A perspective on fractional Laplace transforms and fractional generalized Hankel-Clifford transformation

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Abstract

In this study a relation between the Laplace transform and the generalized Hankel-Clifford transform is established. The relation between distributional generalized Hankel-Clifford transform and distributional one sided Laplace transform is developed. The results are verified by giving illustrations. The relation between fractional Laplace and fractional generalized Hankel-Clifford transformation is also established. Further inversion theorem considering fractional Laplace and fractional generalized Hankel-Clifford transformation is proved in Zemanian space.

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1 Introduction

The Laplace transform of a function of a function $f(t) \in L(0,\infty)$ is defined by the equation [1]

$$L(f;p) = \int_0^\infty e^{-pt} f(t)\, dt\,; \quad (Re\,(p) > 0) \tag{1.1}$$

and Malgonde [4] investigated the variant of the generalized Hankel-Clifford transform defined by

$$(h_{\alpha,\beta} f)(\xi) = F(\xi) = \int_0^\infty (\xi/t)^{-(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{t\xi}\right) f(t)\, dt\,,\ (\alpha-\beta) \geq -1/2 \tag{1.2}$$

where $J_{\alpha-\beta}(x)$ being the Bessel function of the first kind of order $(\alpha-\beta)$, in spaces of generalized functions. In [6] Bhonsle developed a relation between Laplace and Hankel transforms. Panchal [5] has developed a relation between Hankel and Laplace transforms of distributions. Namias [7] introduced a number of fractional integral transforms. His representation in [7] has led to extend fractional Hankel transform to fractional generalized Hankel-Clifford transformation in this paper. The applications of fractional integral transforms in quantum mechanics and optics have been presented in [9,10]. The paper [12] Taywade et.al hasgeneralized fractional Hankel transforms in Zemanian space. In this paper the author develops the relation between Laplace transforms and generalized Hankel-Clifford transformation and extends to fractional Laplace transforms and fractional generalized Hankel-Clifford transformation. The author illustrates few examples to this study. Relation between Laplace transforms and generalized Hankel-Clifford transformation to the space of distributions is established. A study of Fractional Laplace transforms and fractional generalized Hankel-Clifford transformation in Zemanian Space is also developed.

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2 Methodology

2.1 Relation between Laplace transforms and generalized Hankel-Clifford transformation

Firstly, a relation between the Laplace transforms of $t^{\alpha+\beta}f(t)$ and the generalized Hankel-Clifford transform of $f(t)$ has been calculated, when $(Re(\alpha-\beta)>-1)$. The result is stated in the form of a theorem which is then illustrated by an example

Theorem 2.1. *If f and $(h_{\alpha,\beta}f)(\xi)$ belongs to $L(0,\infty)$ and if*

$$Re\ (a)>0\,, Re\,(p)>0, Re\,(\alpha+\beta)>-1,$$

then

$$L\left\{t^{\alpha+\beta}f(t)\,;p\right\}=\int_0^\infty k(p,\xi)\,(h_{\alpha,\beta}f)(\xi)\,d\xi\,,$$

where

$$k(p,\xi)=p^{-1-\alpha}\xi^{\alpha}\Gamma(1+\alpha)\,{}_1F_1\left[1+\alpha,1+\alpha-\beta,-\frac{\xi}{p}\right].$$

Proof. Since $f\in L(0,\infty)$, by the generalized Hankel-Clifford inversion theorem [4], that

$$f(t)=\int_0^\infty \xi^{\alpha+\beta}\,(h_{\alpha,\beta}f)(\xi)\,\mathcal{J}_{\alpha,\beta}(t\xi)\,d\xi.$$

Hence

$$L\left\{t^{\alpha+\beta}f(t)\right\}=\int_0^\infty \xi^{\alpha+\beta}\,(h_{\alpha,\beta}f)(\xi)\,L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\right\}d\xi. \tag{2.1}$$

The change of order of integration is justified because $e^{-pt}t^{\alpha+\beta}\in L(0,\infty)$ if $Re\,(\alpha-\beta)>-1;Re\,(p)>0$ and $(h_{\alpha,\beta}f)(\xi)\in L(0,\infty)$,$\mathcal{J}_{\ \alpha,\beta}(t\xi)$ being a bounded function of both the variables. The theorem then follows from the fact [2] that

$$L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\,;p\right\}=p^{-1-\alpha}\xi^{\alpha}\Gamma(1+\alpha)\,{}_1F_1\left[1+\alpha,1+\alpha-\beta,-\frac{\xi}{p}\right]. \tag{2.2}$$

Example 2.1. Let $f(t)=t^{n-1}e^{-at}$. Then

$$L\left\{t^{\alpha+\beta}f(t)\,;p\right\}=(p+a)^{-n-\alpha-\beta}\,\Gamma(\alpha+\beta+n) \tag{2.3}$$

and

$$(h_{\alpha,\beta}f)(\xi)=F(\xi)=\xi^{-\alpha-\beta}\int_0^\infty e^{-at}t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\,dt=\xi^{-\alpha-\beta}L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\,;a\right\}. \tag{2.4}$$

This integral (2.4) can be evaluated by using (2.2). Substituting these expressions in the theorem the result is given by

$$\begin{aligned}
&\int_0^\infty \xi^{-\alpha-\beta}\,(h_{\alpha,\beta}f)(\xi)\,L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\right\}d\xi\\
&=\int_0^\infty \xi^{-\alpha}\xi^{-\beta}L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\,;a\right\}\ L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\right\}d\xi\\
&=\int_0^\infty \left[\begin{array}{c} p^{-1-\alpha}\xi^{\alpha}\Gamma(1+\alpha)\,{}_1F_1\left[1+\alpha,1+\alpha-\beta,-\frac{\xi}{p}\right]\\ \times\, a^{-1-\alpha}\xi^{\alpha}\Gamma(1+\alpha)\,{}_1F_1\left[1+\alpha,1+\alpha-\beta,-\frac{\xi}{a}\right]\end{array}\right]d\xi\\
&=\frac{(p+a)^{-n-\alpha-\beta}\Gamma(\alpha+\beta+n)}{\Gamma(1+\alpha+n)\Gamma(1+2\alpha+\beta)}
\end{aligned}$$

where $a>0, Re\ (a)>0, Re\ (p)>0\,, Re\,(\alpha-\beta)>-\dfrac{1}{2};\ Re\,(\alpha)>-\dfrac{1}{2}, Re\,(\alpha+n)>-1.$

