DOI 10.1515/tmj-2015-0001

Pitts monads and a lax descent theorem

Marta Bunge

Department of Mathematics and Statistics, McGill University, Burnside Hall, 805 Sherbrooke Street West, Montreal, QC, Canada H3A 2K6

E-mail: marta.bunge@mcgill.ca

Dedicated to Marco Grandis on his 70th Birthday

Abstract

A theorem of A.M.Pitts (1986) states that essential surjections of toposes bounded over a base topos $\mathscr{S}$ are of effective lax descent. The symmetric monad $\mathscr{M}$ on the 2-category of toposes bounded over $\mathscr{S}$ is a KZ-monad (Bunge-Carboni 1995) and the $\mathscr{M}$-maps are precisely the $\mathscr{S}$-essential geometric morphisms (Bunge-Funk 2006). These last two results led me to conjecture[1] and then prove[2] the general lax descent theorem that is the subject matter of this paper. By a 'Pitts KZ-monad' on a 2-category $\mathscr{K}$ it is meant here a locally fully faithful equivariant KZ-monad $\mathscr{M}$ on $\mathscr{K}$ that is required to satisfy an analogue of Pitts' theorem on bicomma squares along essential geometric morphisms. The main result of this paper states that, for a Pitts KZ-monad $\mathscr{M}$ on a 2-category $\mathscr{K}$ ('of spaces'), every surjective $\mathscr{M}$-map is of effective lax descent. There is a dual version of this theorem for a Pitts co-KZ-monad $\mathscr{N}$. These theorems have (known and new) consequences regarding (lax) descent for morphisms of toposes and locales.

2010 Mathematics Subject Classification. **03G**. 18C, 18D, 18F, 55R.
Keywords. lax descent, Kock-Zoeberlein monads, symmetric monad, coherent toposes, powerlocales, Pitts' theorem.

1 Introduction

Let $\mathscr{K}$ be a 2-category. The notion of a KZ (short for 'Kock-Zöberlein')-monad [10] on $\mathscr{K}$ is a special sort of pseudomonad $\mathscr{M}$ on $\mathscr{K}$ that is 'property-like' in the sense that, for its algebras, structure is (a reflective) left adjoint to the unit so that, in particular, the so called structure may instead be regarded as a property. There is a dual notion, to wit, that of a co-KZ-monad $\mathscr{N}$ on $\mathscr{K}$. For its algebras, structure is (a reflective) right adjoint to the unit.

The subject matter of this paper is a general lax descent theorem involving a notion of 'Pitts KZ-monad' $\mathscr{M}$, of which the symmetric monad $\mathscr{M}$ [4] is an instance, as well as a dual lax descent theorem involving a 'Pitts co-KZ-monad' $\mathscr{N}$.

The motivation for our main result came from two sources. First, it had been shown by A.M.Pitts [17] that, for any bicomma square

$$\begin{tikzcd}
\mathscr{Y} \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\geqslant"] & \mathscr{X} \arrow[d, "g"] \\
\mathscr{F} \arrow[r, "f"'] & \mathscr{E}
\end{tikzcd}$$

[1]Conference in Memory of Aurelio Carboni. Universitá di Milano, Milan, Italy, 24-26 June 2013.

[2]AMS Meeting, Session on New Trends in Category Theory, Dalhousie University, Halifax, N.S., Canada, 18-19 October 2014.

Tbilisi Mathematical Journal 8(1) (2015), pp. 1–29.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 28 May 2014.
Accepted for publication: 18 November 2014.

in the 2-category $\mathrm{Top}_{\mathscr{S}}$ of toposes bounded over a base topos $\mathscr{S}$ with a natural numbers object [7], if $f : \mathscr{F} \to \mathscr{E}$ is $\mathscr{S}$-essential then $p : \mathscr{Y} \to \mathscr{X}$ is locally connected (or $\mathscr{X}$-essential [2]), and the canonical 2-cell

$$p_! \cdot q^* \Rightarrow g^* \cdot f_!$$

is an isomorphism. Using it, it is proved in [17] that $\mathscr{S}$-essential surjections of bounded $\mathscr{S}$-toposes are of effective lax descent. Second, a 1-cell $f : F \to E$ in a 2-category $\mathscr{K}$ with a KZ-monad $\mathscr{M}$ defined on it is said to be an $\mathscr{M}$-map [5] if the 1-cell $\mathscr{M}f : \mathscr{M}F \to \mathscr{M}E$ has a right adjoint. It is shown therein that, for $\mathscr{M}$ the symmetric monad on $\mathrm{Top}_{\mathscr{S}}$, the $\mathscr{M}$-maps coincide with the $\mathscr{S}$-essential geometric morphisms.

By a *Pitts KZ-monad* on a 2-category $\mathscr{K}$ we shall understand here an equivariant (in the sense of [5]) KZ-monad $\mathscr{M}$ on $\mathscr{K}$ that satisfies an analogue of Pitts' theorem for bicomma squares along $\mathscr{M}$-maps. Any Pitts KZ-monad $\mathscr{M}$ on $\mathscr{K}$ is in particular a locally faithful (stably) cocompletion monad (in the sense of [5]). It follows from this that the $\mathscr{M}$-algebras are precisely the (stably) $\mathscr{M}$-cocomplete objects, a notion that requires the existence and coherence of certain left Kan extensions.

By a 2-category *of abstract spaces* we shall understand here any 2-category $\mathscr{K}$ with a pseudoterminal object, equipped with a representable contravariant 2-functor on $\mathscr{K}$ with values on Cat. The representing object O is said to be its *objects classifier*. Examples of the latter are the 2-categories $\mathrm{Top}_{\mathscr{S}}$ of toposes bounded over $\mathscr{S}$ and $\mathrm{Loc}(\mathscr{S})$ of locales in $\mathscr{S}$ [9]. In the first case one takes O to be the topos $\mathscr{S}^{\mathscr{S}_{\mathrm{fin}}}$ and, in the second, the Sierpinski locale $\mathbb{S}$.

The main result of this paper is the following *general lax descent theorem*. If $\mathscr{M}$ is any Pitts KZ-monad on a 2-category $\mathscr{K}$ of abstract spaces for which its objects classifier is an $\mathscr{M}$-algebra, then every surjective $\mathscr{M}$-map $f : F \to E$ in $\mathscr{K}$ is of effective lax descent. The proof we give consists of two parts. The first is a lax version of the theorem of Bénabou and Roubaud [3]. The second is a direct application of Beck's Tripleability Theorem [7]. The proof that we give of the lax descent theorem consists in putting together these two parts. There is a dual version of this theorem for a Pitts co-KZ-monad $\mathscr{N}$ on a 2-category $\mathscr{K}$ of spaces whose objects classifier is an $\mathscr{N}$-algebra.

Consequences of the lax descent theorem (or of its dual) for toposes over $\mathscr{S}$, include, in addition to the already mentioned result, due to A. M. Pitts [17], and which states that $\mathscr{S}$-essential surjections are of effective lax descent, also a proof of a conjecture of A. M. Pitts (*op.cit.*), proved independently by M. Zawadowski [21], D. Ballard and W. Boshuck [1], and (twice) by I. Moerdijk and J. J. C. Vermeulen [15, 14], and which states that coherent surjections of coherent toposes are of effective lax descent. The Pitts monads intervening in these two cases are, respectively, the symmetric monad [4] and the coherent monad, the latter introduced here specifically for this purpose.

In the context of locales in $\mathscr{S}$, we obtain two seemingly new lax descent results, namely, that semiopen surjections and perfect surjections are of effective lax descent. In these cases, the Pitts monads involved are respectively the lower and the upper powerlocale monads.

As consequences of the effective lax descent theorems obtained for morphisms of toposes and of locales, we derive the (corresponding) effective descent theorems for locally connected surjections of toposes ([8]), open surjections of locales (A. Joyal and M.Tierney [9]), and proper surjections of locales (J.C.C. Vermeulen [18]). Whereas these last results are known, what is novel is the manner in which they are here obtained, which can be summed up as "unification via monads".

2 Pitts Monads on 2-Categories

A *KZ-doctrine* [10] $\mathscr{M}$ on a 2-category $\mathscr{K}$ is a pseudomonad $\langle \mathscr{M}, \delta, \mu \rangle$ that satisfies the conditions

$$\mathscr{M}\delta_B \dashv \mu_B \dashv \delta_{\mathscr{M}B} \tag{1}$$

for each object B in $\mathscr{B}$.

In what follows, we shall employ the more common terminology 'KZ-monad' in lieu of 'KZ-doctrine'. In addition, and for concreteness, we shall state all definitions and results for the case of a KZ-monad. The duals for co-KZ-monads will be stated without proof and used when they are needed.

For $\mathscr{M}$ a KZ-monad on $\mathscr{K}$, a 1-cell $f : C \to D$ of $\mathscr{K}$ is said to be an *$\mathscr{M}$-map* [5] if $\mathscr{M}f : \mathscr{M}C \to \mathscr{M}D$ has a right adjoint, in which case the adjoint pair will be denoted by $\mathscr{M}f \dashv \rho_f$.

