

New families of alternating harmonic number sums

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Abstract

We develop new closed form representations of sums of alternating harmonic numbers and reciprocal binomial coefficients.

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1 Introduction and Preliminaries

Let \mathbb{R} and \mathbb{C} denote respectively, the sets of real and complex numbers and let $\mathbb{N} := \{1, 2, 3, \dots\}$ be the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A generalized binomial coefficient $\binom{\lambda}{\mu}$ ($\lambda, \mu \in \mathbb{C}$) is defined, in terms of the familiar (Euler's) gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda + 1)}{\Gamma(\mu + 1)\Gamma(\lambda - \mu + 1)}, \quad (\lambda, \mu \in \mathbb{C}),$$

which, in the special case when $\mu = n$, $n \in \mathbb{N}_0$, yields

$$\binom{\lambda}{0} := 1 \quad \text{and} \quad \binom{\lambda}{n} := \frac{\lambda(\lambda - 1) \cdots (\lambda - n + 1)}{n!} = \frac{(-1)^n (-\lambda)_n}{n!} \quad (n \in \mathbb{N}),$$

where $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) is the Pochhammer symbol defined, also in terms of the gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed that the Γ -quotient exists.

Let

$$H_n = \sum_{r=1}^n \frac{1}{r} = \gamma + \psi(n + 1) = \int_0^1 \frac{1 - t^n}{1 - t} dt \quad (H_0 := 0)$$

be the n th harmonic number. Here, as usual, γ denotes the Euler-Mascheroni constant and $\psi(z)$ is the Psi (or Digamma) function defined by

$$\psi(z) := \frac{d}{dz} \{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_1^z \psi(t) dt.$$

A *generalized harmonic number* $H_n^{(m)}$ of order m is defined, for positive integers n and m , as follows:

$$H_n^{(m)} := \sum_{r=1}^n \frac{1}{r^m}, \quad (m, n \in \mathbb{N}) \quad \text{and} \quad H_0^{(m)} := 0 \quad (m \in \mathbb{N})$$

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and

$$\psi^{(n)}(z) = \frac{d^n}{dz^n} \{\psi(z)\} = \frac{d^{n+1}}{dz^{n+1}} \{\log \Gamma(z)\} \quad (n \in \mathbb{N}_0).$$

In the case of non integer values of the argument $z = \frac{r}{q}$, we may write the generalized harmonic numbers, $H_z^{(\alpha+1)}$, in terms of polygamma functions

$$H_{\frac{r}{q}}^{(\alpha+1)} = \zeta(\alpha + 1) + \frac{(-1)^\alpha}{\alpha!} \psi^{(\alpha)}\left(\frac{r}{q} + 1\right), \quad \frac{r}{q} \neq \{-1, -2, -3, \dots\},$$

where $\zeta(z)$ is the Riemann zeta function. When we encounter harmonic numbers at possible rational values of the argument of the form $H_{\frac{r}{q}}^{(\alpha)}$, they may be evaluated by an available relation in terms of the polygamma function $\psi^{(\alpha)}(z)$ or, for rational arguments $z = \frac{r}{q}$, and we also define

$$H_{\frac{r}{q}}^{(1)} = \gamma + \psi\left(\frac{r}{q} + 1\right), \text{ and } H_0^{(\alpha)} = 0.$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{q}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [8], or Choi and Cvijović [2] in terms of the polylogarithmic or other special functions. Some specific values are given as follows:

$$H_{-\frac{1}{2}}^{(1)} = -2 \ln 2, \quad H_{-\frac{1}{2}}^{(2)} = -2\zeta(2),$$

$$H_{-\frac{3}{4}}^{(1)} = -\frac{\pi}{2} - 3 \ln 2, \text{ and } H_{-\frac{3}{4}}^{(2)} = -8G - 5\zeta(2).$$

Many others are listed in the books[15], [22] and [23]. In this paper we will develop identities, closed form representations of alternating harmonic numbers and reciprocal binomial coefficients of the form

$$S_k(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n^p \binom{n+k}{k}}, \tag{1.1}$$

for $p = 0, 1$ and 2 .