2.2 Relation between fractional generalized Hankel-Clifford transformation and fractional Laplace transforms

Theorem 2.2 Define a one-dimensional continuous fractional generalized Hankel-Clifford transformation with parameter θ of $f(x)$ for $(\alpha-\beta)\geq -1/2$ and $0<\theta<\pi$as follows:

$$h_{\alpha,\beta}^{\theta}\left(f\left(y\right)\right)=F\left(y\right)=y^{-\alpha-\beta}\int_{0}^{\infty}C_{\alpha,\beta}^{\theta}\left(x,y\right)f\left(x\right)dx\ , \tag{2.4}$$

where the kernel

$$C_{\alpha,\beta}^{\theta}\left(x,y\right)=\begin{cases} e^{\left|\cot\theta\right|\ i\ \left(\frac{y^2}{2}+\frac{x^2}{2}\right)}c_{\alpha,\beta}^{\theta}\left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right), & \theta\neq n\pi \\ \left(xy\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(xy\right)^{1/2}\right) & ,\theta=\frac{\pi}{2} \\ \delta\left(x-y\right) & ,\theta=n\pi,\forall n\in\mathbb{Z}. \end{cases}$$

and

$$c_{\alpha,\beta}^{\theta}=\frac{e^{i(\alpha-\beta+1)\left(\frac{\pi}{2}-\theta\right)}}{\sin\theta}$$

and fractional Laplace transforms is defined by Sharma [11] as:

$$L^{\theta}\left[f\left(x\right)\right]\left(y\right)=\frac{1}{\sqrt{2\pi\sin\theta}}\int_{0}^{\infty}e^{\frac{i}{2}\left(x^2-y^2\right)\cot\theta-xy\csc\theta}f\left(x\right)dx.$$

If $L^{\theta}\left[f\left(x\right)\right]\left(y\right)$ and $h_{\alpha,\beta}^{\theta}\left[f\left(x\right)\right]\left(y\right)$ belong to $L\left(0,\infty\right)$, and

if $\left(\sqrt{\frac{y}{\left|\sin\theta\right|}}\right)=0$ & $img\left(\sqrt{\frac{y}{\left|\sin\theta\right|}}\right)=0$

& $Re\left(i\cot\theta\right)\geq 0$ & $Re\left(\alpha\right)>-\frac{1}{2}$ & $3Re\left(\alpha\right)+Re\ \left(\beta\right)<-2$,

$h_{\ \alpha,\beta}^{\ \theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right)=\frac{c_{\alpha,\beta}^{\theta}}{\sqrt{2\pi}}\left(\csc\theta\right)^{(\alpha+\beta-1)/2}\int_{0}^{\infty}y^{-\alpha-\beta}L^{\theta}\left[f\left(x\right)\right]\left(y\right)dy.$

If $\quad y\geq 0$ & $img\left(\sqrt{\frac{y}{\left|\sin\theta\right|}}\right)=0$

& $Re\left(i\cot\theta\right)\geq 0$ & $Re\left(\alpha\right)>-\frac{1}{2}$ & $3Re\left(\alpha\right)+Re\ \left(\beta\right)<-2$,

$h_{\ \alpha,\beta}^{\ \theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right)$

$=\frac{c_{\alpha,\beta}^{\theta}2^{-3/2}\cos\theta^{(1+\alpha)}e^{i\pi(1+\alpha)}(\sin\theta)^{\frac{1}{2}(-5-3\alpha+\beta)}}{\sqrt{\pi}\Gamma(1+\alpha-\beta)\Gamma(2+\alpha-\beta)}\int_{0}^{\infty}g\left(y\right)y^{-\alpha-\beta}L^{\theta}\left[f\left(x\right)\right]\left(y\right)dy.$

Proof.

$$\begin{aligned} & h_{\ \alpha,\beta}^{\ \theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right) \\ &=c_{\alpha,\beta}^{\theta}y^{-\alpha-\beta}\int_{0}^{\infty}x^{\alpha}e^{xy\csc\theta}e^{-\left|\cot\theta\right|\ i\ \left(\frac{y^2}{2}+\frac{x^2}{2}\right)}\left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right)f\left(x\right)dx \\ &=c_{\alpha,\beta}^{\theta}\int_{0}^{\infty}x^{\alpha}e^{-\left|\cot\theta\right|\ i\ \left(\frac{y^2}{2}+\frac{x^2}{2}\right)}\left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right) \\ &\quad\times\left(\frac{1}{\sqrt{2\pi\sin\theta}}\int_{0}^{\infty}e^{-\frac{i}{2}\left(x^2-y^2\right)\cot\theta}y^{-\alpha-\beta}L^{\theta}\left[f\left(x\right)\right]\left(y\right)dy\right)dx \end{aligned}$$

