An efficient computational method based on the hat functions for solving fractional optimal control problems

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Abstract

In this paper, an efficient and accurate computational method based on the hat functions (HFs) is proposed for solving a class of fractional optimal control problems (FOCPs). In the proposed method, the fractional optimal control problem under consideration is reduced to a system of nonlinear algebraic equations which can be simply solved. To this end, the fractional state and control variables are expanded by the HFs with unknown coefficients. Then, the operational matrix of fractional integration of the HFs with some properties of these basis functions are employed to achieve a nonlinear algebraic equation, replacing the performance index and a nonlinear system of algebraic equations, replacing the dynamical system in terms of the unknown coefficients. Finally, the method of constrained extremum is applied, which consists of adjoining the constraint equations derived from the given dynamical system to the performance index by a set of undetermined Lagrange multipliers. As a result, the necessary conditions of optimality are derived as a system of algebraic equations in the unknown coefficients of the state variable, control variable and Lagrange multipliers. Furthermore, the efficiency of the proposed method is shown for some concrete examples. The results reveal that the proposed method is very accurate and efficient.

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1 Introduction

Differential equations (DEs) play an important role in mathematical modeling of real-life phenomena in science, engineering and many other fields. Generally speaking the analytical methods are not suitable for large scale problems with complex solution regions. Numerical methods are commonly used to get an approximate solution for the DEs which are non-linear or derivation of the analytical methods is difficult [1]. Optimal control theory is a branch of optimization theory concerned with minimizing a cost or maximizing a payout pertaining. In fact, an optimal control problem is a set of DEs describing the paths of the control variables that minimize a function of state and control variables. A necessary condition for an optimal control problem can be derived using Pontryagins maximum principle and a sufficient condition can be obtained by HamiltonJacobiBellman equation [1]. Recently, fractional order DEs have gained considerable importance due to their application in various sciences, such as physics, chemistry, mechanics and engineering. Fractional order models are more appropriate than conventional integer order to describe physical systems [1]. FOCP refers to the minimization of a performance index subject to dynamic constraints, on state and control

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variables, which have fractional order models. The fractional optimal control theory ia a very new area in mathematics although the optimal control theory is an area in mathematics which has been under development for years [2]. An FOCP can be defined with respect to different definitions of fractional derivatives. However, the most important definitions are the Riemann-Liouville and Caputo fractional derivatives. General necessary conditions of optimality have been developed for fractional optimal control problems. For example in [3,4], the authors have achieved the necessary conditions for optimization of FOCPs with the Caputo fractional derivative. The interested reader can refer to [5–22] for some recent works on FOCPs.

In this paper, we propose a new efficient and accurate computational method based on HFs for solving the following FOCP [6]:

$$min\, J[u] = \int_0^1 f\left(t, x(t), u(t)\right) dt, \tag{1}$$

with the fractional dynamical system:

$$D_*^{\alpha} x(t) = g\left(t, x(t), u(t)\right), \qquad n-1 \leq \alpha \leq n, \;\; n = 2, 3, \ldots, \tag{2}$$

and the initial conditions:

$$x(0) = a_0, \;\; x'(0) = a_1, \ldots, x^{(\lceil \alpha \rceil - 1)}(0) = a_{\lceil \alpha \rceil - 1}. \tag{3}$$

where f and g are continuous functions, $D_*^{\alpha} x(t) \equiv x^{(\alpha)}(t)$ denotes the Caputo fractional derivative of order α of $x(t)$ which will be described next.

Before continuing the discussion, it is necessary to give some definitions and mathematical preliminaries of the fractional calculus theory which are needed for establishing our results.

The Riemann-Liouville fractional integration operator of order $\alpha \geq 0$ of a function f, is defined as [23]:

$$\left(I^{\alpha} f\right)(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{4}$$

Also, the fractional derivative operator of order $\alpha > 0$ in the Caputo sense is defined as [23]:

$$\left(D_*^{\alpha} f\right)(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \alpha = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n. \end{cases} \tag{5}$$

where n is an integer, $t > 0$.

This is a useful relation between the Riemann-Liouvill operator and Caputo operator which is given by the following expression:

$$\left(I^{\alpha} D_*^{\alpha} f\right)(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \;\; t > 0, \qquad n-1 < \alpha \leq n, \tag{6}$$

where n is an integer, $t > 0$.