The left Kan extension $\Sigma_f(\varphi)$ of φ along an $\mathscr{M}$-map f (if it exists) is said to be *pointwise* if, for any bicomma square

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "g"] \arrow[dd, bend left, "\Sigma_p(\varphi q)"] \\ C \arrow[r, "f"] \arrow[dr, "\varphi"'] & D \arrow[d, "\Sigma_f(\varphi)"] \arrow[l, phantom, shift right=4, "\Rightarrow"] \\ & Z \end{tikzcd}$$

with f as its bottom 1-cell, the left Kan extension $\Sigma_p(\varphi q)$ exists, and the canonical 2-cell

$$\Sigma_p(\varphi q) \Rightarrow \Sigma_f(\varphi)g$$

is an isomorphism.

Definition 2.1. *Let $\mathscr{M}$ be a pseudomonad on a 2-category $\mathscr{K}$.*

- *An object Z of $\mathscr{K}$ is said to be* $\mathscr{M}$-cocomplete *[5] if Z admits pointwise left Kan extensions of any map $\varphi : C \to Z$ along any $\mathscr{M}$-map $f : C \to D$.*
- *An object Z of $\mathscr{K}$ is said to be* stably $\mathscr{M}$-cocomplete *if it is $\mathscr{M}$-cocomplete and satisfies the following additional condition. For any diagram*

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "g"] \arrow[dd, bend left, "\Sigma_p(\varphi q)"] \\ C \arrow[r, "f"] \arrow[dr, "\varphi"'] & D \arrow[d, "\Sigma_f(\varphi)"] \arrow[l, phantom, shift right=4, "\Rightarrow"] \\ & Z \end{tikzcd}$$

where f is an $\mathscr{M}$-map and the square is a bicomma square, and any diagram

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\simeq"] & Y \arrow[d, "h"] \arrow[dd, bend left, "\Sigma_s(\psi t)"] \\
A \arrow[r, "p"] \arrow[dr, "\psi"'] & B \arrow[d, "\Sigma_p(\psi)"] \arrow[dl, phantom, "\Rightarrow"] \\
 & Z
\end{tikzcd}$$

where the square is a bipullback and

$$\psi = \varphi q,$$

the left Kan extensions indicated below exist, and the canonical 2-cell

$$\Sigma_s(\psi t) \to \Sigma_p(\psi)h$$

is an isomorphism.

An *algebra* for a KZ-monad $\mathscr{M} = \langle \mathscr{M}, \delta, \mu \rangle$ on $\mathscr{K}$ is given by an object A of $\mathscr{K}$ for which the unit $\delta_A : A \to \mathscr{M}A$ has a reflective left adjoint, i.e., $\theta \dashv \delta_A$ such that $\theta \cdot \delta_A = \mathrm{id}_A$. A morphism $(A, \theta) \to (A', \theta')$ of $\mathscr{M}$-algebras (or an $\mathscr{M}$-*homomorphism*) is any 1-cell $c : A \to A'$ such that the square

$$\begin{tikzcd}
\mathscr{M}A \arrow[r, "\mathscr{M}c"] \arrow[d, "\theta"'] & \mathscr{M}A' \arrow[d, "\theta'"] \\
A \arrow[r, "\varphi"'] & A'
\end{tikzcd}$$

commutes.

From now on we shall assume that $\mathscr{K}$ is a 2-category with bipullbacks and a pseudoterminal object T. Let

$$K_! \dashv K^*$$

be the biadjoint pair where K^* is taking the bipullback along $K \to T$ and $K_!$ is composition with $K \to T$.

Recall from [5] that a KZ-monad $\mathscr{M}$ in $\mathscr{K}$ is said to be *equivariant* if it is given by the following data and conditions:

1. For each object K, a KZ-monad
$$(\mathscr{M}^K, \delta^K, \mu^K)$$
in $\mathscr{K}/K$. (The case $K = T$ gives a KZ-monad $(\mathscr{M}, \delta, \mu)$ in $\mathscr{K}$.)

2. For any object X,
$$\mathscr{M}^K(K \times X) \simeq K \times \mathscr{M}(K)$$
in the precise form that is part of following assumption. For each object K, there is given a pseudo-natural transformation
$$\sigma^K : K_! \cdot \mathscr{M}^K \to \mathscr{M} \cdot K_!$$

such that, for any object X of $\mathscr{K}$, the diagram

$$\begin{tikzcd}
K \times X \arrow[r, "\delta^K_{K^*X}"] \arrow[d, "\pi_2"'] & \mathscr{M}^K(K \times X) \arrow[r, "\pi_1"] \arrow[d, "\pi_X"] & K \arrow[d] \\
X \arrow[r, "\delta_X"'] & \mathscr{M}(X) \arrow[r] & T
\end{tikzcd}$$

where the middle vertical 1-cell π_X denotes the composite 1-cell

$$\mathscr{M}^K(K \times X) \overset{\sigma^K{}_{K^*X}}{\longrightarrow} \mathscr{M}(K \times X) \overset{\mathscr{M}(\pi_2)}{\longrightarrow} \mathscr{M}(X)$$

is such that the right hand square is a pullback and the left hand one commutes. The left hand square is therefore a pullback.

In addition to commuting with the units δ^K and δ as above, the σ's commute with the multiplications μ^K and μ: for any object $X \longrightarrow K$ of $\mathscr{K}/K$, the diagram

$$\begin{tikzcd}
\mathscr{M}^{K^2}(K \times X) \arrow[r, "\mathscr{M}(\sigma^K{}_{K^*X}) \cdot \sigma^K{}_{\mathscr{M}^K(K^*X)}"] \arrow[d, "(\mu^K)_X"'] & \mathscr{M}^2(X) \arrow[d, "\mu_X"] \\
\mathscr{M}^K(X) \arrow[r, "\sigma^K{}_X"'] & \mathscr{M}(X)
\end{tikzcd}$$

commutes in $\mathscr{K}$.

3. If a 1-cell $A \overset{q}{\longrightarrow} Y$ over K admits an $\mathscr{M}^K$-adjoint $\mathscr{M}^K(q) \dashv (\rho^K)_q$, then it admits an $\mathscr{M}$-adjoint $\mathscr{M}q \dashv \rho_q$ in $\mathscr{K}$, and the canonical 2-cell

$$(\sigma^K)_A \cdot (\rho^K)_q \Rightarrow \rho_q \cdot (\sigma^K)_Y$$

is an isomorphism.

Definition 2.2. *An equivariant KZ-monad $\mathscr{M}$ on a 2-category $\mathscr{K}$ is said to be* locally full and faithful *[5] if the KZ-monad $\mathscr{M}$ is locally full and faithful, that is, if for any object X of $\mathscr{K}$, the unit $\delta_X : X \to \mathscr{M}(X)$ is full and faithful. Notice that from assumption (2) of equivariance, it follows that, for any object K of $\mathscr{K}$, the unit $(\delta^K)_{(K^*X)} : (K \times X) \to \mathscr{M}^K(K \times X)$ is fully faithful.*

Definition 2.3. *An equivariant locally full and faithful KZ-monad $\mathscr{M}$ on a 2-category $\mathscr{K}$ with bipullbacks and a pseudoterminal object T will be said to be a* Pitts monad *if, for any object K and an $\mathscr{M}^K$-map $f : C \to D$ in $\mathscr{K}/K$, any bicomma square*

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"'] \arrow[ur, phantom, "\Rightarrow"] & D
\end{tikzcd}$$

exists, for such a bicomma square the 1-cell p is an $\mathscr{M}^B$-map and, regarding both p and f as $\mathscr{M}^K$-maps, the canonical 2-cell

$$\mathscr{M}^K(q) \cdot (\rho^K)_p \Rightarrow (\rho^K)_f \cdot \mathscr{M}^K(g)$$

is an isomorphism.

Proposition 2.4. Let $\mathscr{K}$ be a 2-category with bipullbacks and a pseudoterminal object and let $\mathscr{M}$ a Pitts KZ-monad on $\mathscr{K}$. Then, for each object K of $\mathscr{K}$, $\mathscr{M}^K$ is a locally full and faithful stably cocompletion monad on $\mathscr{K}/K$ in the sense that, for any $\mathscr{M}^K$-map $f : C \to D$ in $\mathscr{K}/K$, any diagram of the form

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\cong"] & Y \arrow[d, "h"] \\ A \arrow[r, "p"'] \arrow[d, "q"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

with the bottom square a bicomma and the top square a bipullback, exists and furthermore, in this diagram, both $p : A \to B$ and $s : X \to Y$ are $\mathscr{M}^K$-maps and the canonical 2-cells

$$\mathscr{M}^K(q) \cdot (\rho^K)_p \Rightarrow (\rho^K)_f \cdot \mathscr{M}^K(g) \tag{2}$$

and

$$\mathscr{M}^K(t) \cdot (\rho^K)_s \Rightarrow (\rho^K)_p \cdot \mathscr{M}^K(h) \tag{3}$$

are isomorphisms.