By change of order of integration,

$$\begin{aligned} & h_{\ \alpha,\beta}^{\ \theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right) \\ &=\frac{c_{\alpha,\beta}^{\theta}}{\sqrt{2\pi\sin\theta}}\left[\int_{0}^{\infty}\left\{\begin{array}{c}\int_{0}^{\infty}y^{-\alpha-\beta}L^{\theta}\left[f\left(x\right)\right]\left(y\right)\times \\ x^{(3\alpha-\beta)/2}e^{-i\left|\cot\theta\right|\ x^2}\times \\ \left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right)\end{array}\right\}dx\right]dy. \end{aligned}$$

$$h^{\theta}_{\alpha,\beta}\left[x^{\alpha}e^{xy\csc\theta}f(x)\right](y)$$
$$= \frac{c^{\theta}_{\alpha,\beta}}{\sqrt{2\pi\sin\theta}}\frac{1}{(2\Gamma(1+\alpha-\beta)\Gamma(2+\alpha-\beta))}\int_0^{\infty} y^{-\alpha-\beta}L^{\theta}[f(x)](y)$$
$$\times\left\{\begin{array}{l} y^{-3+\beta}\left(\frac{y^2}{i\cot\theta}\right)^{(1+\alpha)}(|\sin\theta|)^{\frac{1}{2}(-2-\alpha+\beta)} \\ \times\left(\begin{array}{l}(i\cot\theta)\sqrt{\frac{y^2}{i\cot\theta}}|\sin\theta|\Gamma(2+\alpha-\beta)\Gamma\left(\frac{1}{2}+\alpha\right) \\ \times {}^pF_q\left[\left\{\frac{1}{2}+\alpha\right\},\left\{\frac{1}{2},\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2},1+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\frac{y^2}{16i\cot\theta|\sin\theta|^2}\right] \\ -y^2\Gamma(1+\alpha-\beta)\Gamma(1+\alpha) \\ \times {}^pF_q\left[\{1+\alpha\},\left\{\frac{3}{2},1+\frac{\alpha}{2}-\frac{\beta}{2},\frac{3}{2}+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\frac{y^2}{16i\cot\theta|\sin\theta|^2}\right]\end{array}\right)\end{array}\right\}dy$$

$$\text{Let } g(y) = y^{(2\alpha+\beta-1)}\left(\begin{array}{l}(i\cot\theta)\sqrt{\frac{y^2}{i\cot\theta}}|\sin\theta|\Gamma(2+\alpha-\beta)\Gamma\left(\frac{1}{2}+\alpha\right) \\ \times {}^pF_q\left[\left\{\frac{1}{2}+\alpha\right\},\left\{\frac{1}{2},\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2},1+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\frac{y^2}{16i\cot\theta|\sin\theta|^2}\right] \\ -y^2\Gamma(1+\alpha-\beta)\Gamma(1+\alpha) \\ \times {}^pF_q\left[\{1+\alpha\},\left\{\frac{3}{2},1+\frac{\alpha}{2}-\frac{\beta}{2},\frac{3}{2}+\frac{\alpha}{2}-\frac{\beta}{2}\right\},\frac{y^2}{16i\cot\theta|\sin\theta|^2}\right]\end{array}\right)$$

$$\begin{aligned} & h^{\theta}_{\alpha,\beta}\left[x^{\alpha}e^{xy\csc\theta}f(x)\right](y) \\ = \; & \frac{c^{\theta}_{\alpha,\beta}2^{-3/2}\cos\theta^{(1+\alpha)}e^{i\pi(1+\alpha)}(\sin\theta)^{\frac{1}{2}(-5-3\alpha+\beta)}}{\sqrt{\pi}\Gamma(1+\alpha-\beta)\Gamma(2+\alpha-\beta)}\int_0^{\infty} g(y)\,y^{-\alpha-\beta}L^{\theta}[f(x)](y)\,dy.\end{aligned}$$

Thus
If $\left(\sqrt{\frac{y}{|\sin\theta|}}\right) = 0$ & $img\left(\sqrt{\frac{y}{|\sin\theta|}}\right) = 0$
& $Re(i\cot\theta) \geq 0$ & $Re(\alpha) > -\frac{1}{2}$ & $3Re(\alpha) + Re(\beta) < -2$,
$h^{\theta}_{\alpha,\beta}\left[x^{\alpha}e^{xy\csc\theta}f(x)\right](y)$
$= \frac{c^{\theta}_{\alpha,\beta}}{\sqrt{2\pi}}(\csc\theta)^{(\alpha+\beta-1)/2}\int_0^{\infty} y^{-\alpha-\beta}L^{\theta}[f(x)](y)\,dy.$
If $y \geq 0$ & $img\left(\sqrt{\frac{y}{|\sin\theta|}}\right) = 0$
& $Re(i\cot\theta) \geq 0$ & $Re(\alpha) > -\frac{1}{2}$ & $3Re(\alpha) + Re(\beta) < -2$,
$h^{\theta}_{\alpha,\beta}\left[x^{\alpha}e^{xy\csc\theta}f(x)\right](y)$
$= \frac{c^{\theta}_{\alpha,\beta}2^{-3/2}\cos\theta^{(1+\alpha)}e^{i\pi(1+\alpha)}(\sin\theta)^{\frac{1}{2}(-5-3\alpha+\beta)}}{\sqrt{\pi}\Gamma(1+\alpha-\beta)\Gamma(2+\alpha-\beta)}\int_0^{\infty} g(y)\,y^{-\alpha-\beta}L^{\theta}[f(x)](y)\,dy.$
Thus proved the theorem.