Now, we can continue the discussion to our method. In the proposed method, the fractional optimal control problem under consideration is reduced to a system of nonlinear algebraic equations which can be simply solved. To this end, we expand the fractional state variable, i.e. $D_*^\alpha x(t)$, and the control variable, i.e. $u(t)$ by the HFs with unknown coefficients. Then, the operational matrix of fractional integration of the HFs with some useful properties of these basis functions are utilized to achieve a nonlinear algebraic equation, instead of the performance index (1) and a nonlinear system of algebraic equations, instead of the dynamical system (7) in terms of the unknown coefficients. Finally, the method of constrained extremum is applied, which consists of adjoining the constraint equations derived from the given dynamical system to the performance index by a set of undetermined Lagrange multipliers. As a result, the necessary conditions of optimality are derived as a system of algebraic equations in the unknown coefficients of $D_*^\alpha x(t)$, $u(t)$ and the Lagrange multipliers.

This paper is organized as follows: In section 2, the HFs and some of their properties are investigated. In section 3, the proposed method is described for solving the underlying fractional optimal control problem. In section 4, the proposed method is applied to solve some numerical examples. Finally a conclusion is drawn in section 5.

2 The HFs and their properties

The HFs are continuous real functions (with shape of hats), which are defined on the interval $[0, 1]$. A set of these basis functions is usually defined on the interval $[0, 1]$ by [24–27], e. g.

$$\psi_0(t) = \begin{cases} \dfrac{h-t}{h}, & 0 \le t < h, \\ 0, & o.w, \end{cases} \tag{7}$$

$$\psi_i(t) = \begin{cases} \dfrac{t-(i-1)h}{h}, & (i-1)h \le t < ih, \\ \dfrac{(i+1)h-t}{h}, & ih \le t < (i+1)h, \\ 0, & o.w, \end{cases} \qquad i = 1, 2, \ldots, m-1, \tag{8}$$

and

$$\psi_m(t) = \begin{cases} \dfrac{t-(1-h)}{h}, & 1-h \le t \le 1, \\ 0, & o.w, \end{cases} \tag{9}$$

where $h = \frac{1}{m}$ and m is an arbitrary positive integer.

From the definition of the generalized hat basis functions, we have:

$$\psi_i(jh) = \begin{cases} 1, & i = j, \\ 0, & i \ne j, \end{cases} \tag{10}$$

and

$$\psi_i(t)\psi_j(t) = 0, \ |i-j| \ge 2. \tag{11}$$

An arbitrary continuous function $u(t)$ which is defined over the interval $[0, 1]$ can be expanded by the HFs as:

$$u(t) \simeq \sum_{i=0}^{m} u_i \psi_i(t) \triangleq U^T \Psi(t), \tag{12}$$

where

$$U \triangleq [u_0, u_1, \dots, u_m]^T, \tag{13}$$

and

$$\Psi(t) \triangleq [\psi_0(t), \psi_1(t), \dots, \psi_m(t)]^T. \tag{14}$$

An important aspect of using the generalized hat basis functions in approximating a function $u(t)$ lies in the fact that the coefficients u_i in (12) are given by:

$$u_i = u(ih),\ i = 0, 1, \dots, m. \tag{15}$$

The fractional integration of order α of the vector $\Psi(t)$ defined in (13) can be expressed as [24]:

$$I^\alpha(\Psi)(t) \simeq P^\alpha \Psi(t), \tag{16}$$

where the $(m+1) \times (m+1)$ matrix P^α is called the operational matrix of fractional integration for the generalized hat functions and is given in [28] as:

$$P^\alpha = \frac{h^\alpha}{\Gamma(\alpha+2)} \begin{pmatrix} 0 & \zeta_1 & \zeta_2 & \dots & \zeta_{n-1} & \zeta_n \\ 0 & 1 & \xi_1 & \dots & \xi_{n-2} & \xi_{n-1} \\ 0 & 0 & 1 & \dots & \xi_{n-3} & \xi_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

where

$$\begin{cases} \zeta_i = i^\alpha(\alpha - i + 1) + (i-1)^{\alpha+1}, & i = 1, 2, \dots, m, \\ \xi_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}, & i = 1, 2, \dots, m-1. \end{cases} \tag{18}$$