While there are many results for sums of harmonic numbers with positive terms, there are fewer results for sums of the type (1.1). Let us define the alternating zeta function

$$\bar{\zeta}(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = (1 - 2^{1-z}) \zeta(z)$$

with $\bar{\zeta}(1) = \ln 2$, and

$$S_{p,q}^{+-} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(p)}}{n^q}.$$

Sitaramachandrarao [11] gave, for $1 + q$ an odd integer,

$$2S_{1,q}^{+-} = (1 + q) \bar{\zeta}(1 + q) - \zeta(1 + q) - 2 \sum_{j=1}^{\frac{q}{2}-1} \bar{\zeta}(2j) \zeta(1 + q - 2j)$$

and, in another special case, gave the integral

$$S_{1,1+2q}^{+-} = \int_0^1 \frac{\ln^{2q}(x) \ln(1+x)}{x(1+x)} dx.$$

In the case where p and q are both positive integers and $p+q$ is an odd integer, Flajolet and Salvy [7] gave the identity:

$$\begin{aligned} 2S_{p,q}^{+-} &= (1 - (-1)^p) \zeta(p) \bar{\zeta}(q) + 2 \sum_{i+2k=q} \binom{p+i-1}{p-1} \zeta(p+i) \bar{\zeta}(2k) \\ &+ \bar{\zeta}(p+q) - 2 \sum_{j+2k=p} \binom{q+j-1}{q-1} (-1)^j \bar{\zeta}(q+j) \bar{\zeta}(2k), \end{aligned}$$

where $\bar{\zeta}(0) = \frac{1}{2}$, $\bar{\zeta}(1) = \ln 2$, $\zeta(1) = 0$, and $\zeta(0) = -\frac{1}{2}$ in accordance with the analytic continuation of the Riemann zeta function. Flajolet and Salvy [7], further gave some specific examples, such as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{2n+1} = \frac{\pi \ln 2}{2} - G$$

where

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx .91596 \text{ is Catalan's constant.}$$

Some other interesting cases are given by Coffey [5]

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{(n+1)^2} = \frac{65}{16} \zeta(4) + \zeta(2) \ln^2 2 - \frac{1}{6} \ln^4(2) - \frac{7}{2} \zeta(3) \ln 2 - 4L_{i_4}\left(\frac{1}{2}\right)$$

where $L_{i_4}(\cdot)$ is the polylogarithm function, Coffey [6] also gave the expression

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \psi^{(p)}(n+a+1)}{n^q} = (-1)^q \int_0^1 \int_0^1 \frac{x^{a+1} \ln^{q-1}(y) \ln^p(x)}{(1+xy)(1-x)} dx dy,$$

where $\psi^{(p)}(\cdot)$ is the polygamma function. Some results for finite sums of alternating harmonic numbers may be seen in the works of [1], [3], [4], [9], [12], [13], [14], [16], [24], [25], [26] and references therein. For results on alternating quadratic harmonic number sums see [17]. Some results for sums of positive terms may be seen in the works [10], [18], [19], [20] and [21].

The following lemma will be useful in the development of the main theorems.

Lemma 1.1. Let r be a positive integer. Then:

$$\sum_{j=1}^r \frac{(-1)^j}{j} = H_{\lfloor \frac{r}{2} \rfloor} - H_r \tag{1.2}$$

where $[x]$ is the integer part of x . We also have the known results. For $0 < t \leq 1$

$$\ln^2(1+t) = 2 \sum_{n=1}^{\infty} \frac{(-t)^{n+1} H_n}{n+1}$$

and when $t = 1$,

$$\begin{aligned} \ln^2 2 &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+1} = \zeta(2) - 2L_{i_2}\left(\frac{1}{2}\right), \\ &= 2 \sum_{n=1}^{\infty} \frac{s(n, 2)}{n!}, \end{aligned} \tag{1.3}$$

where $s(n, 2)$ are Stirling numbers of the first kind.

