies $U_\varphi {U_\varphi}^* = M_\psi$, where $\psi = W_k(|\varphi|^2) = \sum\limits_{m\in\mathbb{Z}} \langle \psi, e_m\rangle e_m \in L^\infty$ (see [1]). This gives $U_{\varphi,\lambda}U^*_{\varphi,\lambda} = D_{\overline{\lambda}} U_\varphi {U_\varphi}^* D_\lambda = D_{\overline{\lambda}} M_\psi D_\lambda$. Now for each $n \in \mathbb{Z}$,

$$\begin{aligned} D_{\overline{\lambda}} M_\psi D_\lambda e_n &= \overline{\lambda}^n D_{\overline{\lambda}} \sum_{m\in\mathbb{Z}} \langle \psi, e_m\rangle e_{m+n} \\ &= (\sum_{m\in\mathbb{Z}} \langle \psi, e_m\rangle \lambda^m e_m) e_n \\ &= M_{\psi_\lambda} e_n. \end{aligned}$$

Therefore $U_{\varphi,\lambda}U^*_{\varphi,\lambda}$ is a Laurent operator induced by the symbol ψ_λ in L^∞ given by $\psi_\lambda(z) = \sum\limits_{n\in\mathbb{Z}} \langle \psi, e_n\rangle \lambda^n z^n$. Q.E.D.

From Lemma 2.5, we have the following.

Theorem 2.6. For $\varphi \in L^\infty$, $\|U_{\varphi,\lambda}\| = \sqrt{\|\psi_\lambda\|_\infty}$, where $\psi_\lambda(z) = \sum\limits_{n\in\mathbb{Z}} \langle \psi, e_n\rangle \lambda^n z^n$, $\psi = W_k(|\varphi|^2)$.

For $\varphi = \sum\limits_{n\in\mathbb{Z}} a_n e_n$ in L^∞, the matrix representation of generalized $\lambda-$slant Toeplitz operator $U_{\varphi,\lambda}$ with respect to the standard orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2$is

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \lambda^{-1}a_{-k+1} & \lambda^{-1}a_{-k} & \lambda^{-1}a_{-k+1} & \lambda^{-1}a_{-k-2} & \cdots & \lambda^{-1}a_{-2k} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & \cdots & a_{-k} & \cdots \\ \cdots & \lambda a_{k+1} & \lambda a_k & \lambda a_{k-1} & \lambda a_{k-2} & \cdots & \lambda a_0 & \cdots \\ \cdots & \lambda^2 a_{2k+1} & \lambda^2 a_{2k} & \lambda^2 a_{2k-1} & \lambda^2 a_{2k-2} & \cdots & \lambda^2 a_k & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Operators like Toeplitz [4] and $k^{th}-$order slant Toeplitz operators [1], are characterized in terms of their respective named matrices. In order to do the same for generalized $\lambda-$slant Toeplitz operators, we define the following notion.

Definition 2.7. For a fixed integer $k \geq 2$, a generalized $\lambda-$slant Toeplitz matrix is a two way infinite matrix (a_{ij}) such that $a_{i+1,j+k} = \lambda a_{i,j}$ for $i, j \in \mathbb{Z}$.

This notion helps us to obtain the following.

Theorem 2.8. A necessary and sufficient condition for an operator S on L^2 to be a generalized $\lambda-$slant Toeplitz operator, $|\lambda| = 1$, is that its matrix (with respect to the standard orthonormal basis $\{e_n : n \in \mathbb{Z}\}$) is a generalized $\lambda-$slant Toeplitz matrix.

Proof. It is clear that the matrix of $U_{\varphi,\lambda}$, $\varphi = \sum\limits_{n\in\mathbb{Z}} a_n e_n \in L^\infty$ is always a generalized $\lambda-$slant Toeplitz matrix.