From now on, for any three constant vectors $U^T = [u_0, u_1, \dots, u_m]$, $V^T = [v_0, v_1, \dots, v_m]$ and $W^T = [w_0, w_1, \dots, w_m]$, and any continuous function $f : \mathbb{R}^3 \to \mathbb{R}$, we define:

$$f(U^T, V^T, W^T) = [f(u_0, v_0, w_0), f(u_1, v_1, w_1), \dots, f(u_m, v_m, w_m)]. \tag{19}$$

Lemma 2.1. *Suppose $U^T\Psi(t)$, $V^T\Psi(t)$ and $W^T\Psi(t)$ be the expansions of $u(t)$, $v(t)$ and $w(t)$ by the HFs, respectively. Then, we have:*

$$u(t)v(t)w(t) \simeq H^T\Psi(t), \tag{20}$$

where $H = U \odot V \odot W$, and $\odot$ denotes element-wise product of U, V and W, i.e., $(H)_{ij} = (U)_{ij}(V)_{ij}(W)_{ij}$.

Proof. From (12) and (15), we have:

$$u(t) \simeq \sum_{i=0}^{m} u(ih)\psi_i(t) = U^T\Psi(t),$$

$$v(t) \simeq \sum_{i=0}^{m} v(ih)\psi_i(t) = V^T\Psi(t),$$

$$w(t) \simeq \sum_{i=0}^{m} w(ih)\psi_i(t) = W^T\Psi(t),$$

and

$$u(t)v(t)w(t) \simeq \sum_{i=0}^{m} u(ih)v(ih)w(ih)\psi_i(t) = H^T\Psi(t),$$

which completes the proof. Q.E.D.

Lemma 2.2. *Suppose that $U^T\Psi(t)$ be the expansion of $u(t)$ by the HFs. Then, for any continuous function $g : \mathbb{R} \to \mathbb{R}$, we have:*

$$g(u(t)) \simeq g(U^T)\Psi(t), \tag{21}$$

where $g(U^T) = [g(u_0), g(u_1), \dots, g(u_m)]$.

Proof. By considering (12) and (15), we have:

$$g(u(t)) \simeq \sum_{i=0}^{m} g(u(ih))\psi_i(t) = g(U^T)\Psi(t), \tag{22}$$

which completes the proof. Q.E.D.

Corollary 2.3. Suppose $U^T\Psi(t)$, $V^T\Psi(t)$ and $W^T\Psi(t)$ be the expansions of $u(t)$, $v(t)$ and $w(t)$ by the HFs, respectively. Then, for any continuous function $f : \mathbb{R}^3 \to \mathbb{R}$, we have:

$$f(u(t), v(t), w(t)) \simeq f(U^T, V^T, W^T)\Psi(t). \tag{23}$$

Proof. By considering Lemma 2.1 and Lemma 2.2, the proof will be straightforward. Q.E.D.

3 Description of the proposed computational method

Consider the following FOCP:

$$min\, J[u] = \int_0^1 f\left(t, x(t), u(t)\right) dt, \tag{24}$$

with the fractional dynamical system:

$$D_*^{\alpha} x(t) = g\left(t, x(t), u(t)\right), \qquad n-1 \le \alpha \le n,\ \ n = 2, 3, \dots, \tag{25}$$

and the initial conditions:

$$x(0) = a_0,\ \ x'(0) = a_1, \dots, x^{(\lceil\alpha\rceil-1)}(0) = a_{\lceil\alpha\rceil-1}. \tag{26}$$

To solve the above FOCP, we approximate the fractional state variable, i.e. $D_*^{\alpha}x(t)$ and the control variable, i.e. $u(t)$ by the HFs as:

$$D_*^{\alpha} x(t) \simeq C^T\Psi(t), \tag{27}$$

$$u(t) \simeq U^T \Psi(t), \tag{28}$$

where

$$C^T = [c_0, c_1, \ldots, c_m], \tag{29}$$

$$U^T = [u_0, u_1, \ldots, u_m], \tag{30}$$

are unknown vectors which we need to compute them.