Proof. The assertion about the bicomma square is precisely the condition of Definition 2.3. For the assertion about the bipullback, notice that $p : A \to B$ is an $\mathscr{M}^B$-map in $\mathscr{K}/B$ and so, since B is pseudoterminal in $\mathscr{K}/B$, the bipullback on top of p is a bicomma square. This part too then follows from Definition 2.3. □

Proposition 2.5. Let $\mathscr{M}$ be a Pitts KZ-monad on a 2-category $\mathscr{K}$. Then, for any object Z of $\mathscr{K}/K$, the following are equivalent:

1. Z is an $\mathscr{M}^K$-algebra.

2. Z is an $\mathscr{M}^K$-cocomplete object.

3. Z is a stably $\mathscr{M}^K$-cocomplete object.

Proof. Since $\mathscr{M}^K$ is a locally full and faithful cocompletion KZ-monad, it follows fromTheorem 4.3.14 [5] that (1) and (2) are equivalent. Also, (3) implies (2). We need only show that (2) implies

(3). Using that the composite of a bicomma square with a bipullback on top of it is a bicomma square, as is the case of the diagram

$$
\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\cong"] & Y \arrow[d, "h"] \\
A \arrow[r, "p"'] \arrow[d, "q"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "g"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}
$$

we derive first that, for a 1-cell $\varphi : C \to Z$, all the left Kan extensions involved in Definition 2.1 exist and tthe canonical 2-cells

$$\alpha : \Sigma_p(\varphi q) \Rightarrow \Sigma_f(\varphi)g$$

and

$$\beta : \Sigma_s(\varphi qt) \Rightarrow \Sigma_f(\varphi)gh$$

are isos. From these in turn we obtain that the canonical 2-cell

$$\gamma : \Sigma_s(\varphi qt) \Rightarrow \Sigma_p(\varphi q)h$$

is an iso since, by uniqueness, we have

$$\gamma = (\alpha^{-1})h \cdot \beta$$

□

The notion of a *co-KZ-monad* $\langle \mathscr{N}, \eta, \nu \rangle$ on $\mathscr{K}$ has the adjunctions (1) reversed, that is, it is required that

$$\eta_{\mathscr{N}(B)} \dashv \nu_B \dashv \mathscr{N}(\eta_B) \tag{4}$$

We state, without proof, the analogues for co-KZ-monads of the results obtained so far for the case of KZ-monads, omitting some obvious definitions. For $\mathscr{N}$ a co-KZ-monad on $\mathscr{K}$, a 1-cell $f : C \longrightarrow D$ of $\mathscr{K}$ is said to be an *$\mathscr{N}$-map* if $\mathscr{N}(f)$ has a left adjoint, in which case it will be denoted by λ_f.

Definition 2.6. *A* Pitts co-KZ-monad *on a 2-category $\mathscr{K}$ is a locally fully faithful equivariant co-KZ-monad $\mathscr{N}$ on $\mathscr{K}$, for which the dual conditions to those of Definition 2.3 hold.*

The right Kan extension $\Pi_f(\varphi)$ of $\varphi : C \to Z$ along an $\mathscr{N}$-map f (if it exists) is said to be *pointwise* if for any bicomma square as above with f the right vertical 1-cell, the canonical 2-cell

$$\Pi_f(\varphi)q \Rightarrow \Pi_p(\varphi g)$$

is an iso.

An object Z is said to be *$\mathscr{N}$-complete* [5] if Z admits pointwise right Kan extensions of any map $\varphi : C \to Z$ along any $\mathscr{N}$-map $f : C \to D$. We leave it to the reader to state the obvious definition of a *stably $\mathscr{N}$-complete object* and of the dual of Proposition 2.5 for a Pitts co-KZ-monad. Notice, when writing down the definitions, that the bipullback condition is self-dual but that the bicomma condition is not.

3 The Lax Descent Theorem for an $\mathscr{M}$-map

We begin with the definition of the lax kernel of a an arbitrary 1-cell in a 2-category $\mathscr{K}$ with the necessary bicomma squares.

Definition 3.1. *Let $f : F \to E$ be a 1-cell in $\mathscr{K}$. The* lax kernel *of f is given by a diagram*

$$\begin{tikzcd} F_2 \arrow[r, bend left, "p_{01}"] \arrow[r, "p_{02}"] \arrow[r, bend right, "p_{12}"'] & F_1 \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] & F \arrow[r, two heads, "f"] & E \end{tikzcd}$$

where

$$\begin{tikzcd} F_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\Downarrow\lambda"] & F \arrow[d, "f"] \\ F \arrow[r, "f"'] & E \end{tikzcd}$$

and

$$\begin{tikzcd} F_2 \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] \arrow[dr, phantom, "\Downarrow\varphi_1"] & F_1 \arrow[d, "p_0"] \\ F_1 \arrow[r, "p_1"'] & F \end{tikzcd}$$

are both bicomma squares.

Denote by d the associated diagonal

$$d : F \to F_1.$$

There are given natural isomorphisms

$$\delta_0 : 1_F \cong p_0 d,$$

and

$$\delta_1 : 1_F \cong p_1 d,$$

such that

$$\lambda d \cdot f\delta_0 = f\delta_1,$$

and, for the projection

$$p_{02} : F_2 \to F_1,$$

natural isomorphisms

$$\varphi_0 : p_0 p_{02} \cong p_0 p_{01},$$

and

$$\varphi_2 : p_1 p_{02} \cong p_1 p_{12},$$

such that

$$\lambda p_{12} \cdot \lambda p_{01} :=: \lambda p_{02},$$

where the symbol $:=:$ *indicates equality after inserting the appropriate isomorphisms φ_i, $i = 0, 1, 2$.*

Proposition 3.2. Let $\mathscr{K}$ be a 2-category with bipullbacks and a pseudoterminal object. Let $\mathscr{M}$ be a Pitts KZ-monad on $\mathscr{K}$ and let $f : F \to E$ an $\mathscr{M}$-map in $\mathscr{K}$. Then, the lax kernel of f reduces to the diagram

$$\begin{tikzcd} F_2 \arrow[r, bend left, "p_{01}"] \arrow[r, "p_{02}" description] \arrow[r, bend right, "p_{12}"'] & F_1 \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] & F \arrow[r, two heads, "f"] & E \end{tikzcd}$$

where

$$\begin{tikzcd} F_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & F \arrow[d, "f"] \\ F \arrow[r, "f"'] & E \end{tikzcd}$$

is a bicomma square, and

$$\begin{tikzcd} F_2 \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] \arrow[dr, phantom, "\overset{\varphi_1}{\cong}"] & F_1 \arrow[d, "p_0"] \\ F_1 \arrow[r, "p_1"'] & F \end{tikzcd}$$

is a bipullback with all other data and conditions unchanged from Definition 3.1.

Proof. Since $\mathscr{M}$ is a Pitts KZ-monad on $\mathscr{K}$, and so $p_1 : F_1 \to F$ is an $\mathscr{M}^F$-map, the bicomma square

$$\begin{tikzcd} F_2 \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] \arrow[dr, phantom, "\overset{\varphi_1}{\Rightarrow}"] & F_1 \arrow[d, "p_0"] \\ F_1 \arrow[r, "p_1"'] & F \end{tikzcd}$$

of Definition 3.1 is a bipullback since F is a pseudoterminal object in $\mathscr{K}/F$.

□

Remark 3.3. *Proposition 3.2 may serve to explain the choice of the notion of lax kernel of a geometric morphism in [17]. The lax descent theorem proved therein is about $\mathscr{S}$-essential surjections $f : \mathscr{F} \to \mathscr{E}$ in* $\mathrm{Top}_{\mathscr{S}}$*. This, in turn, relies on the following theorem (referred to here as 'Pitts' theorem'): the top morphism $p : \mathscr{Y} \to \mathscr{X}$ in a bicomma square*

$$\begin{tikzcd} \mathscr{Y} \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathscr{X} \arrow[d] \\ \mathscr{F} \arrow[r, "f"'] & \mathscr{E} \end{tikzcd}$$

in $\mathrm{Top}_{\mathscr{S}}$ *(exists and) is locally connected, and so, by a theorem of Myles Tierney (unpublished), it is stable under bipullbacks. Pitts' theorem on the nature of any bicomma along an essential geometric morphism is obtained as consequence of a general construction of bicomma squares in* $\mathrm{Top}_{\mathscr{S}}$ *via lex sites, performed in detail therein.*

Definition 3.4. *A 2-category* $\mathscr{K}$ *with a pseudoterminal object will be called a 2-category* of abstract spaces *if it is equipped with a representable contravariant 2-functor*

$$\mathrm{Hom}_{\mathscr{K}}(-,O) : \mathscr{K}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$$

such that, for each 0-cell E *of* $\mathscr{K}$ *(a 'space') the category* $\mid E \mid = \mathrm{Hom}_{\mathscr{K}}(E,O)$ *(the 'underlying category' of* E*) has finite limits and stable finite colimits and is such that, for each 1-cell* $f : F \to E$ *in* $\mathscr{K}$ *(a 'continuous map'), the functor*

$$\mid E \mid \xrightarrow{f^{\#}} \mid F \mid,$$

notationally identified with

$$\mathrm{Hom}_{\mathscr{K}}(f,O) : \mathrm{Hom}_{\mathscr{K}}(E,O) \longrightarrow \mathrm{Hom}_{\mathscr{K}}(F,O),$$

preserves finite colimits. The object O *of* $\mathscr{K}$ *is said to be an 'objects classifier'.*

Remark 3.5. *The two main examples of 2-categories of abstract spaces that will be considered in the applications are :*

- $\mathscr{K} = \mathrm{Top}_{\mathscr{S}}$*, the 2-category of toposes bounded over a base topos* $\mathscr{S}$ *with a natural numbers object [7]. The pseudoterminal object is* $\mathscr{S}$*. The objects classifier* O *is identified with the topos* $\mathscr{S}^{\mathscr{S}_{\mathrm{fin}}}$*.*
- $\mathscr{K} = \mathrm{Loc}(\mathscr{S})$*, the 2-category of locales (called 'spaces' in [9]) in* $\mathscr{S}$*. The pseudoterminal object is the terminal locale* $\mathbf{1}$*.* $\mathscr{K}$ *becomes a 2-category of abstract spaces with* O *the Sierpinski locale* $\mathbb{S}$*, and with the functor* $\mathrm{Hom}_{\mathscr{K}}(-,O) : \mathscr{K}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ *having values in* Posets.