$$t \ln(1+t) = \sum_{n=1}^{\infty} \frac{(-t)^{n+1}}{n},$$

hence

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{2^n}. \tag{1.4}$$

Proof. To prove (1.2) we write,

$$\begin{aligned} \sum_{j=1}^r \frac{(-1)^j}{j} &= \sum_{j=1}^r \frac{1}{j} + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} \frac{1}{j} - 2 \sum_{j=1}^{\lfloor \frac{2r+1}{2} \rfloor} \frac{1}{j} \\ &= H_r + H_{\lfloor \frac{r}{2} \rfloor} - 2H_{\lfloor \frac{2r+1}{2} \rfloor}, \text{ since } r \text{ is a positive integer} \\ &= H_{\lfloor \frac{r}{2} \rfloor} - H_r. \end{aligned}$$

■

Lemma 1.2. Let r be a positive integer. Then

$$\begin{aligned} S_r &: = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+r} = \frac{(-1)^{r+1}}{2} \ln^2 2 + (-1)^r \left(2H_{r-1} - H_{\lfloor \frac{r-1}{2} \rfloor} \right) \ln 2 \\ &\quad + (-1)^r \sum_{j=1}^{r-1} \frac{1}{j} \left(H_{\lfloor \frac{j}{2} \rfloor} - H_j \right), \text{ for } r \geq 2, \end{aligned} \tag{1.5}$$

and

$$S_0 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n} = \frac{1}{2} \zeta(2) - \frac{1}{2} \ln^2 2, \quad S_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+1} = \frac{1}{2} \ln^2 2.$$

Proof. By a change of counter

$$\begin{aligned}
 S_r & : = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+r} = \sum_{n=2}^{\infty} \frac{(-1)^n H_{n-1}}{n+r-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+r-1} \left(H_n - \frac{1}{n} \right) \\
 & = - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+r-1} + \sum_{n=1}^{r-1} \frac{(-1)^{n+1}}{n(n+r-1)} \\
 & = -S_{r-1} + \sum_{n=1}^{r-1} \frac{(-1)^{n+1}}{r-1} \left(\frac{1}{n} - \frac{1}{n+r-1} \right).
 \end{aligned}$$

From Lemma 1.1 and using the known results

$$\begin{aligned}
 S_r & = -S_{r-1} + \frac{1}{r-1} \left(\ln 2 - (-1)^r \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{r-1} \frac{(-1)^n}{n} \right) \\
 & = -S_{r-1} + \frac{\ln 2}{r-1} (1 + (-1)^r) + \frac{(-1)^r}{r-1} \left(H_{\lfloor \frac{r-1}{2} \rfloor} - H_{r-1} \right),
 \end{aligned}$$

which, for $r \geq 2$, yields the recurrence relation

$$S_r + S_{r-1} = \frac{\ln 2}{r-1} (1 + (-1)^r) + \frac{(-1)^r}{r-1} \left(H_{\lfloor \frac{r-1}{2} \rfloor} - H_{r-1} \right). \tag{1.6}$$

By the subsequent reduction of the $S_r, S_{r-1}, S_{r-2}, \dots, S_1$ terms in (1.6), we arrive at the identity (1.5). ■

It is of some interest to note that S_r may be expanded in a slightly different way so that it gives rise to another unexpected harmonic series identity. This is pursued in the next lemma.

Lemma 1.3. For $r \in \mathbb{N}_0$, we have the identity

$$\begin{aligned}
 V_r & : = \sum_{j=1}^{\infty} \frac{H_{j+\frac{r-2}{2}} - H_{j+\frac{r-3}{2}}}{2(2j-1)} = \frac{(-1)^{r+1}}{2} \ln^2 2 + (-1)^r \left(2H_{r-1} - H_{\lfloor \frac{r-1}{2} \rfloor} \right) \ln 2 \\
 & + \frac{H_{\frac{r-1}{2}}}{2(r-1)} + (-1)^r H_{r-1} \left(H_{\lfloor \frac{r}{2} \rfloor} - H_r \right) - \frac{1}{8} \left(H_{\frac{r-2}{2}}^2 - H_{\frac{r-3}{2}}^2 + H_{\frac{r-2}{2}}^{(2)} - H_{\frac{r-3}{2}}^{(2)} \right) \\
 & + (-1)^r \sum_{j=1}^{r-1} \frac{(-1)^{j+1}}{j} \left(H_{\lfloor \frac{r-j}{2} \rfloor} - H_{r-j} + \frac{jH_j}{j+1} \right). \tag{1.7}
 \end{aligned}$$