By applying the Riemann-Liouville fractional integration of order α on both sides of (27) and considering relation (6) and the initial conditions (26), we have:

$$x(t) \simeq C^T P^\alpha \Psi(t) + \sum_{j=0}^{\lceil n \rceil - 1} a_j \frac{t^j}{j!} = \left(C^T P^\alpha + d^T\right) \Psi(t) \triangleq X^T \Psi(t), \tag{31}$$

where d is the coefficients vector for $\sum_{j=0}^{\lceil n \rceil - 1} a_j \frac{t^j}{j!}$.

We also approximate t as a function by the HFs as:

$$t \simeq Q^T \Psi(t), \tag{32}$$

where Q is a known vector.

Next, by Corollary 2.3, we have:

$$f(t, x(t), u(t)) \simeq f\left(Q^T, X^T, U^T\right) \Psi(t), \tag{33}$$

and

$$g(t, x(t), u(t)) \simeq g\left(Q^T, X^T, U^T\right) \Psi(t). \tag{34}$$

Then, using (33), the performance index J is approximated as:

$$J[u] \simeq J[X, U] = f\left(Q^T, X^T, U^T\right) \Lambda, \tag{35}$$

where Λ is a known column vector given by:

$$\Lambda = \left[\int_0^1 \psi_0(t)dt, \int_0^1 \psi_1(t)dt, \ldots, \int_0^1 \psi_m(t)dt\right]^T.$$

and the dynamical system (25) is approximated by the following system of algebraic equations:

$$C^T - g\left(Q^T, X^T, U^T\right) \simeq 0. \tag{36}$$

Now, assume that

$$J^*[X, U, \lambda] = J[X, U] + \left(C^T - g\left(Q^T, X^T, U^T\right)\right) \lambda, \tag{37}$$

where

$$\lambda = [\lambda_0, \lambda_1, \ldots, \lambda_m]^T,$$

is the unknown Lagrange multiplier.
Finally, the necessary conditions for the extremum are

$$\frac{\partial J^*}{\partial X} = 0, \qquad \frac{\partial J^*}{\partial U} = 0, \qquad \frac{\partial J^*}{\partial \lambda} = 0. \tag{38}$$

The above equations can be solved for X, U and λ using Maple or Matlab packages. By determining X and U, we can determine the approximate solutions of $u(t)$ and $x(t)$ from (28) and (31), respectively.
The algorithm for the proposed method is expressed as:

Algorithm

Input: m, $n \in \mathbb{N}$, $n-1 \leq \alpha \leq n$, a_0, a_1, $\ldots$, $a_{\lceil \alpha \rceil -1}$, and the functions f and g.

Step 1: Define the HFs $\psi_j(t)$ by (7)-(9).

Step 2: Compute the fractional operational matrices P^α using (17) and (18).

Step 3: Define the unknown vectors C and U in (27) and (28).

Step 4: Compute the vectors d and X in (31).

Step 5: Compute the vector Q in (32) using (15).

Step 6: Compute the vectors $f\left(Q^T, X^T, U^T\right)$ and $g\left(Q^T, X^T, U^T\right)$ using (19).

Step 7: Put $\Lambda = \left[\int_0^1 \psi_0(t)dt,\, \int_0^1 \psi_1(t)dt,\, \ldots,\, \int_0^1 \psi_m(t)dt\right]^T$.

Step 8: Compute the equation $J[X,U] = f\left(Q^T, X^T, U^T\right)\Lambda$ using (35).

Step 9: Compute the system of algebraic equations $C^T - g\left(Q^T, X^T, U^T\right) \simeq 0$ in (36).

Step 10: Put $\lambda = [\lambda_0, \lambda_1, \ldots, \lambda_m]^T$.

Step 11: Put $J^*[X,U,\lambda] = J[X,U] + \left(C^T - g\left(Q^T, X^T, U^T\right)\right)\lambda$.

Step 12: Solve the system of algebraic equations: $\frac{\partial J^*}{\partial X} = 0$, $\frac{\partial J^*}{\partial U} = 0$, $\frac{\partial J^*}{\partial \lambda} = 0$, for the vectors X and U.

Output: The approximate solutions: $x(t) \simeq X^T\Psi(t)$, $u(t) \simeq U^T\Psi(t)$ and $\mu \simeq \mu_m$.

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4 Illustrative test problems

In this section, some numerical examples will be provided to demonstrate the efficiency and reliability of the proposed method.