2.3 Relation between fractional Laplace transform and fractional generalized Hankel-Clifford transformation

Theorem 2.3: *If* $f(x)$ *and* $h^{\theta}_{\alpha,\beta}[f(x)](y)$ *belong to* $L(0,\infty)$ *and if* $\sqrt{\frac{y}{|\sin\theta|}} \in R$ *and* $\sqrt{\frac{y}{|\sin\theta|}} = 0$ $Re(\alpha) > -\frac{1}{2}$ *, then*

$$L^{\theta}\left[x^{\alpha}e^{xy\csc\theta}f(x)\right](y) = A^{\theta}_{1,\alpha,\beta}\int_0^{\infty} g_1(y)\left(h^{\theta}_{\alpha,\beta}f\right)(y)\,dy,$$

$$\text{where } g_1(y) = y^{-\alpha-\beta}e^{-iy^2\cot\theta} \text{ and } A^{\theta}_{1,\alpha,\beta} = \frac{\overline{c^{\theta}_{\alpha,\beta}}}{\sqrt{2\pi}}(\csc\theta)^{(\alpha+\beta-1)/2}.$$

And

$$if-1<Re\left(\alpha-\beta\right)<\frac{1}{2},\ if\ \sqrt{\frac{y}{\left|\sin\theta\right|}}\in R\ \&\ \sqrt{\frac{y}{\left|\sin\theta\right|}}=0\ Re\left(\alpha\right)>-\frac{1}{2}\ \&\, y\geq 0,$$

then

$L^{\theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right)=\ A_{2,\alpha,\beta}^{\theta}\int_0^{\infty}g_2\left(y\right)\left(h_{\alpha,\beta}^{\theta}f\right)\left(y\right)dy,$

where $g_2\left(y\right)=y^{1-2\alpha-\beta}e^{-iy^2\cot\theta}$ *and* $A_{2,\alpha,\beta}^{\theta}=\frac{\overline{c_{\alpha,\beta}^{\theta}}}{\sqrt{2\pi}}\left(\frac{\left|\sin\theta\right|^{(3+3\alpha+\beta)/2}\Gamma(1+2\alpha)}{\Gamma(-\alpha-\beta)}\right).$

Proof. By definition,

$L^{\theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right)$

$=\frac{1}{\sqrt{2\pi\sin\theta}}\int_0^{\infty}e^{\frac{i}{2}\left(x^2-y^2\right)\cot\theta-xy\csc\theta}x^{\alpha}e^{xy\csc\theta}f\left(x\right)dx$

$=\frac{1}{\sqrt{2\pi\sin\theta}}\int_0^{\infty}e^{\frac{i}{2}\left(x^2-y^2\right)\cot\theta}x^{\alpha}f\left(x\right)dx.$

By definition of inverse fractional generalized Hankel-Clifford transformation,

$$f\left(x\right)=\left(\left(h_{\alpha,\beta}^{\theta}\right)^{-1}F\right)\left(x\right)=y^{-\alpha-\beta}\int_0^{\infty}\overline{C_{\alpha,\beta}^{\theta}\left(x,y\right)}\left(h_{\alpha,\beta}^{\theta}f\right)\left(y\right)dy,\ y\in\mathbb{R}_+$$

where

$$\overline{C_{\alpha,\beta}^{\theta}\left(x,y\right)}=e^{-\left|\cot\theta\right|\,i\,\left(\frac{y^2}{2}+\frac{x^2}{2}\right)}e^{-i(\alpha-\beta+1)\left(\frac{\pi}{2}-\theta\right)}\left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right)$$

$$=\overline{c_{\alpha,\beta}^{\theta}}\sin\theta e^{-\left|\cot\theta\right|\,i\,\left(\frac{y^2}{2}+\frac{x^2}{2}\right)}\left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right).$$

$L^{\theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right)$

$=\frac{\overline{c_{\alpha,\beta}^{\theta}}\sin\theta}{\sqrt{2\pi\sin\theta}}\int_0^{\infty}x^{\alpha}e^{\frac{i}{2}\left(x^2-y^2\right)\cot\theta}$

$\times\left(\int_0^{\infty}y^{-\alpha-\beta}e^{-\left|\cot\theta\right|\,i\,\left(\frac{y^2}{2}+\frac{x^2}{2}\right)}\left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right)\left(h_{\alpha,\beta}^{\theta}f\right)\left(y\right)dy\right)dx.$

By change of order of integration,

$L^{\theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right)$

$=\frac{\overline{c_{\alpha,\beta}^{\theta}}\sin\theta}{\sqrt{2\pi\sin\theta}}\int_0^{\infty}\left(h_{\alpha,\beta}^{\theta}f\right)\left(y\right)$

$\times\left(\int_0^{\infty}x^{\alpha}e^{\frac{i}{2}\left(x^2-y^2\right)\cot\theta}y^{-\alpha-\beta}e^{-\left|\cot\theta\right|\,i\,\left(\frac{y^2}{2}+\frac{x^2}{2}\right)}\left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right)dx\right)dy$

$=\frac{\overline{c_{\alpha,\beta}^{\theta}}\sin\theta}{\sqrt{2\pi\sin\theta}}\int_0^{\infty}\left(h_{\alpha,\beta}^{\theta}f\right)\left(y\right)\left(\int_0^{\infty}x^{\alpha}y^{-\alpha-\beta}e^{-iy^2\cot\theta}\left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right)dx\right)dy.$

In extension to [8], the result obtained is

$L^{\theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right)$

$=\frac{\overline{c_{\alpha,\beta}^{\theta}}\sin\theta}{\sqrt{2\pi\sin\theta}}\int_0^{\infty}\left(h_{\alpha,\beta}^{\theta}f\right)\left(y\right)y^{-\alpha-\beta}e^{-iy^2\cot\theta}\left(\int_0^{\infty}x^{\alpha}\left(\frac{xy}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\left(\frac{xy}{\left|\sin\theta\right|}\right)^{1/2}\right)dx\right)dy.$

If $\sqrt{\frac{y}{\left|\sin\theta\right|}}\in\mathbb{R}\ \&\ \sqrt{\frac{y}{\left|\sin\theta\right|}}=0\ Re\left(\alpha\right)>-\frac{1}{2}$