Proof. By expansion

$$S_r := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+r} = \sum_{n=1}^{\infty} \left(\frac{H_{2n}}{(2n+r)(2n+r-1)} - \frac{1}{2n(2n+r-1)} \right),$$

since we know that $H_{2n} = \frac{1}{2}H_n + \sum_{j=1}^n \frac{1}{2j-1}$ then

$$S_r = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{(2n+r)(2n+r-1)} + \sum_{n=1}^{\infty} \sum_{j=1}^n \frac{1}{(2j-1)(2n+r)(2n+r-1)} - \sum_{n=1}^{\infty} \frac{1}{2n(2n+r-1)} \quad (1.8)$$

$$= \frac{1}{8} \left(H_{\frac{r-2}{2}}^2 - H_{\frac{r-3}{2}}^2 + H_{\frac{r-2}{2}}^{(2)} - H_{\frac{r-3}{2}}^{(2)} \right) - \frac{H_{\frac{r-1}{2}}}{2(r-1)} + \sum_{n=1}^{\infty} \sum_{j=1}^n \frac{1}{(2j-1)(2n+r)(2n+r-1)}. \quad (1.9)$$

For an arbitrary double sequence $Y_{k,l}$ we have that

$$\sum_{k=0}^{\infty} \sum_{l=0}^k Y_{k,l} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} Y_{k,l+k},$$

and from (1.9)

$$\begin{aligned} V_r &= \sum_{n=1}^{\infty} \sum_{j=1}^n \frac{1}{(2j-1)(2n+r)(2n+r-1)} \\ &= \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2j-1)(2n+2j+r)(2n+2j+r-1)} \\ &= \sum_{j=1}^{\infty} \frac{1}{2(2j-1)} \left(H_{j+\frac{r-2}{2}} - H_{j+\frac{r-3}{2}} \right). \end{aligned}$$

Upon utilizing (1.9) and the known result (1.5) for S_r , we are able to write

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{1}{2(2j-1)} \left(H_{j+\frac{r-2}{2}} - H_{j+\frac{r-3}{2}} \right) &= S_r + \frac{H_{\frac{r-1}{2}}}{2(r-1)} \\ &\quad - \frac{1}{8} \left(H_{\frac{r-2}{2}}^2 - H_{\frac{r-3}{2}}^2 + H_{\frac{r-2}{2}}^{(2)} - H_{\frac{r-3}{2}}^{(2)} \right). \end{aligned}$$

Substituting for S_r and upon simplification we have the result (1.7) for V_r . ■

Remark 1.4. We note that Lemma 1.3 states the difference of two diverging harmonic series

produce a converging series:

$$\begin{aligned}
 V_0 &= \sum_{j=1}^{\infty} \frac{H_j - H_{j-\frac{1}{2}}}{2j+1} = \zeta(2) - 2 \ln 2, \\
 V_1 &= \sum_{j=1}^{\infty} \frac{H_{j+\frac{1}{2}} - H_j}{2j+1} = \frac{1}{2} \zeta(2) + 2 \ln 2 - 2, \\
 V_2 &= \sum_{j=1}^{\infty} \frac{H_{j+1} - H_{j+\frac{1}{2}}}{2j+1} = 1 - \frac{1}{2} \zeta(2), \text{ and} \\
 V_3 &= \sum_{j=1}^{\infty} \frac{H_{j+\frac{3}{2}} - H_{j+1}}{2j+1} = \frac{1}{2} \zeta(2) - \frac{2}{3}
 \end{aligned}$$

The next several theorems relate the main results of this investigation, namely the closed form and integral representation of (1.1).

2 Closed form and Integral identities

We now prove the following theorems.