Example 4.1. Consider the following FOCP [6]:

$$\min J[u] = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt,$$

subject to the dynamical system:

$$D_*^\alpha x(t) = -x(t) + u(t),$$

and the initial condition $x(0) = 1$.
Our aim is to find $u(t)$ such that the quadratic performance index J is minimized. For $\alpha = 1$, the exact solution of this problem is

$$\begin{cases} x(t) = \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t), \\ u(t) = \left(1 + \sqrt{2}\beta\right) \cosh(\sqrt{2}t) + \left(\sqrt{2} + \beta\right) \sinh(\sqrt{2}t). \end{cases}$$

where

$$\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})} \simeq -0.98.$$

Note that the minimum value of J is $\mu = 0.192909$.
Now, we will solve this problem again by the proposed method for $m = 50$. Fig. 1 demonstrates the behavior of the numerical solutions for the state variable $x(t)$ (left side) and the control variable $u(t)$ (right side). From this figure, it can be simply seen that the numerical solution is in a very good agreement with the exact solution for $\alpha = 1$. Moreover, it is clear that as α increases, the approximated values of $x(t)$ and $u(t)$ converge uniformly to the exact solution with $\alpha = 1$.

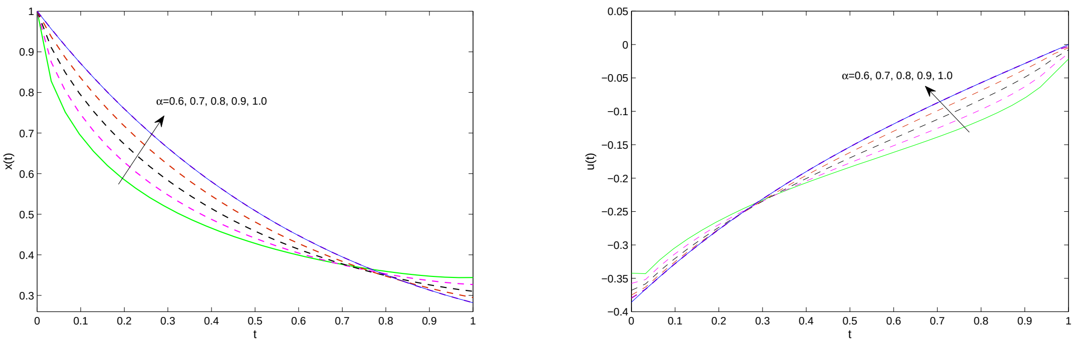

FIG. 1. The graphs of the approximate solutions for state variable (left side) and control variable (right side) of Example 4.1 for some different values of $0 < \alpha \leq 1$.

Example 4.2. Consider the following FOCP [6]:

$$\min J[u] = \int_0^1 \left(\left(x(t) - t^2\right)^2 + \left(u(t) + t^2 - \frac{20\, t^{\frac{9}{10}}}{9\Gamma(\frac{9}{10})}\right)\right) dt,$$

subject to the dynamical system:

$$D_*^{1.1} x(t) = t^2 x(t) + u(t),$$

and the initial conditions $x(0) = x'(0) = 0$.

For this problem, the solution $x(t) = t^2$ and $u(t) = \dfrac{20\,t^{\frac{9}{10}}}{9\Gamma(\frac{9}{10})} - t^4$ minimizes the performance index J and its minimum value is $\mu = 0$. Fig. 2 compares the exact and approximate solution for state variable $x(t)$ and control variable $u(t)$. Table 1 contains the absolute errors in the state variable $x(t)$ and control variable $u(t)$, and also CPU time for some different values of m. By this information, it can be stated that the numerical results are in a good agreement with the exact solutions. Moreover, it can be seen that the proposed method requires a less time to compute the solution. It is also worth to mention that the computational efforts and CPU time of the proposed method are less than the method in [6]. To conclude, it can be seen that the proposed method is very simple in implementation and also leads to accurate solutions with little computational work.