Definition 3.6. *Let* $\mathscr{K}$ *be a 2-category of abstract spaces with* O *its objects classifier. Let* $f : F \to E$ *be a 1-cell of* $\mathscr{K}$*.*

1. *The category* $\mathrm{LDes}(f)$ *of* lax descent data *has*

   - *objects that are are pairs* (Y,a) *where* $F \xrightarrow{Y} O$ *is a 1-cell in* $\mathscr{K}$*, and*

     $$a : Yp_0 \Rightarrow Yp_1$$

     is a 2-cell between the pair (Yp_0, Yp_1) *of 1-cells* $F_1 \longrightarrow O$ *in* $\mathscr{K}$*, satisfying ("the unit condition")*

     $$ad :=: 1_Y$$

     where the symbol $:=:$ *means that isomorphisms* δ_i *(for* $i = 0,1$*) should be inserted as appropriate to render it into an equality, and ("the cocycle condition")*

     $$(ap_{12}) \cdot (ap_{01}) :=: (ap_{02})$$

     where isomorphisms φ_i *(for* $i = 0,1,2$*) should be inserted as appropriate to render it into an equality.*

- *and arrows $b : (Y, a) \to (Y', a')$ given by 2-cells $b : Y \to Y'$ in $\mathscr{K}$ such that the square*

$$\begin{tikzcd} Yp_0 \arrow[r, Rightarrow, "a"] \arrow[d, Rightarrow, "bp_0"'] & Yp_1 \arrow[d, Rightarrow, "bp_1"] \\ Y'p_0 \arrow[r, Rightarrow, "a'"'] & Y'p_1 \end{tikzcd}$$

commutes.

2. *A* comparison functor
$$\kappa_f : \mathrm{Hom}_{\mathscr{K}}(E, O) \longrightarrow \mathrm{LDes}(f)$$
is defined as follows. For a 1-cell $X : E \to O$ of $\mathscr{K}$,
$$\kappa_f(X) = (Xf, X\lambda : Xfp_0 \Rightarrow Xfp_1),$$
and with an obvious definition of κ_f on a morphism $X \xrightarrow{c} X'$ of $\mathrm{Hom}_{\mathscr{K}}(E, O)$*, that is, on a 2-cell $c : X \Rightarrow X'$ where $X, X' : E \longrightarrow O$.*

3. *A morphism $f : F \longrightarrow E$ is said to be of* effective lax descent *if*
$$\kappa_f : \mathrm{Hom}_{\mathscr{K}}(E, O) \longrightarrow \mathrm{LDes}(f)$$
is an equivalence of categories.

Theorem 3.7. Let $\mathscr{K}$ be a 2-category of abstract spaces. Let $\mathscr{M}$ be a Pitts KZ-monad on $\mathscr{K}$ such the objects classifier O is an $\mathscr{M}$-algebra. Let $f : F \to E$ an $\mathscr{M}$-map in $\mathscr{K}$ and let $\mathbb{T}_f$ be the monad on $\mathrm{Hom}_{\mathscr{K}}(F, O)$ induced by the adjoint pair

$$\Sigma_f \dashv f^{\#} : \mathrm{Hom}_{\mathscr{K}}(E, O) \longrightarrow \mathrm{Hom}_{\mathscr{K}}(F, O),$$

where $f^{\#}$ denotes 'composition with f'.

Under these assumptions, there exists an equivalence of categories

$$\mathrm{LDes}(\mathbb{C}_f) \xrightarrow{\Psi_f} \mathrm{Alg}^{\mathbb{T}_f}$$

where $\mathrm{Alg}^{\mathbb{T}_f}$ is the category of $\mathbb{T}_f$-algebras, and the triangle

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{K}}(E, O) \arrow[rr, "\kappa_f"] \arrow[dr, "F^{\mathbb{T}_f}"'] & & \mathrm{LDes}(f) \arrow[dl, "\Psi_f"] \\ & \mathrm{Alg}^{\mathbb{T}_f} & \end{tikzcd}$$

commutes.

Proof. Let
$$\Psi_f : \mathrm{LDes}(f) \longrightarrow \mathrm{Alg}^{\mathbb{T}_f}$$
assign, to an object (Y, a) of $\mathrm{LDes}(f)$, the object (Y, θ) of $\mathrm{Alg}^{\mathbb{T}_f}$ defined as follows. Recall that $Y : F \to O$ is a 1-cell in $\mathscr{K}$, together with a 2-cell $a : Yp_0 \Rightarrow Yp_1$, satisfying the unit and cocycle conditions. Since O is an $\mathscr{M}$-algebra and $\mathscr{M}$ is a Pitts KZ-monad, O is stably $\mathscr{M}$-cocomplete by Theorem 2.5. In particular, the left Kan extension of $Y : F \to O$ along the $\mathscr{M}$-map f exists and is pointwise so that, from the diagram

$$\begin{tikzcd}
F_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\Rightarrow"] & F \arrow[d, "f"] \arrow[dd, bend left, "\Sigma_{p_1}(Yp_0)"] \\
F \arrow[r, "f"] \arrow[dr, "Y"'] \arrow[r, phantom, shift right=4, "\Rightarrow"] & E \arrow[d, "\Sigma_f(Y)"] \\
& O
\end{tikzcd}$$

where the top 2-cell is the given λ, and the bottom 2-cell is the unit η_Y of the adjointness $\Sigma_f \dashv f^{\#}$, we derive that the canonical 2-cell
$$\alpha : \Sigma_{p_1}(Yp_0) \Rightarrow \Sigma_f(Y)f$$
is an isomorphism.

Consider now the diagram

$$\begin{tikzcd}
F_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\Rightarrow a"] & F \arrow[d, "Y"] \\
F \arrow[r, "Y"'] & O
\end{tikzcd}$$

which is part of the lax descent data (Y, a).

From the bicomma square

$$\begin{tikzcd}
F_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\Rightarrow \lambda"] & F \arrow[d, "f"] \\
F \arrow[r, "f"'] & E
\end{tikzcd}$$

and the universal property of $\Sigma_{p_1}(Yp_0)$ follows the existence of a unique 2-cell
$$\bar{\theta} : \Sigma_{p_1}(Yp_0) \Rightarrow Y$$
such that
$$(\bar{\theta}p_1) \cdot (\alpha^{-1}p_1) \cdot \Sigma_f(Y)\lambda \cdot (\eta_Y)p_0 = a.$$

Define
$$\theta = \bar{\theta} \cdot \alpha^{-1} : \Sigma_f(Y)f \Rightarrow Y.$$

It follows that

$$\theta p_1 \cdot \Sigma_f(Y)\lambda \cdot (\eta_Y)p_0 = a$$

We claim that from the *unit condition*

$$ad :=: \mathrm{id}_Y,$$

explicitly expressed via the commutative square

$$\begin{tikzcd}
Y \arrow[r, Rightarrow, "\mathrm{id}_Y"] \arrow[d, Rightarrow, "Y\delta_0"'] & Y \arrow[d, Rightarrow, "Y\delta_1"] \\
Yp_0d \arrow[r, Rightarrow, "ad"'] & Yp_1d
\end{tikzcd}$$

follows that

$$\theta \cdot \eta_Y = \mathrm{id}_Y,$$

which is the *first condition* for the pair (Y, θ) to be an algebra for the monad induced by the adjoint pair $\Sigma_f \dashv f^{\#}$.