Theorem 2.1. Let k be real positive integer. Then from (1.1) with $p = 0$ we have

$$\begin{aligned}
 S_k(0) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{\binom{n+k}{k}} = 2^{k-2} k \ln^2 2 \tag{2.1} \\
 &- \sum_{r=1}^k r \binom{k}{r} \left(\left(2H_{r-1} - H_{\lfloor \frac{r-1}{2} \rfloor} \right) \ln 2 + H_{r-1} \left(H_{\lfloor \frac{r}{2} \rfloor} - H_r \right) \right. \\
 &\quad \left. - \sum_{j=1}^{r-1} \frac{(-1)^j}{j} \left(H_{\lfloor \frac{r-j}{2} \rfloor} - H_{r-j} + \frac{jH_j}{j+1} \right) \right).
 \end{aligned}$$

Proof. Consider the expansion

$$\begin{aligned}
 S_k(0) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{\binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} k! H_n}{(n+1)_k} \\
 &= \sum_{n=1}^{\infty} (-1)^{n+1} k! H_n \sum_{r=1}^k \frac{\Lambda_r}{n+r},
 \end{aligned}$$

where

$$\Lambda_r = \lim_{n \rightarrow -r} \left\{ \frac{n+r}{\prod_{r=1}^k (n+r)} \right\} = \frac{(-1)^{r+1} r}{k!} \binom{k}{r}.$$

Hence

$$\begin{aligned}
 S_k(0) &= \sum_{r=1}^k (-1)^{r+1} r \binom{k}{r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+r} \\
 &= \sum_{r=1}^k (-1)^{r+1} r \binom{k}{r} S_r.
 \end{aligned}
 \tag{2.2}$$

From Lemma 1.2, (1.5) substituting into (2.2),

$$\begin{aligned}
 S_k(0) &= \sum_{r=1}^k (-1)^{r+1} r \binom{k}{r} \left(\begin{aligned} &\frac{(-1)^{r+1}}{2} \ln^2 2 + (-1)^r \left(2H_{r-1} - H_{\lfloor \frac{r-1}{2} \rfloor} \right) \ln 2 \\ &+ (-1)^r H_{r-1} \left(H_{\lfloor \frac{r}{2} \rfloor} - H_r \right) \\ &- (-1)^r \sum_{j=1}^{r-1} \frac{(-1)^j}{j} \left(H_{\lfloor \frac{r-j}{2} \rfloor} - H_{r-j} + \frac{jH_j}{j+1} \right) \end{aligned} \right) \\
 &= 2^{k-2} k \ln^2 2 \\
 &\quad - \sum_{r=1}^k r \binom{k}{r} \left(\begin{aligned} &\left(2H_{r-1} - H_{\lfloor \frac{r-1}{2} \rfloor} \right) \ln 2 + H_{r-1} \left(H_{\lfloor \frac{r}{2} \rfloor} - H_r \right) \\ &- \sum_{j=1}^{r-1} \frac{(-1)^j}{j} \left(H_{\lfloor \frac{r-j}{2} \rfloor} - H_{r-j} + \frac{jH_j}{j+1} \right) \end{aligned} \right),
 \end{aligned}$$

and Theorem 2.1 follows. ■

The other two cases of $S_k(1)$, $S_k(2)$ can be evaluated in a similar fashion. We list the results in the next corollary.

Corollary 2.2. Under the assumptions of Theorem 2.1, we have,

$$S_k(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n \binom{n+k}{k}} = \frac{1}{2} \zeta(2) + (2^{k-1} - 1) \ln^2 2
 \tag{2.3}$$

$$+ \sum_{r=1}^k r \binom{k}{r} \left(\begin{aligned} &\left(2H_{r-1} - H_{\lfloor \frac{r-1}{2} \rfloor} \right) \ln 2 + H_{r-1} \left(H_{\lfloor \frac{r}{2} \rfloor} - H_r \right) \\ &- \sum_{j=1}^{r-1} \frac{(-1)^j}{j} \left(H_{\lfloor \frac{r-j}{2} \rfloor} - H_{r-j} + \frac{jH_j}{j+1} \right) \end{aligned} \right),$$