$L^{\theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right)=\frac{\overline{c_{\alpha,\beta}^{\theta}}\sin\theta}{\sqrt{2\pi\sin\theta}}\int_0^{\infty}\left(h_{\alpha,\beta}^{\theta}f\right)\left(y\right)y^{-\alpha-\beta}e^{-iy^2\cot\theta}\left(\frac{1}{\left|\sin\theta\right|}\right)^{(\alpha+\beta)/2}dy;$

and if $\sqrt{\frac{y}{\left|\sin\theta\right|}}\in\mathbb{R}\ \&\ \sqrt{\frac{y}{\left|\sin\theta\right|}}=0\ Re\left(\alpha\right)>-\frac{1}{2}\ \&\, y\geq 0,$ then

$L^{\theta}\left[x^{\alpha}e^{xy\csc\theta}f\left(x\right)\right]\left(y\right)=\frac{\overline{c_{\alpha,\beta}^{\theta}}\sin\theta}{\sqrt{2\pi\sin\theta}}\int_0^{\infty}\left(h_{\alpha,\beta}^{\theta}f\right)\left(y\right)y^{-\alpha-\beta}e^{-iy^2\cot\theta}\left(\frac{y^{1-\alpha}\left|\sin\theta\right|^{(3\alpha+\beta+2)/2}\Gamma(1+2\alpha)}{\Gamma(-\alpha-\beta)}\right)dy.$

If $\sqrt{\frac{y}{|\sin\theta|}} \in \mathbb{R}$ & $\sqrt{\frac{y}{|\sin\theta|}} = 0$ $Re(\alpha) > -\frac{1}{2}$

$L^{\theta}\left[x^{\alpha} e^{xy\csc\theta} f(x)\right](y)$

$= \frac{\overline{c^{\theta}_{\alpha,\beta}}}{\sqrt{2\pi}} (\csc\theta)^{(\alpha+\beta-1)/2} \int_0^\infty y^{-\alpha-\beta} e^{-iy^2\cot\theta} \left(h^{\theta}_{\alpha,\beta} f\right)(y)\, dy.$

Let $g_1(y) = y^{-\alpha-\beta} e^{-iy^2\cot\theta}$ and $A^{\theta}_{1,\alpha,\beta} = \frac{\overline{c^{\theta}_{\alpha,\beta}}}{\sqrt{2\pi}} (\csc\theta)^{(\alpha+\beta-1)/2}$; then

$L^{\theta}\left[x^{\alpha} e^{xy\csc\theta} f(x)\right](y) = A^{\theta}_{1,\alpha,\beta} \int_0^\infty g_1(y) \left(h^{\theta}_{\alpha,\beta} f\right)(y)\, dy.$

and if $\sqrt{\frac{y}{|\sin\theta|}} \in \mathbb{R}$ & $\sqrt{\frac{y}{|\sin\theta|}} = 0$ $Re(\alpha) > -\frac{1}{2}$ & $y \geq 0$, then

$L^{\theta}\left[x^{\alpha} e^{xy\csc\theta} f(x)\right](y)$

$= \frac{\overline{c^{\theta}_{\alpha,\beta}}}{\sqrt{2\pi}} \left(\frac{|\sin\theta|^{(3+3\alpha+\beta)/2}\Gamma(1+2\alpha)}{\Gamma(-\alpha-\beta)}\right) \int_0^\infty \left(h^{\theta}_{\alpha,\beta} f\right)(y)\, y^{-\alpha-\beta} e^{-iy^2\cot\theta} y^{1-\alpha} dy$

Let $g_2(y) = y^{1-2\alpha-\beta} e^{-iy^2\cot\theta}$ and $A^{\theta}_{2,\alpha,\beta} = \frac{\overline{c^{\theta}_{\alpha,\beta}}}{\sqrt{2\pi}} \left(\frac{|\sin\theta|^{(3+3\alpha+\beta)/2}\Gamma(1+2\alpha)}{\Gamma(-\alpha-\beta)}\right)$; then

$L^{\theta}\left[x^{\alpha} e^{xy\csc\theta} f(x)\right](y) = A^{\theta}_{2,\alpha,\beta} \int_0^\infty g_2(y) \left(h^{\theta}_{\alpha,\beta} f\right)(y)\, dy.$

Thus the proof.

3 Testing function space

3.1 Relation between Laplace transforms and generalized Hankel-Clifford transformation to the space of distributions

Let $(h_{\alpha,\beta} f)(\xi)$ is a testing function space for generalized Hankel-Clifford transform and $\left(h'_{\alpha,\beta} f\right)(\xi)$ is its dual. $L(w,z)$ and $L(w)$ are testing function spaces for Laplace transform and $L'(w,z)$ and $L'(w)$ are their duals respectively. Since the testing function space $(h_{\alpha,\beta} f)(\xi)$, $L(w,z)$ and $L(w)$ are subspace of E, the space of distributions of compact support E' is a subspace of all the generalized function space $\left(h'_{\alpha,\beta} f\right)(\xi)$,$L'(w,z)$ and $L'(w)$. The restriction $f \in \left(h'_{\alpha,\beta} f\right)(\xi) \bigcap L'(w,z)$ to E is a member of E'[4]. In order to extend the relation (2.2) to the space of distributions, considered a lemma to prove

Lemma 3.1. *If $f \in L'(w,z)$ then the mapping $f(x) \to x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) f(x)$ is a linear and continuous from $L'(w)$ into itself.*

Proof. For each integer $k \geq 0$ there exists an integer n_k such that

$\left|(1+x)^{N_k} D^k \left[x^{-\alpha-\beta} \mathrm{J}_{\alpha,\beta}(x\lambda_n)\right]\right|$

$= \left|(1+x)^{N_k}\right| \left|\sum_{j=0}^{m} \binom{k}{m} D^{k-m} D^m \left[x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n)\right]\right|$