Indeed, consider the following commutative diagram

$$\begin{tikzcd}
Y \arrow[dr, Rightarrow, "Y\delta_0"] \arrow[drr, Rightarrow, "(\Sigma_f(Y)f\delta_0)\cdot\eta_Y"] \arrow[ddr, Rightarrow, "Y\delta_1"'] & & \\
& Yp_0d \arrow[r, Rightarrow, "(\eta_Y p_0 d)"'] \arrow[d, Rightarrow, "ad"] & \Sigma_f(Y)fp_0d \arrow[d, Rightarrow, "\Theta"] \\
& Yp_1d \arrow[r, Rightarrow, "\mathrm{id}_{(Yp_1d)}"'] & Yp_1d
\end{tikzcd}$$

where Θ is defined by the commutative triangle

$$\begin{tikzcd}
\Sigma_f(Y)fp_0d \arrow[rr, Rightarrow, "\Sigma_f(Y)\lambda d"] \arrow[dr, Rightarrow, "\Theta"'] & & \Sigma_f(Y)fp_1d \arrow[dl, Rightarrow, "\theta p_1 d"] \\
& Yp_1d &
\end{tikzcd}.$$

Using, in addition, the data and condition

$$\delta_0 : 1_F \cong p_0 d,$$

$$\delta_1 : 1_F \cong p_1 d,$$

$$\lambda d \cdot f\delta_0 = f\delta_1,$$

we derive the identities

$$\mathrm{id}_Y = Y\delta_1{}^{-1} \cdot Y\delta_1$$

$$= Y\delta_1{}^{-1} \cdot \Theta \cdot \Sigma_f(Y)f\delta_0 \cdot \eta_Y$$
$$= Y\delta_1{}^{-1} \cdot (\theta p_1 d) \cdot \Sigma_f(Y)\lambda d \cdot \Sigma_f(Y)f\delta_0 \cdot \eta_Y$$
$$= \theta \cdot \Sigma_f(Y)f\delta_1{}^{-1} \cdot \Sigma_f(Y)\lambda d \cdot \Sigma_f(Y)f\delta_0 \cdot \eta_Y$$
$$= \theta \cdot \eta_Y.$$

Consider now the *cocycle condition* for (Y, a), namely

$$(ap_{12}) \cdot (ap_{01}) :=: (ap_{02}),$$

explicitly given by the identity

$$(Y\varphi_2)(ap_{12}) \cdot (Y\varphi_1)(ap_{01}) = (ap_{02})(Y\varphi_0).$$

We now claim that from it, the given isos, and the assumption that O is an $\mathscr{M}$-algebra (hence stably $\mathscr{M}$-cocomplete), one can derive the second condition for the pair (Y, θ) to be an algebra for the monad induced by the adjoint pair $\Sigma_{f)} \dashv f^{\#}$, namely that the square

$$\begin{tikzcd}
\Sigma_f(\Sigma_f(Y)f))f \arrow[r, Rightarrow, "\Sigma_f(\theta)f"] \arrow[d, Rightarrow, "\mu_Y"'] & \Sigma_f(Y)f \arrow[d, Rightarrow, "\theta"] \\
\Sigma_f(Y)f \arrow[r, Rightarrow, "\theta"'] & Y
\end{tikzcd}$$

commutes.

This is done in two steps. First, we claim that taking transposes of both sides of the cocycle condition gives a commutative square

$$\begin{tikzcd}
\Sigma_{p_1}(\Sigma_{p_1}(Yp_0)p_0) \arrow[r, Rightarrow, "\Sigma_{p_1}(\bar{\theta})p_0"] \arrow[d, Rightarrow, "(\star)"'] & \Sigma_{p_1}(Yp_0) \arrow[d, Rightarrow, "\bar{\theta}"] \\
\Sigma_{p_1}(Yp_0) \arrow[r, Rightarrow, "\bar{\theta}"'] & Y
\end{tikzcd}$$

where

$$(\star) :=: \Sigma_{p_1}(\bar{\varepsilon}_{Yp_0})$$

is an identity when inserting the iso 2-cells

$$\beta : \Sigma_{p_1}(\Sigma_{p_1}(Yp_0)p_0) \cong \Sigma_{p_1}(\Sigma_{p_{12}}(Yp_0p_{01}))$$

and

$$\varphi_0 : \Sigma_{p_1}(\Sigma_{p_{12}}(Yp_0p_{01})) \cong \Sigma_{p_1}(\Sigma_{p_{02}}(Yp_0p_{01})).$$

Let us first establish the following *subclaim*. Denoting by

$$\hat{a} : \Sigma_{p_1}(Yp_0) \gtrdot Y$$

the transpose of

$$a : Yp_0 \rightarrowtail Yp_1$$

under the adjointness $\Sigma_{p_1} \dashv (-)p_1$, the identity

$$\hat{a} = \bar{\theta}$$

holds. To prove it we need to recall that $\bar{\theta}$ was defined as the unique 2-cell for which

$$\bar{\theta}p_1 \cdot \alpha^{-1}p_1 \cdot \Sigma_f(Y)\lambda \cdot (\eta_Y)p_0 = a.$$

We then check that (also)

$$\hat{a}p_1 \cdot \alpha^{-1}p_1 \cdot \Sigma_f(Y)\lambda \cdot (\eta_Y)p_0 = a$$

holds. There is a string of easily justified identities:

$$\begin{aligned} &\bar{\varepsilon}_{(Yp_1)} \cdot \Sigma_{p_1}(a)p_1 \cdot \alpha^{-1}p_0 \cdot \Sigma_f(Y)\lambda \cdot (\eta_Y)p_0 \\ &= a \cdot \bar{\varepsilon}_{(Yp_0)} \cdot \alpha^{-1}p_0 \cdot \Sigma_f(Y)\lambda \cdot (\eta_Y)p_0 \\ &= a \cdot \bar{\varepsilon}_{(Yp_0)} \cdot \alpha p_0 \cdot (\eta_Y)p_0 \\ &= a \cdot \bar{\varepsilon}_{(Yp_0)} \cdot (\bar{\eta}_Y)p_0 = a. \end{aligned}$$

This ends the proof of the *subclaim*.

We show next that

$$\widehat{ap_{02}} :=: \hat{a} \cdot \Sigma_{p_1}(\bar{\varepsilon}_{(Yp_0)})$$

modulo φ_0. We have each side of the claimed equation given in terms of the following strings of identities:

1.
$$\widehat{ap_{02}} = \bar{\varepsilon}_Y \cdot \Sigma_{p_1}\bar{\varepsilon}_{(Yp_1)} \cdot \Sigma_{p_1}(\Sigma_{p_{02}}(ap_{02}))$$
and

2.
$$\hat{a} \cdot \Sigma_{p_1}(\bar{\varepsilon}_{(Yp_0)}) = \bar{\varepsilon}_Y \cdot \Sigma_{p_1}(a) \cdot \Sigma_{p_1}(\bar{\varepsilon}_{(Yp_0)})$$

from which it follows that it is enough to check that the identity

$$\bar{\varepsilon}_{(Yp_1)} \cdot \Sigma_{p_{02}}(ap_{02}) = a \cdot \bar{\varepsilon}_{(Yp_0)}$$

holds. This is so by naturality of the counit $\bar{\varepsilon}$ of the adjointness

$$\Sigma_{p_{02}} \dashv (-)p_{02}.$$

Next, we show that

$$a\widehat{p_{12} \cdot a}p_{01} :=: \hat{a} \cdot \Sigma_{p_1}(\hat{a})p_0,$$

is an identity modulo the isomorphism

$$\Sigma_{p_1}(\beta) : \Sigma_{p_1}(\Sigma_{p_{12}}(Yp_0p_{01})) \cong \Sigma_{p_1}\Sigma_{p_1}(Yp_0)p_0$$

as well as φ_1 and φ_2.

To prove it, apply Σ_{p_1} to the string of identities:

$$\bar{\varepsilon}_{(Yp_1)} \cdot \Sigma_{p_{12}}(ap_{12}) \cdot \Sigma_{p_{12}}(Y\varphi_1) \cdot \Sigma_{p_{12}}(ap_{01}) =$$

$$a \cdot \hat{a}\bar{\varepsilon}_{(Yp_0)} \cdot \Sigma_{p_{12}}(Y\varphi_1) \cdot \Sigma_{p_{12}}(ap_{01})) =$$

$$a \cdot \hat{a}p_0 \cdot \beta.$$

where we also use that

$$\hat{a} = \bar{\varepsilon}_Y \cdot \Sigma_{p_1}(a).$$

To conclude the argument we resort to the following commutative cube:

$$\begin{tikzcd}
\Sigma_{p_1}(\Sigma_{p_1}(Yp_0)p_0) \arrow[rr, Rightarrow, "\Sigma_{p_1}(\bar{\theta})p_0"] \arrow[dr, Rightarrow, "\alpha\cdot\alpha"'] \arrow[dd, Rightarrow, "(\star)"'] & & \Sigma_{p_1}(Yp_0) \arrow[dr, Rightarrow, "\alpha"] \arrow[dd, Rightarrow, "\bar{\theta}" near end] & \\
& \Sigma_f(\Sigma_f(Y)f))f \arrow[rr, Rightarrow, "\Sigma_f(\theta)f" near start] \arrow[dd, Rightarrow, "(\varepsilon_{\Sigma_f(Y)})f"' near end] & & \Sigma_f(Y)f \arrow[dd, Rightarrow, "\theta"] \\
\Sigma_{p_1}(Yp_0) \arrow[rr, Rightarrow, "\bar{\theta}"' near start] \arrow[dr, Rightarrow, "\alpha"'] & & Y \arrow[dr, Rightarrow, "\mathrm{id}_Y"] & \\
. & \Sigma_f(Y)f \arrow[rr, Rightarrow, "\theta"'] & & Y
\end{tikzcd}$$

Since the back and front faces are connected by iso 2-cells, to have the front face commutative it is enough to have the back face commutative. The latter has already been shown. This ends the proof of the cocycle *versus* second algebra condition.