and

$$S_k(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n^2 \binom{n+k}{k}} = \frac{5}{8} \zeta(3) - \frac{1}{2} (\zeta(2) - \ln^2 2) H_k
 \tag{2.4}$$

$$+ \sum_{r=1}^k \frac{1}{r} \binom{k}{r} \left(\begin{aligned} &\frac{1}{2} \ln^2 2 - \left(2H_{r-1} - H_{\lfloor \frac{r-1}{2} \rfloor} \right) \ln 2 - H_{r-1} \left(H_{\lfloor \frac{r}{2} \rfloor} - H_r \right) \\ &+ \sum_{j=1}^{r-1} \frac{(-1)^j}{j} \left(H_{\lfloor \frac{r-j}{2} \rfloor} - H_{r-j} + \frac{jH_j}{j+1} \right) \end{aligned} \right).$$

Proof. The proof follows directly from Theorem 2.1 and using the same technique. ■

It is possible to represent the alternating harmonic number sums (2.1), (2.3) and (2.4) in terms of an integral, this is developed in the next theorem.

Theorem 2.3. Let k be a positive integer. Then we have:

$$S_k(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{\binom{n+k}{k}} = -k \int_0^1 \int_0^1 \frac{xy(1-x)^{k-1}}{(1-y)(1+xy)} dx dy, \tag{2.5}$$

$$= -k \int_0^1 \int_0^1 \frac{x(1-x)^{k-1} \ln(1-y)}{(1+xy)^2} dx dy \tag{2.6}$$

$$= \left(\frac{1-k}{2} \right) \ln(2) - \frac{k}{1+k} \int_0^1 \frac{(k+y) {}_2F_1 \left[\begin{matrix} 1, 1 \\ 2+k \end{matrix} \middle| -y \right] \ln(1+y)}{(1+y)^2} dy. \tag{2.7}$$

Proof. Let $j \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $|t| \leq 1$. Consider the expansion

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{t^n}{\binom{n+k}{k} \binom{n+j}{j}} &= kj \sum_{n=1}^{\infty} \frac{t^n \Gamma(k) \Gamma(n+1) \Gamma(j) \Gamma(n+1)}{\Gamma(n+k+1) \Gamma(n+j+1)} \\ &= kj \sum_{n=1}^{\infty} t^n B(k, n+1) B(j, n+1), \end{aligned}$$

where $B(\cdot, \cdot)$ is the classical Beta function. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{t^n}{\binom{n+k}{k} \binom{n+j}{j}} &= kj \int_0^1 \int_0^1 (1-x)^{k-1} (1-y)^{j-1} \sum_{n=1}^{\infty} (txy)^n dx dy. \\ &= kj \int_0^1 \int_0^1 \frac{(1-x)^{k-1} (1-y)^{j-1} txy}{1-txy} dx dy. \end{aligned}$$

The next step is to differentiate both sides with respect to j then put $j = 0$ and $t = -1$, from which we obtain

$$S_k(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{\binom{n+k}{k}} = -k \int_0^1 \int_0^1 \frac{xy(1-x)^{k-1}}{(1-y)(1+xy)} dx dy := J_k(0). \tag{2.8}$$

Hence (2.5) follows. The identity (2.6) follows from the expansion

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{t^n}{\binom{n+k}{k} \binom{n+j}{j}} &= k \int_0^1 \int_0^1 \frac{(1-x)^{k-1} (1-y)^j}{y} \sum_{n=1}^{\infty} n (txy)^n dx dy \\ &= k \int_0^1 \int_0^1 \frac{(1-x)^{k-1} (1-y)^j}{(1-txy)^2} tx dx dy \end{aligned}$$

differentiating and making the appropriate substitutions (2.6) follows. Moreover from (2.6)

$$\begin{aligned} S_k(0) &= -k \int_0^1 \int_0^1 \frac{x(1-x)^{k-1} \ln(1-y)}{(1+xy)^2} dx dy = (k-1) \int_0^1 \frac{\ln(1-y)}{(1+y)^2} dy \\ &\quad - \frac{k}{1+k} \int_0^1 \frac{(k+y) {}_2F_1 \left[\begin{matrix} 1, 1 \\ 2+k \end{matrix} \middle| -y \right] \ln(1+y)}{(1+y)^2} dy, \end{aligned}$$