Conversely, let the matrix (α_{ij}) of an operator S on L^2 be a generalized $\lambda-$slant Toeplitz matrix. Then for all $i, j \in \mathbb{Z}$

$$\lambda\langle Se_j, e_i\rangle = \lambda\alpha_{i,j} = \alpha_{i+1,j+k} = \langle Se_{j+k}, e_{i+1}\rangle = \langle M_z^* S M_{z^k} e_j, e_i\rangle.$$

Thus $\lambda M_z S e_i = S M_{z^k} e_i$ for each $i \in \mathbb{Z}$. Therefore $\lambda M_z S = S M_{z^k}$ and hence S is a generalized $\lambda-$slant Toeplitz operator. Q.E.D.

It is apparent to see that the sum of two generalized $\lambda-$slant Toeplitz operators is a generalized $\lambda-$slant Toeplitz operator. However, the following properties of generalized $\lambda-$slant Toeplitz operators, which are known for $k^{th}-$order slant Toeplitz operators (see [1]), can be proved without any extra efforts.

Proposition 2.9. Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

1. The mapping $\varphi \mapsto U_{\varphi,\lambda}$ from L^∞ into $\mathfrak{B}(L^2)$ is linear and one-one.
2. The set of all generalized $\lambda-$slant Toeplitz operators is weakly closed and hence strongly closed.
3. For an unimodular complex number μ, $D_{\overline{\mu}\lambda} U_{\varphi,\lambda}$ is a generalized $\mu-$slant Toeplitz operator.
4. A generalized $\lambda-$slant Toeplitz operator $U_{\varphi,\lambda}$ for $\varphi \in L^\infty$ is compact if and only if $\varphi = 0$.
5. Let $\lambda = e^{\iota\hat{\theta}}$, $\hat{\theta} \in [0, 2\pi[$. Then $U_{\varphi,\lambda}$ is co-isometry if and only if $|\varphi(\frac{\theta}{k})|^2 + |\varphi(\frac{\theta+2\pi}{k})|^2 + \cdots$ $+|\varphi(\frac{\theta+(k-1)2\pi}{k})|^2 = k$ for a.e. $\theta \in [0, 2\pi[$.
6. For unimodular $\varphi \in L^\infty$, $U_{\varphi,\lambda}$ is always a co-isometry.
7. A generalized λ-slant Toeplitz operator $U_{\varphi,\lambda}$ is a partial isometry if and only if $\varphi = \varphi W_k^*(W_k|\varphi|^2)$

Now we find that the only hyponormal generalized $\lambda-$slant Toeplitz operator on L^2 is the zero operator.

Theorem 2.10. A generalized $\lambda-$slant Toeplitz operator $U_{\varphi,\lambda}$ is hyponormal if and only if $\varphi = 0$.

Proof. Suppose generalized $\lambda-$slant Toeplitz operator $U_{\varphi,\lambda}$ is hyponormal. Then for all $f \in L^2$, $||U_{\varphi,\lambda} f|| \geq ||U^*_{\varphi,\lambda} f||$. On substituting $f = e_0$ in above inequality, we have $\sum\limits_{n\in\mathbb{Z}} |a_{kn}|^2 \geq \sum\limits_{n\in\mathbb{Z}} |\overline{a}_n|^2$, which implies that $a_{kn-m} = 0, m = 1, 2, \cdots, k-1$ for all $n \in \mathbb{Z}$. Now on substituting $f = e_1$ in the inequality, we find $\sum\limits_{n\in\mathbb{Z}} |a_{kn-1}|^2 \geq \sum\limits_{n\in\mathbb{Z}} |\overline{a}_{k-n}|^2$, which yields that $a_{k-n} = 0$ for all $n \in \mathbb{Z}$. Thus $\varphi = 0$.
Converse is obvious. Q.E.D.

We know the fact that an isometry is always hyponormal, so in view of Theorem 2.10, the set of generalized $\lambda-$slant Toeplitz operators does not contain an isometry.

Proposition 2.11. For $\varphi \in L^\infty$, $W_k U_{\varphi,\lambda}$ is a generalized $\lambda-$slant Toeplitz operator if and only if $\varphi = 0$.