That the assignment

$$\Psi_f : (Y, a) \mapsto (Y, \theta)$$

is functorial, an equivalence of categories, and compatible with the comparison maps, are all easily established facts and left to the reader. □

Definition 3.8. *Let $\mathscr{K}$ be any 2-category. A 1-cell $f : F \to E$ in $\mathscr{K}$ is said to be* surjective *(or* a surjection*) if given any 2-cell $\alpha : g \Rightarrow h : E \to E'$ in $\mathscr{K}$, such that the 2-cell $\alpha f : gf \Rightarrow hf$, obtained by composing α with f, is an iso 2-cell, then α is an iso 2-cell.*

Theorem 3.9. (Beck's Theorem) Let $\mathscr{K}$ be a 2-category of abstract spaces and let $\mathscr{M}$ be a Pitts KZ-monad on $\mathscr{K}$. Assume that the objects classifier O in $\mathscr{K}$ is an $\mathscr{M}$-algebra. Then, for any surjective $\mathscr{M}$-map $f : F \longrightarrow E$, the adjoint pair

$$\Sigma_f \dashv f^{\#}$$

is monadic.

Proof. Since O is an $\mathscr{M}$-algebra, it is $\mathscr{M}$-cocomplete by Proposition 2.5. In particular, since f is an $\mathscr{M}$-map, the left Kan extension Σ_f exists. Since, by assumption,

$$\mid E \mid \xrightarrow{f^{\#}} \mid F \mid$$

preserves finite (limits and finite) colimits, and reflects isomorphisms, the result now follows from an application of Beck's 'Weak Tripleability Theorem' [13] (Exercise VI 7.2). $\square$

Lemma 3.10. Let $\mathscr{M}$ be a Pitts monad on a 2-category $\mathscr{K}$ of abstract spaces with O as its objects classifier. Then, for every K in $\mathscr{K}$,

1. $\mathscr{M}^K$ is a stably cocompletion monad on $\mathscr{K}/K$.
2. $\mathscr{K}/K$ is a 2-category of spaces with $(K \times O) \xrightarrow{\pi_1} K$ as its objects classifier.
3. If (Z, θ) is an $\mathscr{M}$-algebra in $\mathscr{K}$, then $(K \times Z, K \times \theta)$ is an $\mathscr{M}^K$ algebra in $\mathscr{K}/K$.

Proof.

1. The first assertion is the contents of Proposition 2.4.
2. The contravariant functor

   $$\mathrm{Hom}_{\mathscr{K}/K}(-, K \times O) : \mathscr{K}/K^{\mathrm{op}} \longrightarrow \mathrm{Cat}$$

   is such that, for each 0-cell E of $\mathscr{K}/K$ the category

   $$\mid E \mid = \mathrm{Hom}_{\mathscr{K}/K}(E, K \times O)$$

   has finite limits and stable finite colimits and is such that, for each 1-cell $f : F \to E$ in $\mathscr{K}/K$, the functor

   $$\mid E \mid \xrightarrow{f^{\#}} \mid F \mid,$$

   notationally identified with

   $$\mathrm{Hom}_{\mathscr{K}/K}(f, K \times O) : \mathrm{Hom}_{\mathscr{K}/K}(E, K \times O) \longrightarrow \mathrm{Hom}_{\mathscr{K}/K}(F, K \times O),$$

   preserves the finite colimits that exist since they are stable.

3. Since O is an $\mathscr{M}$-algebra, there is a left adjoint $\theta \dashv \delta_O$ such that $\theta \cdot \delta_O = \mathrm{id}_O$. We claim that $K \times O$ is an $\mathscr{M}^K$-algebra in $\mathscr{K}/K$. This follows from condition (2) in the notion of an equivariant KZ-monad $\mathscr{M}$ on $\mathscr{K}$ applied to O, letting

$$\theta^K = (K \times O) \cdot ({\sigma^K}_{K^*O}).$$

First, the adjointness $\theta \dashv \delta_O$ implies the adjointness $(K \times \theta) \dashv (K \times \delta_O)$. Second, in the commutative diagram

$$\begin{tikzcd} K \times O \arrow[r, "\delta^K_{K^*O}"] \arrow[d, "\pi_2"'] & \mathscr{M}^K(K \times O) \arrow[r, "\theta^K"] \arrow[d, "\pi_O"] & K \times O \arrow[d, "\pi_2"] \\ O \arrow[r, "\delta_O"'] & \mathscr{M}(O) \arrow[r, "\theta"'] & O \end{tikzcd}$$

where the middle vertical 1-cell π_O is the composite

$$\mathscr{M}^K(K \times O) \overset{{\sigma^K}_{K^*O}}{\longrightarrow} \mathscr{M}(K \times O) \overset{\mathscr{M}(\pi_2)}{\longrightarrow} \mathscr{M}(O)\ ,$$

the left square is commutative (actually a bipullback) by equivariance, and the right square is commutative. Therefore, since the bottom composite is the identity on O and since the diagram where the vertical arrows are the projections $\pi_!$ is also commutative (as it simply expresses that the top composite is a 1-cell in $\mathscr{K}/K$), the top composite is the identity on $K \times O$. This concludes the verification.

□

Theorem 3.11. Let $\mathscr{K}$ be a 2-category of abstract spaces. Let $\mathscr{M}$ be a Pitts KZ-monad on $\mathscr{K}$. Assume that the objects classifier O is an $\mathscr{M}$-algebra. Then, the following hold:

1. For any object K of $\mathscr{K}$, any surjective $\mathscr{M}^K$-map $f : F \twoheadrightarrow E$ in $\mathscr{K}/K$ is of effective lax descent.

2. For any object K of $\mathscr{K}$, any surjective $\mathscr{M}^K$-map $f : F \twoheadrightarrow K$ in $\mathscr{K}/K$ is of effective descent.

Proof.

1. Theorem 3.7 and Theorem 3.9 hold for any stably cocompletion monad such as $\mathscr{M}^K$ for any object K. Furthermore, the assumptions made in those theorems are stable under localization by Lemma 3.10.

2. For an $\mathscr{M}^K$-map $f : F \twoheadrightarrow K$, the lax kernel of f reduces (by Proposition 3.2) to a diagram

$$\begin{tikzcd} F_2 \arrow[r, bend left, "p_{01}"] \arrow[r, "p_{02}"] \arrow[r, bend right, "p_{12}"'] & F_1 \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] & F \arrow[r, two heads, "f"] & E \end{tikzcd}$$

where

$$\begin{tikzcd} F_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & F \arrow[d, "f"] \\ F \arrow[r, "f"'] & E \end{tikzcd}$$

is a bipullback and

$$\begin{tikzcd} F_2 \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] \arrow[dr, phantom, "\overset{\varphi_1}{\Rightarrow}"] & F_1 \arrow[d, "p_0"] \\ F_1 \arrow[r, "p_1"'] & F \end{tikzcd}$$

is a bicomma. Furthermore, since K is pseudoterminal in $\mathscr{K}/K$, also the second bicomma is a bipullback. Hence lax descent reduces to descent in this case.

□

Remark 3.12. *For $\mathscr{K}$ a 2-category of abstract spaces we may consider a Pitts co-KZ-monad $\mathscr{N}$ on $\mathscr{K}$. Theorem 3.7 admits a dual version with virtually no changes except that the characterization of the $\mathscr{N}$-algebras (as stable $\mathscr{N}$-complete objects) now involves pointwise right Kan extensions Π_f, Π_{p_1} and $\Pi_{p_{12}}$. As for Theorem 3.9, it is its dual version for coalgebras of the comonad induced by the adjoint pair $f^{\#} \dashv \Pi_f$ that is needed. Using both we derive the following Theorem 3.13.*

Theorem 3.13. Let $\mathscr{K}$ be a 2-category of abstract spaces. Let $\mathscr{N}$ be a Pitts co-KZ-monad on $\mathscr{K}$. Assume that the objects classifier O is an $\mathscr{N}$-algebra. Then, the following hold:

1. For any object K of $\mathscr{K}$, any surjective $\mathscr{N}^K$-map $f : F \twoheadrightarrow E$ in $\mathscr{K}/K$ is of effective lax descent.

2. For any object K of $\mathscr{K}$, any surjective $\mathscr{N}^K$-map $f : F \twoheadrightarrow K$ is of effective descent.

4 The Symmetric Monad

The first example is that of the *symmetric monad* $\mathscr{M}$ on the 2-category $\mathrm{Top}_{\mathscr{S}}$ of bounded $\mathscr{S}$-toposes, geometric morphisms, and 2-cells [7]. We refer the reader to [4, 5] for the construction and basic properties of $\mathscr{M}$.

Theorem 4.1. The symmetric monad $\mathscr{M}$ on $\mathrm{Top}_{\mathscr{S}}$ is a Pitts KZ-monad.