which is the identity (2.7). If we let

$$I_k(0) := \frac{k}{1+k} \int_0^1 \frac{(k+y) {}_2F_1 \left[\begin{matrix} 1, 1 \\ 2+k \end{matrix} \middle| -y \right] \ln(1+y)}{(1+y)^2} dy$$

we note that $I_k(0)$ may be written explicitly in terms of the right hand side of (2.1), moreover we note that

$$I_k(0) + J_k(0) = \left(\frac{1-k}{2} \right) \ln 2.$$

■

Similar integral representations can be evaluated for $S_k(1)$ and $S_k(2)$, the results are recorded in the next theorem.

Theorem 2.4. Let the conditions of Theorem 2.3 hold. Then we have:

$$S_k(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n \binom{n+k}{k}} = - \int_0^1 \int_0^1 \frac{(1-x)^k \ln(1-y)}{1+xy} dx dy,$$

$$\begin{aligned}
 &= \int_0^1 \frac{(1-x)^k \ln(1+x)}{x(1+x)} dx \\
 &= \frac{1}{1+k} {}_3F_2 \left[\begin{matrix} 1, 1, 1 \\ 2, 2+k \end{matrix} \middle| -1 \right] \\
 &+ 2^{k-1} \left(2 \ln 2 H_k - \ln^2 2 - H_k^2 - H_k^{(2)} + k {}_4F_3 \left[\begin{matrix} 1, 1, 1, 1-k \\ 2, 2, 2 \end{matrix} \middle| \frac{1}{2} \right] \right).
 \end{aligned}$$

Also

$$S_k(1) = -k \int_0^1 \int_0^1 \frac{x(1-x)^k \ln(1-y)}{1+xy} dx dy.$$

For $S_k(2)$,

$$S_k(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n^2 \binom{n+k}{k}} = -k \int_0^1 \int_0^1 \frac{(1-x)^{k-1} \ln(1-y) \ln(1+xy)}{y} dx dy,$$

and also,

$$\begin{aligned}
 S_k(2) &= - \int_0^1 \int_0^1 \frac{(1-x)^k \ln(1-y)}{1+xy} dx dy \\
 &= - \frac{1}{1+k} \int_0^1 {}_2F_1 \left[\begin{matrix} 1, 1 \\ 2+k \end{matrix} \middle| -y \right] \ln(1+y) dy.
 \end{aligned}$$

Proof. The proof follows from the same pattern as that employed in Theorem 2.3. ■

Example 2.5. Some illustrative examples are given as follows:

$$\begin{aligned}
 S_4(0) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{\binom{n+4}{4}} = 16 \ln^2 2 - \frac{176}{3} \ln 2 + \frac{298}{9} \\
 &= -\frac{3}{2} \ln(2) - \frac{4}{5} \int_0^1 \frac{(4+y) {}_2F_1 \left[\begin{matrix} 1, 1 \\ 6 \end{matrix} \middle| -y \right] \ln(1+y)}{(1+y)^2} dy, \\
 S_4(1) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n \binom{n+4}{4}} = \frac{1}{2} \zeta(2) - 8 \ln^2 2 + \frac{68}{3} \ln 2 - \frac{451}{36} \\
 &= \int_0^1 \frac{(1-x)^4 \ln(1+x)}{x(1+x)} dx, \text{ and}
 \end{aligned}$$

$$\begin{aligned}
S_4(2) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n^2 \binom{n+4}{4}} = \frac{5}{8} \zeta(3) - \frac{25}{24} \zeta(2) + \frac{16}{3} \ln^2 2 - \frac{28}{3} \ln 2 + \frac{727}{144} \\
&= -\frac{1}{5} \int_0^1 {}_2F_1 \left[\begin{matrix} 1, 1 \\ 6 \end{matrix} \middle| -y \right] \ln(1+y) dy.
\end{aligned}$$