$= \left|(1+x)^{N_k}\right| \left|\sum_{j=0}^{m} \binom{k}{m} D^{k-m} \left[\sum_{j=0}^{m} a_j(\alpha)\, y^{-\left(\frac{\alpha+\beta+j}{2}\right)} \lambda_n^{j-m} x^{\left(\frac{\alpha+\beta+j}{2}\right)} J_{\alpha-\beta-j}\left(2\sqrt{x\lambda_n}\right)\right]\right|$

$= \left|(1+x)^{N_k}\right| \left|\sum_{j=0}^{m} \binom{k}{m} \begin{bmatrix} \sum_{j=0}^{m} a_j(\alpha)\, \lambda_n^{-\left(\frac{\alpha+\beta+j}{2}\right)} \lambda_n^{j-m} x^{\left(\frac{\alpha+\beta+j}{2}\right)} J_{\alpha-\beta-j}\left(2\sqrt{x\lambda_n}\right) \\ \times(-1)^{k-m} x^{(\alpha+\beta+k-m)/2} \left[\lambda_n^{-(\alpha-\beta-j+k-m)/2} J_{\alpha-\beta-j+k-m}\left(2\sqrt{x\lambda_n}\right)\right] \end{bmatrix}\right|$

where $0 < x < \infty$ and where the $a_j(\alpha)$ are constants depending on α only. Therefore $x^{-(\alpha-\beta)/2} J_{\alpha-\beta}\left(2(\lambda_n x)^{1/2}\right) \in \theta_M$. As $e^{-px} L(w,z)$ the mapping

$$e^{-px} \to x^{-(\alpha-\beta)/2} J_{\alpha-\beta}\left(2(\lambda_n x)^{1/2}\right) e^{-px}$$

is linear and continuous from $L(w,z)$ into itself and the adjoint mapping

$$f(x) \to x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n) f(x)$$

defined by

$$\left\langle x^{-\alpha-\beta}\mathrm{J}_{\alpha,\beta}(x\lambda_n) f(x), e^{-px}\right\rangle = \left\langle f(x), x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n) e^{-px}\right\rangle \tag{3.1}$$

is a linear and continuous from $L'(w,z)$ into itself [3] [Theorem 1.10 and Sec. 2.5], where

$$f \in L'(w,z)\ e^{-px} \in L(w,z),\ x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n) \in \theta_M.$$

Lemma 3.2. *Let* $f \in \left(h'_{\alpha,\beta}f\right)(\xi)$, *then mapping* $f(x) \to e^{-px}f(x)$ *is a linear and continuous* $\left(h'_{\alpha,\beta}f\right)(\xi)$ *into itself.*

Proof. Since for each nonnegative integer k there exists an integer N_k such that $\left|\frac{\left(x^{-\alpha}D\right)^k e^{-px}}{1+x^{N_k/2}}\right| < \infty$ for $0 < x < \infty$. Thus $e^{-px} \in \theta_M$ the space of multipliers for $(h_{\alpha,\beta}f)(\xi)$ [4]. As $e^{-px} \in \theta_M$, the mapping

$$x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n) \to e^{-px}x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)$$

is a linear and continuous from $(h_{\alpha,\beta}f)(\xi)$ into itself and the adjoint mapping

$$f(x) \to e^{-px}f(x), f \in \left(h'_{\alpha,\beta}f\right)(\xi)$$

defined by

$$\left\langle e^{-px}f(x), x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)\right\rangle = \left\langle f(x), x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n) e^{-px}\right\rangle \tag{3.2}$$

is a linear and continuous from $\left(h'_{\alpha,\beta}f\right)(\xi)$ into itself where

$$f \in \left(h'_{\alpha,\beta}f\right)(\xi), x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n) \in (h_{\alpha,\beta}f)(\xi).$$

Theorem 3.1. If $f \in E'$ then

$$\left\langle e^{-px}f(x), x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)\right\rangle = \left\langle x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n) f(x), e^{-px}\right\rangle$$

for $Re(p) < \infty, \lambda_n \geq 0; \alpha-\beta \geq -\frac{1}{2}$.

Proof. Since the testing function space $(h_{\alpha,\beta}f)(\xi), L(w,z)$ and $L(w)$are subspace of E, the space of distributions of compact support $E'$is a subspace of all the generalized function space $\left(h'_{\alpha,\beta}f\right)(\xi), L'(w,z)$ and $L'(w)$ [4]. Therefore the restriction of $f \in L'(w)$ to $L(w,z)$ is in $L'(w,z)$ and the restriction of $f \in \left(h'_{\alpha,\beta}f\right)(\xi) \bigcap L'(w,z)$ to E is a member of E'. In view of above the result is obvious from Lemma 2.1 and Lemma 2.2 for every $f \in E'$, since the right hand sides of the equations (3.1) and (3.2) are equal. Thus the equality

$$\left\langle e^{-px}f(x), x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)\right\rangle = \left\langle x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n) f(x), e^{-px}\right\rangle$$

holds good for $0 < Re(p) < \infty, \lambda_n \geq 0; \alpha-\beta \geq -\frac{1}{2}$.