Proof. If part of the result is obvious. We prove the reverse part. For, suppose $\varphi = \sum_{n\in\mathbb{Z}} a_n e_n \in L^\infty$ is such that $W_k U_{\varphi,\lambda}$ is a generalized $\lambda-$slant Toeplitz operator. Then $\lambda\langle W_k U_{\varphi,\lambda} e_j, e_i\rangle = \langle W_k U_{\varphi,\lambda} e_{j+k}, e_{i+1}\rangle$, which yields that

$$a_{k^2 i-j} = \lambda^{k-1} a_{k^2 i+k^2-j-k}$$

for each $i, j \in \mathbb{Z}$. This helps us in concluding that for each $t \in \mathbb{Z}$, $a_t = \lambda^{n(k-1)} a_{n(k^2-k)-t} \to 0$ as $n \to \infty$. Therefore $\varphi = 0$. Q.E.D.

Proposition 2.11 helps us to provide a characterization for the product $U_\psi U_{\varphi,\lambda}$ to be a generalized $\lambda-$slant Toeplitz operator, where $U_\psi (= W_k M_\psi)$ is a slant Toeplitz operator and $\varphi, \psi \in L^\infty$.

Theorem 2.12. Let $\varphi, \psi \in L^\infty$. Then $U_\psi U_{\varphi,\lambda}$ is a generalized $\lambda-$slant Toeplitz operator if and only if $\psi(\overline{\lambda} z^k)\varphi(z) = 0$.

Proof. Suppose $\varphi, \psi \in L^\infty$. Then $U_\psi U_{\varphi,\lambda} = W_k M_\psi D_{\overline{\lambda}} W_k M_\varphi = W_k D_{\overline{\lambda}} M_{\psi(\overline{\lambda} z)} W_k M_\varphi = W_k D_{\overline{\lambda}} W_k M_{\psi(\overline{\lambda} z^k)} M_{\varphi(z)} = W_k U_{\psi(\overline{\lambda} z^k)\varphi(z),\lambda}$. On applying the lemma, we have the result. Q.E.D.

It can be easily shown that $W_k M_\varphi W_k M_\psi = W_k M_{\varphi\psi}$ for φ, ψ in the space generated by $\{e_{kn} : n \in \mathbb{Z}\}$. This serves a great tool to show the following.

Theorem 2.13. Let $\varphi, \psi \in L^\infty$ be such that either φ or ψ is $h(z^k)$ for some $h \in L^\infty$. Then $U_{\varphi,\lambda} U_\psi = U_{\varphi\psi,\lambda}$.

Proof. Suppose φ (*or* ψ) $= h(z^k)$ for some $h \in L^\infty$. Then $W_k M_\varphi W_k M_\psi = W_k M_{\varphi\psi}$, which serves that $U_{\varphi,\lambda} U_\psi = D_{\overline{\lambda}} W_k(\varphi\psi) = D_{\overline{\lambda}} W_k M_{\varphi\psi} = D_{\overline{\lambda}} U_{\varphi\psi} = U_{\varphi\psi,\lambda}$. Q.E.D.

As a consequence of Theorem 2.13, we see that the product of a generalized $\lambda-$slant Toeplitz operator with a $k^{th}-$order slant Toeplitz operator induced by a symbol in the space generated by $\{e_{kn} : n \in \mathbb{Z}\}$ becomes a generalized $\lambda-$slant Toeplitz operator. However, in the next result we show that the product of a generalized $\lambda-$slant Toeplitz operator with a multiplication operator is always a generalized $\lambda-$slant Toeplitz operator.

Theorem 2.14. $M_\varphi U_{\psi,\lambda}$ and $U_{\psi,\lambda} M_\varphi$ are always generalized $\lambda-$slant Toeplitz operators for $\varphi, \psi \in L^\infty$. Further, $M_\varphi U_{\psi,\lambda} = U_{\psi,\lambda} M_\varphi$ if and only if $\varphi(\overline{\lambda} z^k)\psi(z) = \varphi(z)\psi(z), z \in \mathbb{T}$.