Generally speaking it should be possible to obtain explicit identities of

$$\sum_{n=1}^{\infty} \frac{(-t)^{n+1} H_n}{n^p \binom{n+4}{4}}, \quad 0 < t \leq 1.$$

For $t = \frac{1}{2}$,

$$\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^{n+1} H_n}{\binom{n+4}{4}} = 54 (\ln^2 3 + \ln^2 2) - 108 \ln 2 \ln 3 - 198 \ln \frac{3}{2} + \frac{643}{9}.$$

3 Some Observations and Concluding remarks

The alternating sums of harmonic numbers $S_k(p)$, for $p = 0, 1$ and 2 have been successfully represented in integral form and in terms of zeta functions, harmonic numbers and ln functions. It may also be possible to represent the sums

$$S_k(p, q, r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(r)}}{n^p \binom{n+k}{k}^q}$$

in closed form, this work is currently under investigation. It does appear however, that there is an impasse with the representation $S_k(3)$ in closed form. This is related to the fact that

$$V(0) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n^3}$$

has, currently no closed form representation. In one scenario, $S_k(3)$ necessitates the evaluation of the difference of two diverging sums

$$\Theta(\alpha) = \sum_{n=1}^{\infty} \frac{H_n^{(\alpha)} - H_{n-\frac{1}{2}}^{(\alpha)}}{2n+1}$$

for $\alpha = 3$. Currently we have the known identities

$$\Theta(1) = \zeta(2) - 2 \ln 2, \quad \Theta(2) = (3 \ln 2 - 2) \zeta(2),$$

$$\Theta(3) = 17 \zeta(4) - (6 + 7 \ln 2) \zeta(3) + 4 \ln^2 2 \zeta(2) - \frac{2}{3} \ln^4 2 - 16 L_{i_4} \left(\frac{1}{2} \right),$$

$$\Theta(4) = 2 \zeta(2) \zeta(3) - (15 \ln 2 - 14) \zeta(4)$$

and $\Theta(2m)$ for $m \in \mathbb{N}$. Similarly

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n^3} &= \frac{1}{8} (5\zeta(4) + (6 - 7 \ln 2)\zeta(3) + \Theta(3)) \\ &= \frac{11}{4}\zeta(4) - \frac{7}{4} \ln 2 \zeta(3) + \frac{1}{2} \ln^2 2 \zeta(2) \\ &\quad - \frac{1}{12} \ln^4 2 - 2L_{i_4} \left(\frac{1}{2} \right). \end{aligned}$$

While $V(0)$ appears to have no "closed" form representation, quite remarkably two subsequent terms $V(r - 1)$ and $V(r)$ can be represented in closed form in the following way.

Lemma 3.1. The following representation follows,

$$\begin{aligned} V(r - 1) + V(r) &:= \sum_{n=1}^{\infty} (-1)^{n+1} H_n \left(\frac{1}{(n + r - 1)^3} + \frac{1}{(n + r)^3} \right) \\ &= \frac{\ln 2}{(r - 1)^3} + \sum_{j=1}^3 \frac{(-1)^j}{2^j (r - 1)^{4-j}} \left(H_{\frac{r-1}{2}}^{(j)} - H_{\frac{r-2}{2}}^{(j)} \right), \text{ for } r \geq 2, \end{aligned}$$

and for $r = 1$

$$V(0) + V(1) := \sum_{n=1}^{\infty} (-1)^{n+1} H_n \left(\frac{1}{n^3} + \frac{1}{(n + 1)^3} \right) = \frac{7}{8} \zeta(4).$$

Proof. Let

$$\begin{aligned} V(r - 1) + V(r) &= \sum_{n=1}^{\infty} (-1)^{n+1} H_n \left(\frac{1}{(n + r - 1)^3} + \frac{1}{(n + r)^3} \right) \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(H_n - H_{n+1})}{(n + r)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + 1)(n + r)^3} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(r - 1)^3} \left(\frac{1}{n + 1} - \frac{1}{n + r} - \frac{r - 1}{(n + r)^2} - \frac{(r - 1)^2}{(n + r)^3} \right) \end{aligned}$$

and the identity of Lemma 3.1 follows. ■

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