Example 3.1. For $0 < Re\,(p) < \infty, \lambda_n \geq 0; \alpha - \beta \geq -\frac{1}{2}$, from (2.4) as illustrated in [5],

$$\begin{aligned}\left\langle e^{-px} f(x), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) \right\rangle &= \left\langle x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) f(x), e^{-px} \right\rangle \\ &= a^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(a\lambda_n) e^{-pa} \text{ for } f(x) \\ &= \delta(x-a).\end{aligned}$$

Put $f(x) = \delta(x-a)$, then

$$e^{-px}\delta(x-a) \in \left(h'_{\alpha,\beta} f\right)(\xi), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n)\,\delta(x-a) \in L'(w,z).$$

$\left\langle e^{-px} f(x), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) \right\rangle$
$= \left\langle f(x), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) e^{-px} \right\rangle$
$= \left\langle x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n)\ f(x), e^{-px} \right\rangle.$
For $f(x) = \delta(x-a),$
$\left\langle e^{-px}\delta(x-a), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) \right\rangle$
$= \left\langle \delta(x-a), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) e^{-px} \right\rangle$
$= \left\langle x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}\ (x\lambda_n)\,\delta(x-a), e^{-px} \right\rangle$
$= a^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}\ (a\lambda_n) e^{-pa}.$

3.2 Fractional Laplace transforms and fractional generalized Hankel-Clifford transformation in Zemanian Space

An infinitely differentiable complex-valued function φ on R^n belongs to $E(R^n)$ or E if for each compact set $K \subset S_a$, where

$$S_a = \{x : x \in R,\ |x| < a, a > 0\}$$

and

$$\gamma_{K,k}(\varphi) = \sup_{x\in K} \left|D^k \varphi(x)\right| < \infty.$$

If $f \in E'(R)$,sup $f \subset S_a$ and a one-dimensional continuous fractional generalized Hankel-Clifford transformation with parameter θ of $f(x)$ for $(\alpha - \beta) \geq -1/2$ and $0 < \theta < \pi$, $h^\theta_{\alpha,\beta}(f(y))$ and $L^\theta[f(x)](y)$ is considered

Lemma 3.3. *For $\varphi(x) \in E$, a one-dimensional continuous fractional generalized Hankel-Clifford transformation with parameter θ of $f(x)$ for $(\alpha-\beta) \geq -1/2$ and $0 < \theta < \pi$ as follows:*

$$\left[h^\theta_{\alpha,\beta} f(x)\right](y) = h^\theta_{\alpha,\beta}(y), \psi(y) = \int_0^\infty \varphi(x)\, \overline{C^\theta_{\alpha,\beta}(x,y)}\, dx.$$

Then for any fixed number r, where $0 < r < \infty$,

$$\int_0^r \left\langle \widetilde{f^\theta(\varsigma)}, C^\theta_{\alpha,\beta}(\varsigma,y) \right\rangle \psi(y)\, d\tau = \left\langle \widetilde{f^\theta(\varsigma)}, \int_0^r C^\theta_{\alpha,\beta}(\varsigma,y)\ \psi(y)\, d\tau \right\rangle. \tag{3.3}$$

where fractional Laplace transforms is represented as $L^\theta[f(x)](y) = \widetilde{f^\theta(\varsigma)}$ and $y = \sigma + i\tau \in C^n$ and ς is restricted to a compact subset of R.

Proof. If $\varphi(x) = 0$ the case is trivial. If $\varphi(x) \neq 0$, consider a Rienmann-sum for describing $\int_0^r \left\langle \widetilde{f^\theta(\varsigma)}, C^\theta_{\alpha,\beta}(\varsigma,y) \right\rangle \psi(y)\, d\tau.$

$\int_0^r \left\langle \widetilde{f^{\theta}(\varsigma)}, C^{\theta}_{\alpha,\beta}(\varsigma,y)\right\rangle \psi(y)\,d\tau$
$= \lim\limits_{m\to\infty} \left\langle \widetilde{f^{\theta}(\varsigma)}, \sum_{n=0}^{m-1} C^{\theta}_{\alpha,\beta}(\varsigma,y)\,\psi(y)\,\Delta y\right\rangle$
$= \lim\limits_{m\to\infty} \left\langle \widetilde{f^{\theta}(\varsigma)}, \sum_{n=0}^{m-1} C^{\theta}_{\alpha,\beta}(\varsigma,\sigma+i\tau_{n,m})\,\psi(\sigma+i\tau_{n,m})\,\Delta\tau_{n,m}\right\rangle.$
Consider
$\gamma_{k,K}\left\{\sum_{n=0}^{m-1} C^{\theta}_{\alpha,\beta}(\varsigma,\sigma+i\tau_n)\,\psi(\sigma+i\tau_n)\,\Delta\tau_n - \int_0^r C^{\theta}_{\alpha,\beta}(\varsigma,y)\ \psi(y)\,d\tau\right\}$
$= \underset{\varsigma\in K}{Sup}\left[\sum_{n=0}^{m-1} D^k_{\varsigma} C^{\theta}_{\alpha,\beta}(\varsigma,\sigma+i\tau_{n,m})\,\Delta\tau_{n,m} - \int_0^r C^{\theta}_{\alpha,\beta}(\varsigma,y)\ \psi(y)\,d\tau\right]$
$\lim\limits_{m\to\infty} \sum_{n=0}^{m-1} D^k_{\varsigma} C^{\theta}_{\alpha,\beta}(\varsigma,\sigma+i\tau_{n,m})\,\Delta\tau_{n,m} - \int_0^r C^{\theta}_{\alpha,\beta}(\varsigma,y)\ \psi(y)\,d\tau \to 0$
as $r\to\infty$ for all $\varsigma\in K$.

$$\lim_{m\to\infty} \sum_{n=0}^{m-1} D^k_{\varsigma} C^{\theta}_{\alpha,\beta}(\varsigma,\sigma+i\tau_{n,m})\,\Delta\tau_{n,m} = \int_0^r C^{\theta}_{\alpha,\beta}(\varsigma,y)\ \psi(y)\,d\tau \tag{3.4}$$

It follows that for every m, the summation is a member of E and it converges in E. Hence the proof.

Lemma 3.4. For $\varphi(x)\in E$, and $y\in C$, ς is restricted to a compact subset of R then,

$$M_r(\varsigma) = \int_0^r C^{\theta}_{\alpha,\beta}(\varsigma,y)\left(\int_0^{\infty} \varphi(x)\,\overline{C^{\theta}_{\alpha,\beta}(x,y)}\,dx\right)d\varsigma, \tag{3.5}$$

converges in E to $\varphi(\varsigma)$ as $r\to\infty$.