Proof. Let $\mathscr{K} = \mathrm{Top}_{\mathscr{S}}$. The symmetric monad $\mathscr{M}$ on $\mathscr{K}$ is a locally fully faithful equivariant KZ-monad [5]. Moreover, a geometric morphism f is an $\mathscr{M}$-map if and only if it is $\mathscr{S}$-essential [5] That in a bicomma square with bottom map an essential geometric morphism the opposite map is locally connected and that the BCC holds, is a theorem due to A. M. Pitts [17]. These imply that $\mathscr{M}$ is a Pitts KZ-monad. □

Theorem 4.2. [17] If $f : \mathscr{F} \twoheadrightarrow \mathscr{E}$ is an $\mathscr{S}$-essential surjection in $\mathrm{Top}_{\mathscr{S}}$, then f is of effective lax descent.

Proof. By construction, for $\mathscr{M}$ the symmetric monad on $\mathscr{K} = \mathrm{Top}_{\mathscr{S}}$, $M\mathscr{S} = \mathscr{S}^{\mathscr{S}_{\mathrm{fin}}}$ is the objects classifier. It is an $\mathscr{M}$-algebra –in fact, the free $\mathscr{M}$-algebra on the pseudoterminal object $\mathscr{S}$. The conclusion now follows from Theorem 3.11(1). □

Theorem 4.3. [7] If $f : \mathscr{F} \twoheadrightarrow \mathscr{E}$ is a locally connected morphism in $\mathrm{Top}_{\mathscr{S}}$, then f is of effective descent.

Proof. The symmetric monad $\mathscr{M}$ on $\mathrm{Top}_{\mathscr{S}}$ is a Pitts KZ-monad. The result now follows from Theorem 3.11(2) applied to the $\mathscr{M}^{\mathscr{E}}$-map $\mathscr{F} \twoheadrightarrow \mathscr{E}$ in $\mathrm{Top}_{\mathscr{E}}$. □

5 The Lower Powerlocale Monad

Related to the symmetric monad on toposes is the *lower powerlocale monad* $\mathscr{L}$ on the 2-category $\mathrm{Loc}(\mathscr{S})$. Whereas the symmetric topos $\mathscr{M}\mathscr{E}$ classifies distributions ($\mathscr{S}$-cocontinuous functors) on an $\mathscr{S}$-bounded topos $\mathscr{E}$, the lower powerlocale $\mathscr{L}X$ classifies distributions (sup-preserving maps) on a locale X.

Remark 5.1. *The power locales are best known in the context of theoretical computer science [19, 6] and are usually defined in terms of generators and relations. One of the novelties introduced in [4] is the characterization of (frames of) locales as the algebras for a KZ-monad on the posetal 2-category* **sl** *of suplattices, a characterization which, unlike the one used in [9] for proving that open surjections of locales are of effective descent, does not make use of the tensor product. We recall it in what follows.*

It is well-known [9] that the free frame on a poset Z is given by the set $D(Z)$ of downward closed subsets of Z, with union the supremum, and down-segment

$$Z \overset{\downarrow}{\twoheadrightarrow} D(Z)$$

the universal map. Let X be a locale in $\mathscr{S}$. Denote its corresponding frame as $\mathscr{O}(X)$. Any frame $\mathscr{O}(X)$, regarded as a suplattice, is canonically presented as a coinverter

$$\begin{tikzcd} D(Q) \arrow[r, shift left=1.5ex, "d_0"] \arrow[r, shift right=1.5ex, "d_1"'] \arrow[r, phantom, "\Downarrow"] & D(\mathscr{O}(X)) \arrow[r, two heads] & \mathscr{O}(X) \end{tikzcd}$$

in **sl**, where Q is the poset whose elements are pairs (R, u) such that $R \subseteq \downarrow u$, $u \in \mathscr{O}(X)$, and $\bigvee R = u$. The maps d_0 and d_1 are induced by the assignments $(R, u) \mapsto R$ and $(R, u) \mapsto (\downarrow u)$, respectively, where $\geqslant$ is the unique 2-cell from d_0 to d_1, i.e., $d_0 \leq d_1$.

One now defines the locale $\mathscr{L}X$ via its symmetric frame $\Sigma(\mathscr{O}(X))$, that is, so that

$$\mathscr{O}(\mathscr{L}X) = \Sigma(\mathscr{O}(X)).$$

The finite inf-completion $Q^\bullet$ of a poset Q can be given as the collection of equivalence classes $[S]$, where S is a (Kuratowski) finite subset of Q, and where $[S] = [S']$ if and only if S and S'

generate the same upper set in Q. As a poset, $Q^\bullet$ has the partial order given by $[S] \leq [T]$ if and only if T is contained in the upper set generated by S.

Consider the coinverter diagram

$$\begin{tikzcd} D(Q^\bullet) \arrow[r, shift left=2, "(d_0)^\bullet"] \arrow[r, shift right=2, "(d_1)^\bullet"'] \arrow[r, phantom, "\Downarrow"] & D(\mathscr{O}(X)^\bullet) \arrow[r, two heads] & \Sigma(\mathscr{O}(X)) \end{tikzcd}$$

in **Fr** where the parallel arrows ${d_0}^\bullet$, ${d_1}^\bullet$ are induced from the canonical suplattice presentation of $\mathscr{O}(X)$ via *finite inf-completions* at the level of the posets.

That the Σ construction is part of a co-KZ-monad $\mathscr{L}$ on the posetal 2-category **sl** of suplattices is shown in the same way as in the proof that the symmetric topos (also called Σ) construction is a co-KZ-monad on the 2-category **A** of locally presentable categories [4]. In the former, it was used that the finite limits completion is a co-KZ-monad at the level of categories, whereas in the latter, this refers to the finite inf-completion on posets. By turning around $\Sigma \dashv U$ and taking opposites, one gets a KZ-monad $\mathscr{L}$ on $\mathrm{Loc}(\mathscr{S})$. In order to apply the lax descent theorem, we first need to identify the $\mathscr{L}$-maps, that is, those morphisms $f : X \longrightarrow Y$ of locales such that $\mathscr{L}(f) : \mathscr{L}(X) \longrightarrow \mathscr{L}(Y)$ has a right adjoint.

Definition 5.2. *A morphism $f : X \longrightarrow Y$ of locales is said to be* semiopen *if $f^- : \mathscr{O}(Y) \longrightarrow \mathscr{O}(X)$ has a left adjoint $\exists_f : \mathscr{O}(X) \longrightarrow \mathscr{O}(Y)$. It is said to be* open *[9] if it is semiopen and Frobenius Reciprocity holds, in the sense that for all $y \in \mathscr{O}(Y)$, $x \in \mathscr{O}(X)$,*

$$\exists_f(f^-(y) \wedge x) = y \wedge \exists_f(x).$$

It is well known that open maps of locales are pullback stable and that the BCC holds [9]. For arbitrary semiopen maps we have the following result.

Theorem 5.3. Let

$$\begin{tikzcd} B \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\leq"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a bicomma square in $\mathrm{Loc}(\mathscr{S})$, where f is semiopen. Then p is open and the BCC holds, that is, the inequality

$$\exists_q \cdot p^- \leq g^- \cdot \exists_f$$

is an equality.

Proof. The proof is entirely similar to that of [17] for the case of $\mathscr{S}$-essential geometric morphisms, but in the posetal case. □

Theorem 5.4. $\mathscr{L}$ is a Pitts KZ-monad on $\mathrm{Loc}(\mathscr{S})$.

Proof. $\mathscr{L}$ is a locally full and faithful equivariant KZ-monad [5]. That in a bicomma square with bottom map semiopen the opposite map is open and the BCC holds, is the content of Theorem 5.3. It says that $\mathscr{L}$ is a Pitts KZ-monad. □

Proposition 5.5. A morphism of locales is an $\mathscr{L}$-map if and only if it is semiopen.

Proof. Let $f : X \to Y$ be a morphism of locales in $\mathscr{S}$. Assume that f is semiopen, that is, that $f^- : \mathscr{O}(Y) \to \mathscr{O}(X)$ has a left adjoint $\exists_f : \mathscr{O}(X) \to \mathscr{O}(Y)$. In terms of distributions, $\mathscr{L}f$ corresponds to composition with f^-, so that the right adjoint ρ_f corresponds to composition with $\exists_f$. Conversely, assume that $f : X \to Y$ admits a right $\mathscr{L}$-adjoint ρ_f, that is,

$$(\rho_f)^- \dashv (\mathscr{L}f)^-.$$

Then,

$$(\rho_f)^- \cdot \exists_{(\delta_X)} \dashv (\delta_X)^- \cdot (\mathscr{L}f)^-.$$

By naturality,

$$\begin{array}{ccc} \mathscr{L}Y & \xrightarrow{(\mathscr{L}f)^-} & \mathscr{L}X \\ {\scriptstyle (\delta_Y)^-}\downarrow & & \downarrow{\scriptstyle (\delta_X)^-} \\ Y & \xrightarrow[f^-]{} & X \end{array}$$

commutes. Therefore $f^- \cdot (\delta_Y)^-$ has a left adjoint. Since δ_Y is an inclusion, it follows that f^- has a left adjoint. Thus, f is semiopen. □

It is well known that the algebras for the lower powerlocale monad are characterized as the injectives with respect to semiopen embeddings [6]. The following proposition is a related fact and a consequence of Theorem 5.4.