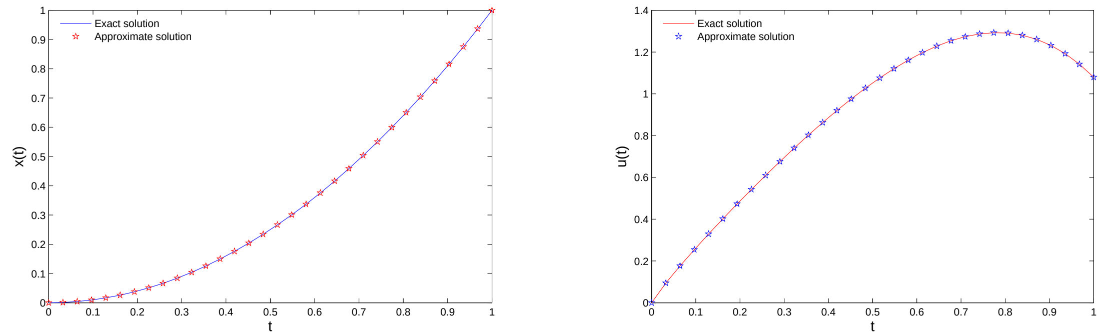

FIG. 2. The graphs of the exact and approximate solutions for state variable (left side) and control variable (right side) for Example 4.2.

TABLE 1. The absolute errors in the state and control variables and CPU time for different values of m for Example 4.2.

m	$\lvert x(t) - x_m(t)\rvert$	$\lvert u(t) - u_m(t)\rvert$	$\lvert \mu - \mu_m\rvert$	CPU Time (s)
10	2.25×10^{-3}	1.35×10^{-2}	6.86×10^{-7}	03.81 (s)
20	5.51×10^{-4}	3.52×10^{-3}	6.19×10^{-8}	04.10 (s)
30	2.62×10^{-4}	1.62×10^{-3}	1.49×10^{-8}	08.25 (s)
40	1.40×10^{-4}	9.12×10^{-4}	5.40×10^{-9}	08.80 (s)
50	8.22×10^{-5}	5.62×10^{-4}	2.44×10^{-9}	09.53 (s)

Example 4.3. Consider the following FOCP [6]:

$$\min J[u] = \int_0^1 \left(e^t \left(x(t) - t^4 + t - 1\right)^2 + (1+t^2)\left(u(t) + 1 - t + t^4 - \frac{8000\,t^{\frac{21}{10}}}{77\Gamma(\frac{1}{10})}\right)^2 \right) dt,$$

subject to the dynamical system:

$$D_*^{1.9}x(t) = x(t) + u(t),$$

and the initial conditions $x(0) = 1,\ x'(0) = -1$.
For this problem, the solution $x(t) = 1 - t + t^4$ is the minimizing state variable and the performance index J has the minimum value of $\mu = 0$. Fig. 3 demonstrates the exact and approximate solution for the state variable $x(t)$ and control variable $u(t)$. Table 2 contains the absolute errors in the state variable $x(t)$ and control variable $u(t)$, and also CPU time for some different values of m. Table 2 shows that the numerical results are in a good agreement with the exact solutions. Moreover, it can be seen that the proposed method requires little time to perform. It is also worth to mention that the computational efforts and CPU time of the proposed method are less than the method in [6].

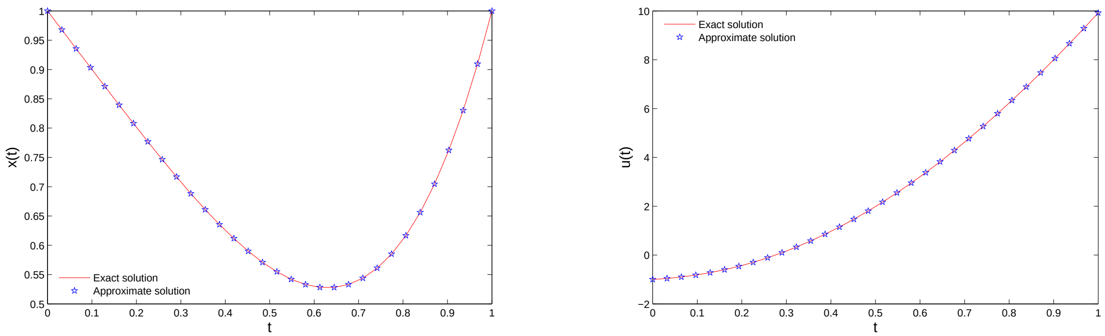

FIG. 3. The graphs of the exact and approximate solutions for state variable (left side) and control variable (right side) for Example 4.3.