Proof. In it is shown that $M_r\to\varphi(\varsigma)$ as $r\to\infty$.Also it is shown that

$$\gamma_{K,k}\left[M_r-\varphi(\varsigma)\right] = \underset{\varsigma\in K}{Sup}\left[D^k_{\varsigma}\{M_r\}-\varphi(\varsigma)\right]\to 0 \ \text{ as } \ r\to\infty.$$

$$\gamma_{K,k}\left[M_r-\varphi(\varsigma)\right] = \int_0^r C^{\theta}_{\alpha,\beta}(\varsigma,y)\left(\int_0^{\infty} \varphi(x)\,\overline{C^{\theta}_{\alpha,\beta}(x,y)}\,dx\right)d\varsigma - \varphi(\varsigma)\to 0 \ \text{ as } \ r\to\infty.$$

This is to say that $\lim\limits_{r\to\infty} M_r = \varphi(\varsigma)$.

Since the integrand is a C^{∞}function of ς and $\varphi(x)\in E$, repetitively differentiating under integral sign in (3.3) and the integrals are uniformly convergent. Thus
$\underset{\varsigma\in K}{Sup}\left[D^k_{\varsigma}\{M_r\}-\varphi(\varsigma)\right]$
$= \int_0^r D^k_{\varsigma} C^{\theta}_{\alpha,\beta}(\varsigma,y)\left(\int_0^{\infty} \varphi(x)\,\overline{C^{\theta}_{\alpha,\beta}(x,y)}\,dx\right)d\varsigma - \varphi(\varsigma)\to 0$
as $r\to\infty$ for all $\varsigma\in K$.
Hence the claim.

Theorem 3.2 (Inversion theorem) Let $\varphi(x)\in E$. Show that

$$\left\langle \int_0^{\infty} \frac{1}{\sqrt{2\pi\sin\theta}} e^{\frac{i}{2}\left(x^2-\varsigma^2\right)\cot\theta - x\varsigma\csc\theta} \int_0^{\infty} \left\{ y^{-\alpha-\beta}\overline{C^{\theta}_{\alpha,\beta}(x,y)}\left(h^{\theta}_{\alpha,\beta}f\right)(y)\,dy\right\}d\tau, \varphi(x)\right\rangle$$

$$\to \left\langle \widetilde{f^{\theta}(x)}, \varphi(x)\right\rangle \ \text{ as } r\to\infty. \tag{3.6}$$

Proof. From the analyticity of $\left(h^{\theta}_{\alpha,\beta}f\right)(y)$ on $C$and $\varphi(x)$ is a compact support in R, it follows that the left side expression in (3.6) is merely a repeated integral with respect to x and y and the integral in (3.6) is a continuous function of $x$as the closed bound domain of integration. Therefore

$$\int_0^\infty \varphi(x)\left[\int_0^\infty \frac{1}{\sqrt{2\pi\sin\theta}} e^{\frac{i}{2}\left(x^2-\varsigma^2\right)\cot\theta - x\varsigma\csc\theta}\left\{\int_0^r y^{-\alpha-\beta}\overline{C^{\theta}_{\alpha,\beta}(x,y)}\left(h^{\theta}_{\alpha,\beta}f\right)(y)\,dy\right\}d\tau\right]dx$$
$$= \int_0^r \left\langle \widetilde{f^{\theta}(\varsigma)}\ , C^{\theta}_{\alpha,\beta}(\varsigma,y)\right\rangle \psi(y)\,d\tau.$$

Since $\varphi(x)$ is a compact support, and the integrand is a continuous function of (x,y) the order of integration is changed. The change of the order of integration is justified, where

$$\psi(y) = \int_0^\infty \varphi(x)\,\overline{C^{\theta}_{\alpha,\beta}(x,y)}\,dx.$$

$$\int_0^r \left\langle \widetilde{f^{\theta}(\varsigma)}, C^{\theta}_{\alpha,\beta}(\varsigma,y)\right\rangle \psi(y)\,d\tau = \left\langle \widetilde{f^{\theta}(\varsigma)}, \int_0^\infty C^{\theta}_{\alpha,\beta}(\varsigma,y)\,\psi(y)\,d\tau\right\rangle.$$

From Lemma (3.3),

$$\left\langle \widetilde{f^{\theta}(\varsigma)}, \int_0^\infty C^{\theta}_{\alpha,\beta}(\varsigma,y)\,\psi(y)\,d\tau\right\rangle$$

converges

$$\left\langle \widetilde{f^{\theta}(\varsigma)}, \varphi(\varsigma)\right\rangle \text{ as } r\to\infty.$$

This completes the proof.

4 Discussions

In the first section the introduction to the related work is presented. In the second section of this work the classical work related to Laplace transforms and the generalized Hankel-Clifford transform is represented. In this section applying fractional Laplace transforms to fractional generalized Hankel-Clifford transformation and the fractional generalized Hankel-Clifford transformation to fractional Laplace transforms, observations were made. Relation between Laplace transform and generalized Hankel-Clifford transformation to the space of distributions has been derived in the third section. Examples have been demonstrated at the end of each section in the methodology and development. In this section study of fractional Laplace transform and the fractional generalized Hankel-Clifford transformation is done in Zemanian space. The new developments can be used in engineering applications.

5 Conclusion

The reader can further develop a relation between Laplace transform and the finite generalized Hankel-Clifford transform, its relation in distributional sense. The readers can find the relation between fractional Laplace transform and the finite fractional generalized Hankel-Clifford transformation is my next paper. The partial differential equations related to these transforms will be useful for proving the applications of the said transforms. The developed relations in this study would open new areas of applications in mechanics and optics. This study helps reader who would encounter kernels in fractional Laplace transform and fractional generalized Hankel-Clifford transformation during their experiments.

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