Proof. With little efforts, we can prove that $\lambda M_z (M_\varphi U_{\psi,\lambda}) = (M_\varphi U_{\psi,\lambda})\ M_{z^k}$ and $\lambda M_z (U_{\psi,\lambda} M_\varphi) = (U_{\psi,\lambda} M_\varphi)\ M_{z^k}$ for $\varphi, \psi \in L^\infty$. As a consequence, both $M_\varphi U_{\psi,\lambda}$ and $U_{\psi,\lambda} M_\varphi$ are generalized $\lambda-$slant Toeplitz operators.

Furhter, we find that $M_{\varphi(z)} U_{\psi(z),\lambda} = M_{\varphi(z)} D_{\overline{\lambda}} W_k M_{\psi(z)} = D_{\overline{\lambda}} M_{\varphi(\overline{\lambda} z)}\ W_k M_{\psi(z)} = D_{\overline{\lambda}} W_k M_{\varphi(\overline{\lambda} z^k)} M_{\psi(z)} = U_{\varphi(\overline{\lambda} z^k)\psi(z),\lambda}$ and $U_{\psi(z),\lambda}\ M_{\varphi(z)} = D_{\overline{\lambda}} W_k M_{\psi(z)} M_{\varphi(z)} = D_{\overline{\lambda}} W_k M_{\psi(z)\varphi(z)} = U_{\varphi(z)\psi(z),\lambda}$. Now Proposition 2.9 (1) gives the result. Q.E.D.

On looking the $k^{th}-$order slant Toeplitz operators as generalized $1-$slant Toeplitz operators, it becomes genuine to know the product of two generalized $\lambda-$slant Toeplitz operators. In order to answer this query, we first prove the the following.

Lemma 2.15. Let $\varphi \in L^\infty$. Then $D_{\overline{\lambda}}W_kU_{\varphi,\lambda}$ is a generalized $\lambda-$slant Toeplitz operator if and only if $\varphi = 0$.

Proof. We need to prove one way only. For, suppose $D_{\overline{\lambda}}W_kU_{\varphi,\lambda}$ is a generalized $\lambda-$slant Toeplitz operator. Then for integers i,j, we have $\lambda\langle D_{\overline{\lambda}}W_kU_{\varphi,\lambda}\ e_j, e_i\rangle = \langle D_{\overline{\lambda}}W_kU_{\varphi,\lambda}e_{j+k}, e_{i+1}\rangle$. This gives $\langle \sum\limits_n \lambda^n a_{kn-j}e_n,\ e_{ki}\rangle = \langle \sum \lambda^n\ a_{kn-j-k}e_n, e_{ki+k}\rangle$ or $a_{k(ki)-j} = \lambda^k a_{k^2(i+1)-(j+k)}$ for each $i,j \in \mathbb{Z}$. From this, we can prove that $a_t = \lambda^{kn}a_{n(k^2-k)+t}$ for all $n \in \mathbb{Z}$. This provide that $a_t = 0$ for all $t \in \mathbb{Z}$ and hence $\varphi = 0$. Q.E.D.

Now, we answer our query in the following form.

Theorem 2.16. The product of two generalized $\lambda-$slant Toeplitz operators is a generalized $\lambda-$slant Toeplitz operator if and only if the product is zero.