TABLE 2. The absolute errors in the state and control variables and CPU time for different values of m for Example 4.3.

m	$\lvert x(t) - x_m(t)\rvert$	$\lvert u(t) - u_m(t)\rvert$	$\lvert \mu - \mu_m\rvert$	CPU Time (s)
10	2.15×10^{-2}	2.65×10^{-2}	4.46×10^{-5}	04.09 (s)
20	5.55×10^{-3}	5.82×10^{-3}	2.72×10^{-6}	05.88 (s)
30	2.52×10^{-3}	2.51×10^{-3}	5.34×10^{-7}	08.98 (s)
40	1.42×10^{-3}	1.42×10^{-3}	1.68×10^{-8}	16.00 (s)
50	9.22×10^{-4}	9.32×10^{-4}	8.95×10^{-9}	23.33 (s)

Example 4.4. Consider the following FOCP [6]:

$$\min J[u] = \int_0^1 \left(\left(x(t) - t^{\frac{5}{2}} \right)^4 + (1 + t^2) \left(u(t) + t^6 - \frac{15\sqrt{\pi}\, t}{8} \right)^2 \right) dt,$$

subject to the dynamical system:

$$D_*^{1.5}x(t) = t\,x^2(t) + u(t),$$

and the initial conditions $x(0) = x'(0) = 0$.
For this problem, the solution $x(t) = t^{\frac{5}{2}}$ is the minimizing state variable and the performance index J has the minimum value of $\mu = 0$. Fig. 4 displays the exact and approximate solutions for the state variable $x(t)$ and control variable $u(t)$. Table 3 contains the absolute errors in the state variable $x(t)$ and control variable $u(t)$, and also CPU time for some different values of m. Table 3 shows that the numerical results are in a good agreement with the exact solutions. Moreover, it can be seen that the proposed method requires little time to perform. It is also worth mentioning that the computational work of the proposed method is less than the method in [6]. By this Table, it can be concluded that the proposed method is very simple in implementation and also leads to accurate solutions with a little computational work for this problem.

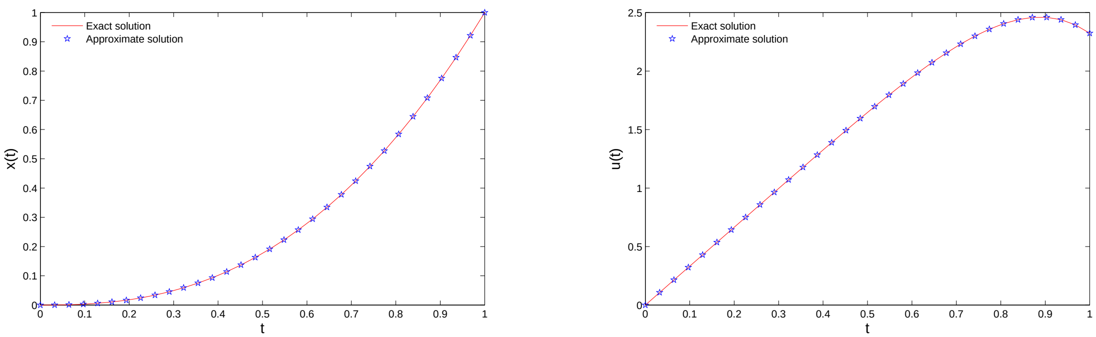

FIG. 4. The graphs of the exact and approximate solutions for the state variable (left side) and control variable (right side) for Example 4.4.

TABLE 3. The absolute errors in the state and control variables and CPU time for different values of m for Example 4.4.

m	$\lvert x(t) - x_m(t)\rvert$	$\lvert u(t) - u_m(t)\rvert$	$\lvert \mu - \mu_m\rvert$	CPU Time (s)
10	4.62×10^{-3}	3.15×10^{-2}	1.76×10^{-12}	003.79 (s)
15	2.12×10^{-3}	1.41×10^{-2}	4.21×10^{-12}	009.11 (s)
20	1.53×10^{-3}	8.22×10^{-3}	2.15×10^{-13}	026.14 (s)
25	7.13×10^{-4}	5.54×10^{-3}	5.70×10^{-13}	085.10 (s)
30	5.12×10^{-4}	3.71×10^{-3}	2.47×10^{-13}	134.00 (s)