Proof. Let $\varphi, \psi \in L^\infty$ and $U_{\varphi,\lambda}$ and $U_{\psi,\lambda}$ be two generalized $\lambda-$slant Toeplitz operators. Now

$$\begin{aligned} U_{\varphi,\lambda}U_{\psi,\lambda} &= D_{\overline{\lambda}}W_kM_\varphi D_{\overline{\lambda}}W_kM_\psi \\ &= D_{\overline{\lambda}}W_kD_{\overline{\lambda}}M_{\varphi(\overline{\lambda}z)}W_kM_{\psi(z)} \\ &= D_{\overline{\lambda}}W_kD_{\overline{\lambda}}W_kM_{\varphi(\overline{\lambda}z^k)\psi(z)} \\ &= D_{\overline{\lambda}}W_kU_{\varphi(\overline{\lambda}z^k)\psi(z),\lambda}. \end{aligned}$$

Now use of Lemma 2.15 completes the proof. Q.E.D.

An immediate information that we receive from Theorem 2.16 is that the class of generalized $\lambda-$slant Toeplitz operators neither forms an algebra nor contains any non-zero idempotent operator.

3 Spectrum of generalized $\lambda-$Slant Toeplitz operators

It is shown in Theorem 2.3 that each generalized $\lambda-$slant Toeplitz operator, $|\lambda| = 1$ is induced on multiplying a generalized slant Toeplitz operator by a unitary composition operator and as a consequence there is a one-one correspondence between the class of generalized $\lambda-$slant Toeplitz operators and the class of generalized slant Toeplitz operators. We use the symbols $\sigma(A)$, $\sigma_p(A)$ and $\Pi(A)$ to denote the spectrum, the point spectrum and the approximate spectrum of an operator A respectively. Motivated by the approach initiated by Ho [8], the following information without any extra efforts can be gathered along the lines of techniques used to obtain the same results in case of $\lambda-$slant Toeplitz operators in [6]. We are giving the outlines of the proof in some cases and refer [8] and [6] for details.

Theorem 3.1. If φ is invertible in L^∞ then $\sigma_p(U_{\varphi,\lambda}) = \sigma_p(U_{\varphi(z^k),\lambda})$, where $\varphi = \sum\limits_{n\in\mathbb{Z}} \langle\varphi, e_n\rangle\lambda^n e_n$.

For $\varphi \in L^\infty$, $M_\varphi = D_{\overline{\lambda}}M_\varphi D_\lambda$ so that $M_\varphi D_{\overline{\lambda}}W_k = D_{\overline{\lambda}}M_\varphi W_k = D_{\overline{\lambda}}W_kM_{\varphi(z^k)} = D_{\overline{\lambda}}U_{\varphi(z^k)} = U_{\varphi(z^k),\lambda}$, where $\varphi = \sum\limits_{n\in\mathbb{Z}} \langle\varphi, e_n\rangle\lambda^n e_n$. We use this observation to obtain the following.

Theorem 3.2. For $\varphi \in L^\infty$, $\sigma(U_{\varphi,\lambda}) = \sigma(U_{\varphi(z^k),\lambda})$, where $\varphi = \sum_{n\in\mathbb{Z}} \langle \varphi, e_n\rangle \lambda^n e_n$.

Proof. Let $\varphi \in L^\infty$ and $\varphi = \sum_{n\in\mathbb{Z}} \langle \varphi, e_n\rangle \lambda^n e_n$. Then $\sigma(U_{\varphi,\lambda}) \cup \{0\} = \sigma(M_\varphi(D_{\overline{\lambda}} W_k)) \cup \{0\} = \sigma(U_{\varphi(z^k),\lambda}) \cup \{0\}$. As $\overline{R(U^*_{\varphi(z^k),\lambda})} = \overline{R(W_k^* D_\lambda M_{\overline{\varphi}})} \subseteq \overline{R(W_k^*)}$, which is the closed linear subspace generated by $\{e_{kn} : n \in \mathbb{Z}\}$, where $R(\cdot)$ stands for the range of the operator $(\cdot)$. Hence $\sigma(U_{\varphi(z^k),\lambda})$ contains 0. Proof completes once we prove that $\sigma(U_{\varphi,\lambda})$ also contains 0. This holds trivially using Theorem 3.1 in case φ is invertible in L^∞. We consider the case when φ is non-invertible element of L^∞. In this case, we get a sequence $< \varphi_n >$ of invertible elements in L^∞ satisfying $\|\varphi_n - \varphi\|_\infty \to 0$ as $n \to \infty$. Then $\|\varphi_n - \varphi\|_\infty = \|\varphi_n - \varphi\|_\infty \to 0$ as $n \to \infty$, where $\varphi_n(z) = \varphi_n(\lambda z)$ and $\varphi(z) = \varphi(\lambda z)$. For each $n \in \mathbb{N}$, choose a non-zero element f_n in L^2 such that $U_{\varphi_n,\lambda} f_n = 0$ with $\|f_n\| = 1$. Now $\|U_{\varphi,\lambda} f_n\| \le \|U_{\varphi,\lambda} f_n - U_{\varphi_n,\lambda} f_n\| \le \|\varphi - \varphi_n\| \to 0$ as $n \to \infty$. This yields that $0 \in \Pi(U_{\varphi,\lambda})$ and hence $0 \in \sigma(U_{\varphi,\lambda})$. This completes the proof. Q.E.D.

Theorem 3.3. For any invertible φ in L^∞, $\sigma(U_{\varphi,\lambda})$ contains a closed disc, where $\varphi = \sum_{n\in\mathbb{Z}} \langle \varphi, e_n\rangle \lambda^n e_n$.

Proof. Let P_k be the projection of L^2 onto the closed span of $\{e_{kn} : n \in \mathbb{Z}\}$. Now, if μ is any non-zero complex number and φ is invertible in L^∞ then for each $f \in L^2$, we have

$$\begin{aligned}(U^*_{\overline{\varphi}^{-1}(z^k),\lambda} - \mu I)f &= M_{\varphi^{-1}(z^k)} W_k^* D_\lambda f - \mu(P_k f \oplus (I-P_k)f)\\ &= \mu W_k^* M_{\varphi^{-1}}(\mu^{-1} D_\lambda - M_\varphi W_k) f \oplus (-\mu(I-P_k)f),\end{aligned}$$

Suppose that $(U^*_{\overline{\varphi}^{-1}(z^k),\lambda} - \mu I)$ is onto. Now, pick $0 \neq g_0$ in $(I-P_k)(L^2)$. Then we find $f \in L^2$ such that

$$g_0 = \mu W_k^* M_{\varphi^{-1}}(\mu^{-1} D_\lambda - M_\varphi W_k) f \oplus (-\mu(I-P_k)f).$$

Since $g_0 \in (I-P_k)(L^2)$, we have $\mu W_k^* M_{\varphi^{-1}}(\mu^{-1} D_\lambda - M_\varphi W_k) f = 0$. This provides $(\mu^{-1} D_\lambda - M_\varphi W_k) f = 0$ as W_k is co-isometry (i.e. $W_k W_k^* = I$). Hence we have $0 = (\mu^{-1} I - D_{\overline{\lambda}} M_\varphi W_k) f = (\mu^{-1} I - D_{\overline{\lambda}} W_k M_{\varphi(z^k)}) f = (\mu^{-1} I - U_{\varphi(z^k),\lambda}) f$. This implies that $\mu^{-1} \in \sigma_p(U_{\varphi(z^k),\lambda})$. Now $(U^*_{\overline{\varphi}^{-1}(z^k),\lambda} - \mu I)$ is onto (in fact invertible) for each μ in the resolvent of $U^*_{\overline{\varphi}^{-1}(z^k),\lambda}$, so on applying Theorem 3.1, we get that

$$\{\mu^{-1} : \mu \in \rho(U^*_{\overline{\varphi}^{-1}(z^k),\lambda})\} \subseteq \sigma_p(U_{\varphi(z^k),\lambda}) = \sigma_p(U_{\varphi,\lambda}) \subseteq \sigma_(U_{\varphi,\lambda}),$$

where $\varphi = \sum_{n\in\mathbb{Z}} \langle \varphi, e_n\rangle \lambda^n e_n$. As spectrum of any operator is compact it follows that $\sigma(U_{\varphi,\lambda})$ contains a disc of eigenvalues of $U_{\varphi,\lambda}$. Q.E.D.