Example 4.5. Consider the following FOCP [6]:

$$\min J[u] = \int_0^1 \Bigg(-2e^{1+t^2+x(t)} + e^{2(1+t^2+x(t))} + \frac{8\sqrt{t}}{\sqrt{\pi}}u(t) - 2\sin(1+t^2)u(t) + u^2(t) + \frac{16t}{\pi}$$

$$-\frac{8\sqrt{t}\sin(1+t^2)}{\sqrt{\pi}} + \sin^2\left(1+t^2\right) + 1\Big)\, dt,$$

subject to the dynamical system:

$$D_*^{1.5}x(t) = \sin\left(x(t)\right) + u(t),$$

and the initial conditions $x(0) = -1,\ x'(0) = 0$.

For this problem we have the solution $x(t) = -1 - t^2$ and $u(t) = -\frac{4\sqrt{t}}{\sqrt{\pi}} + \sin(1+t^2)$ as optimal state and control variables with the minimum value $\mu = 0$ for the performance index J. Fig. 4 demonstrates the exact and approximate solutions for the state variable $x(t)$ and control variable $u(t)$. Table 4 contains the absolute errors in the state variable $x(t)$ and control variable $u(t)$, and also CPU time for some different values of m. Table 4 shows that the numerical results are in a good agreement with the exact solutions. Moreover, it can be seen that the proposed method requires little time to perform. It is also worth mentioning that the computational work of the proposed method is less than the method in [6].

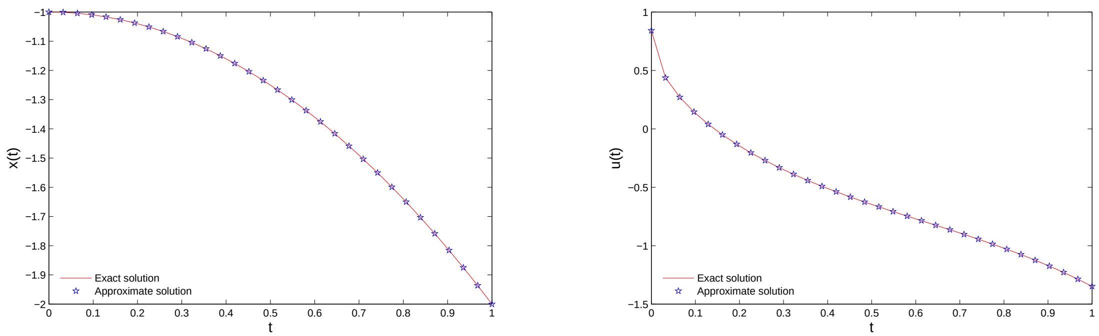

FIG. 5. The graphs of the exact and approximate solutions for the state variable (left side) and control variable (right side) for Example 4.5.

TABLE 4. The absolute errors in the state and control variables and CPU time for different values of m for Example 4.5.

m	$\lvert x(t) - x_m(t)\rvert$	$\lvert u(t) - u_m(t)\rvert$	$\lvert\mu - \mu_m\rvert$	CPU Time (s)
10	1.31×10^{-2}	1.15×10^{-1}	1.00×10^{-4}	04.16 (s)
20	4.53×10^{-3}	6.52×10^{-3}	1.39×10^{-5}	09.03 (s)
30	2.51×10^{-3}	5.51×10^{-3}	4.30×10^{-6}	15.22 (s)
40	1.61×10^{-3}	1.81×10^{-3}	1.86×10^{-6}	22.90 (s)
50	8.96×10^{-4}	9.51×10^{-4}	9.71×10^{-7}	39.18 (s)

5 Conclusion

In this paper, the operational matrix of fractional order integration for the HFs was applied in order to solve a class of fractional optimal control problems. By utilizing the HFs, the operational matrix of fractional integration, some new properties of these basis functions and the Lagrange multiplier method for constrained optimization, the underlying problem was reduced to the problem of solving a system of algebraic equations. Several examples were given to demonstrate the ability of the proposed method. The numerical results illustrates the efficiency of the presented scheme for solving FOCPs.

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