Remark 3.4. We conclude with the following observation.

1. The spectrum $\sigma(U_{\varphi,\lambda})$ of the generalized $\lambda-$slant Toeplitz operator $U_{\varphi,\lambda}$ contains a closed disc of radius is $\frac{1}{r(U_{\overline{\varphi}^{-1},\lambda})}$, where $r(A)$ denotes the spectral radius of the operator A.

2. For unimodular $\varphi \in L^\infty$, $||U^n_{\varphi,\lambda}||^2 = ||U^n_{\varphi,\lambda} U^{*n}_{\varphi,\lambda}|| = ||I|| = 1$, so that $r(U_{\varphi,\lambda}) = 1$ (using Gelfand formula for spectral radius). Hence, if $|\varphi| = 1$, then $\sigma(U_{\psi,\lambda}) = \overline{\mathbb{D}}$, the closed unit disc.

4 Compressions of generalized $\lambda-$slant Toeplitz operators

We denote the compression of a generalized $\lambda-$slant Toeplitz operator $U_{\varphi,\lambda}$, $\varphi \in L^{\infty}$, $|\lambda| = 1$ to H^2 by $V_{\varphi,\lambda}$ or simply by V if there is no confusion about the symbol φ. Then by the definition of compression, we have $V_{\varphi,\lambda} = PU_{\varphi,\lambda}|_{H^2}$, that is, $V_{\varphi,\lambda}P = PU_{\varphi,\lambda}P$, where P is the orthogonal projection of L^2 onto H^2. As $U_{\varphi,\lambda} = D_{\overline{\lambda}}U_{\varphi}$, we have $V_{\varphi,\lambda} = PD_{\overline{\lambda}}U_{\varphi}|_{H^2}$, where U_{φ} denotes the k^{th}-order slant Toeplitz operator. Since $PD_{\overline{\lambda}} = D_{\overline{\lambda}}P$, we further have $V_{\varphi,\lambda} = D_{\overline{\lambda}}V_{\varphi}$, where V_{φ} is the compression of $k^{th}-$order slant Toeplitz operator U_{φ} to H^2. It is straight forward to verify that $\varphi \to V_{\varphi,\lambda}$ is one-one. It is interesting to obtain the following.

Theorem 4.1. An operator V on H^2 is the compression of a generalized $\lambda-$slant Toeplitz operator if and only if $\lambda V = U^*VU^k$, where U is the forward unilateral shift on H^2.

Proof. Suppose V is compression of a generalized $\lambda-$slant Toeplitz operator. Then $V = D_{\overline{\lambda}}V_{\varphi}$ for some φ in L^{∞}. Now $U^*VU^k = U^*D_{\overline{\lambda}}V_{\varphi}U^k = \lambda D_{\overline{\lambda}}U^*V_{\varphi}\ U^k = \lambda D_{\overline{\lambda}}V_{\varphi} = \lambda V$.

Conversely, suppose that V is an operator satisfying $\lambda V = U^*VU^k$. Then $\lambda D_{\lambda}V = D_{\lambda}U^*VU^k = \lambda U^*D_{\lambda}VU^k$. Since $|\lambda| = 1$, we get $D_{\lambda}V = U^*D_{\lambda}VU^k$. So $D_{\lambda}V$ is compression of a $k^{th}-$order slant Toeplitz operator [1]. So $D_{\lambda}V = V_{\varphi}$ for some φ in L^{∞}. Thus $V = D_{\overline{\lambda}}V_{\varphi}$ for some φ in L^{∞}. Q.E.D.

To discuss the compactness of compression of a generalized $\lambda-$slant Toeplitz operators, we first prove the following.

Lemma 4.2. Let $|\lambda| = 1$ and $\varphi \in L^{\infty}$. Then we have the following:

1. $W_kV^*_{\varphi,\lambda} = D_{\lambda}T_{\psi}$, where T_{ψ} is Toeplitz operator induced by $\psi(z) = W_k\overline{\varphi}(\lambda z)$.
2. If $\overline{\varphi}$ (or ψ) is analytic then $V_{\varphi,\lambda}T_{\psi} = V_{\varphi\psi,\lambda}$.
3. If $\overline{\varphi}$ (or $\overline{\psi}$) is analytic then $V_{\varphi,\lambda}V^*_{\psi,\lambda}$ is a Toeplitz operator.
4. If ψ is analytic then $T_{\psi}V_{\varphi,\lambda}$ is again compression of a generalized $\lambda-$slant Toeplitz operator.

Proof. Proof of (1) follows as $W_kV^*_{\varphi,\lambda} = W_kPU^*_{\varphi}D_{\lambda}|_{H^2} = PM_{W_k\overline{\varphi}}D_{\lambda}|_{H^2} = D_{\lambda}PM_{W_k\overline{\varphi}(\lambda z)}|_{H^2} = D_{\lambda}T_{\psi}$, where $\psi = W_k\overline{\varphi}(\lambda z)$.
Proof of (2) follows using the fact that $V_{\varphi}T_{\psi} = V_{\varphi\psi}$ when either of $\overline{\varphi}$ (or ψ) is analytic [1].
A simple computation shows that if $\overline{\varphi}$ (or $\overline{\psi}$) is analytic then $V_{\varphi,\lambda}V^*_{\psi,\lambda} = D_{\overline{\lambda}}T_{W_k\varphi\overline{\psi}}D_{\lambda}|_{H^2} = PD_{\overline{\lambda}}M_{W_k\overline{\varphi}\psi}D_{\lambda}|_{H^2} = PM_{\xi}|_{H^2} = T_{\xi}$, where $\xi(z) = W_k\varphi\overline{\psi}(\lambda z)$. This completes the proof of (3).
Now for (4), if ψ is analytic then $T_{\psi}V_{\varphi,\lambda} = PD_{\overline{\lambda}}M_{\psi(\overline{\lambda}z)}V_{\varphi}|_{H^2} = D_{\overline{\lambda}}\ V_{\psi(\overline{\lambda}z^k)\varphi(z)} = V_{\psi(\overline{\lambda}z^k)\varphi(z),\lambda}$. Hence the result. Q.E.D.

Now we see the following, which is a very common result known for various classes of operators, like, Toeplitz operators [4], slant Toeplitz operators [8].

Theorem 4.3. $V_{\varphi,\lambda}$ is compact if and only if $\varphi = 0$.

Proof. Proof of one part is obvious. For the converse, suppose $\varphi(z) = \sum_{n\in\mathbb{Z}} a_nz^n$ is such that $V_{\varphi,\lambda}$ is compact. By Lemma 4.2(1), $D_{\lambda}T_{\psi}$ is compact, where $\psi(z) = W_k\overline{\varphi}(\lambda z)$. Now D_{λ} being unitary, we have T_{ψ} is compact. Thus $W_k\overline{\varphi}(\lambda z) = \psi = 0$. This means that $W_k\overline{\varphi} = 0$. Therefore $\overline{a}_{-kn} = 0$ for all $n \in \mathbb{Z}$.

Now we use Lemma 4.2(2) that provides the compactness of $V_{\varphi z^m,\lambda}$ for $m = 1, 2, ..., k-1$. As a consequence $W_k V^*_{\varphi z^m,\lambda}$ and hence $D_\lambda T_\psi$ is compact, where $\psi(z) = W_k(\overline{\varphi z^m})(\lambda z)$. This implies $W_k(\overline{\varphi z^m}) = 0$, which means that $\overline{a}_{-kn-m} = 0$ for all $n \in \mathbb{Z}$, $m = 1, 2, ..., k-1$. Hence $\varphi = 0$. Q.E.D.

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