Proposition 5.6. Let X be a locale. Then the following are equivalent:

1. X is an $\mathscr{L}$-algebra.
2. X is injective with respect to semiopen embeddings.
3. X is a stably $\mathscr{L}$-cocomplete object.

The following theorem seems to be new.

Theorem 5.7. Semiopen surjections of locales are of effective lax descent.

Proof. Let $\mathscr{K} = \mathrm{Loc}(\mathscr{S})$. Let $\mathscr{L}$ be the lower powerlocale monad on $\mathscr{K}$. By Proposition 5.5, a morphism $f : Y \to X$ of locales in $\mathscr{S}$ has a (right) $\mathscr{L}$-adjoint iff it is semiopen. The pseudoterminal object in $\mathscr{K}$ is the terminal locale $\underline{1}$. By construction, $\mathscr{L}\underline{1} = \mathbb{S}$, the Sierpinski locale, which classifies open sublocales of locales, the latter identifiable with objects of a locale frame $\mathscr{O}(X)$. It is a $\mathscr{L}$-algebra, in fact a free one. The conclusion now follows from Theorem 3.11(1). □

Theorem 5.8. [9] Any open surjection $f : Y \to X$ of locales is of effective descent.

Proof. The lower power locale monad $\mathscr{L}$ on $\mathrm{Loc}\mathscr{S}$ is a Pitts KZ-monad. The result now follows from Theorem 3.11(2) applied to the $\mathscr{L}^X$-map $f : Y \to X$ in $\mathrm{Loc}(\mathscr{S})/X \cong \mathrm{Loc}(\mathrm{Sh}_{\mathscr{S}}(X))$. □

Remark 5.9. *The KZ-monads considered in this and in the previous section are classifiers of Lawvere distributions – the symmetric monad $\mathscr{M}$ for distributions on toposes, and the lower powerlocale monad $\mathscr{L}$ for distributions on locales. The theory developed in [5] explains why in these cases the object classifiers are given respectively by the free algebras $\mathscr{M}(\mathscr{S})$ and $\mathscr{L}(1)$. Indeed, the objects of a space X (meaning here either a topos or a locale) correspond to the 'discrete opfibrations' over X, and such are classified by the free algebra on the terminal object. However, for the purposes of the general lax descent theorem, all that is needed is that the objects classifier be an algebra for the Pitts KZ-monad/co-KZ-monad in question. This remark is an important consideration in the next two sections.*

6 The Upper Powerlocale Monad

The posetal 2-category Frm of frames of locales is monadic over the posetal 2-category PrFrm of preframes, by means of an adjoint pair $F \dashv G$ where

$$G : \mathrm{Frm} \longrightarrow \mathrm{PrFrm}$$

is the forgetful, with F described in [6]. This adjoint pair induces a comonad on Frm, hence a monad $\mathscr{U}$ on $\mathrm{Loc}(\mathscr{S})$, the upper powerlocale monad.

Power locales have been investigated extensively by Steve Vickers [19, 20] and Martín Escardó [6], among others that they mention in their work. This section relies largely on their results, as well as on those of Jaapie Vermeulen [18] and of M. Korostenski and C.C.A. Labuschagne [11], as indicated in what follows.

Theorem 6.1. The upper powerlocale monad $\mathscr{U}$ is an equivariant co-KZ-monad on $\mathrm{Loc}(\mathscr{S})$.

Proof. That the *upper power locale monad* $\mathscr{U}$ is a co-KZ-monad $\mathscr{U}$ is shown in [19] and [6]. The equivariance is covered by geometricity [20] (Section 7.2). ◻

Recall that a continuous map $f : X \to Y$ of locales is called *perfect* if the right adjoint $f_\star$ of the frame homomorphism $f^{-1} : \mathscr{O}(Y) \to \mathscr{O}(X)$ preserves directed joins.

Theorem 6.2. [6] A continuous map of locales is perfect if and only if it is a $\mathscr{U}$-map.

Remark 6.3. *Notice that the perfect maps are the continuous maps satisfying half of the definition of a proper map of locales [18] where the missing half is Frobenius Reciprocity. Indeed, a map $f : X \to Y$ of locales is* proper *if it is perfect and the condition*

$$f_\star(f^- u \vee v) = u \vee f_\star(v)$$

holds for all $u \in \mathscr{O}(X)$, $v \in \mathscr{O}(Y)$.

Theorem 6.4. [11] Consider a bicomma square

$$\begin{tikzcd} B \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\leq"] & Z \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

in $\mathrm{Loc}(\mathscr{S})$, where f is perfect. Then h is proper and the canonical inequality

$$\mathscr{U}k \cdot l_h \leq l_f \cdot \mathscr{U}g$$

is an equality.

Theorem 6.5. [18] In a bipullback square

$$\begin{tikzcd} B \arrow[r, "k"] \arrow[d, "h"'] & Z \arrow[d, "f"] \arrow[dl, phantom, "\overset{\tau}{\cong}"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

if f is proper then h is proper and the canonical inequality

$$\mathscr{U}k \cdot l_h \leq l_f \cdot \mathscr{U}g$$

is an equality.

Theorem 6.6. The upper powerlocale monad $\mathscr{U}$ on $\mathrm{Loc}(\mathscr{S})$ is a Pitts co-KZ-monad.

Proof. In addition to Theorems 6.1, 6.2, 6.4 and 6.5, we need to know that $\mathscr{U}$ is locally fully faithful. This is indeed the case by construction [6]. □

It is shown in [6] that the $\mathscr{U}$-algebras are the injectives over perfect embeddings. As a consequence of Theorem 6.6, we can now add to this characterization as follows.

Proposition 6.7. Let X be a locale. Then the following are equivalent:

1. X is a $\mathscr{U}$-algebra.
2. X is injective with respect to perfect embeddings.
3. X is a stably $\mathscr{U}$-complete object.

Remark 6.8. *An example of a locally fully faithful co-KZ-monad that is not a completion monad (hence not a Pitts monad) is due to M. H. Escardó and included as Remark 4.2.15 in [5]. Let $\mathscr{F}$ be the filter monad on the category of topological spaces (and analogously on that of locales). The $\mathscr{F}$-algebras are the injective spaces (every continuous map is an $\mathscr{F}$-map) but the injective spaces do not coincide with the $\mathscr{F}$-complete objects. More precisely, the Sierpinski space (or locale, depending where this monad is considered) is injective, therefore an $\mathscr{F}$-algebra, but it is not $\mathscr{F}$-complete.*

The foliowing theorem seems to be new.

Theorem 6.9. Perfect surjections of locales are of effective lax descent.

Proof. The Sierpinski locale $\mathbb{S}$ is an object classifier for $\mathrm{Loc}(\mathscr{S})$ [9]. In order to apply Theorem 3.13(1), we need to show that $\mathbb{S}$ is a $\mathscr{U}$-algebra. That this is the case follows from the fact that $\mathbb{S}$ is an algebra of the filter monad $\mathscr{F}$ on $\mathrm{Loc}(\mathscr{S})$ (using that all maps are $\mathscr{F}$-maps), and the fact that $\mathscr{U}$ is a submonad of $\mathscr{F}$, so that all $\mathscr{F}$-algebras are $\mathscr{U}$-algebras. □

Theorem 6.10. [18] Proper surjections of locales are of effective descent.

Proof. The upper powerlocale monad $\mathscr{U}$ on $\mathrm{Loc}(\mathscr{S})$ is a Pitts co-KZ-monad. The result now follows from Theorem 3.11(2) applied to the $\mathscr{U}^X$-map $f : Y \twoheadrightarrow X$ in $\mathrm{Loc}(\mathscr{S})/X \cong \mathrm{Loc}(\mathrm{Sh}_{\mathscr{S}}(X))$. □

7 The Coherent Monad

A lax descent theorem, originally conjectured by A.M. Pitts [17] and proved independently by M. Zawadowski [21], D. Ballard and W. Boshuck [1], and (twice) by I. Moerdijk and J.J.C. Vermeulen [15, 14], states that coherent surjections of coherent toposes are of lax effective descent. This theorem will be shown to be a corollary of Theorem 3.13.

Let us first recall some notions (see for instance [7]). A bounded $\mathscr{S}$-topos is *coherent* if it is equivalent to a category of sheaves on a finitary site, that is, on a site whose underlying category has finite limits, and where the covering families are finite. For any coherent topos $\mathscr{E}$, there is a *canonical* such site, to wit, the full subcategory $\mathscr{E}_{\mathrm{coh}}$ of coherent objects with the topology of finite coverings. The latter category is a *pretopos*. Any pretopos is the category of coherent objects in a coherent topos. Coherent toposes are the classifiers of finitary geometric (or coherent) logic. A geometric morphism $f : \mathscr{F} \to \mathscr{E}$ of coherent toposes is said to be a *coherent morphism* if f^* restricts to the full subcategories of coherent objects. For such morphisms, f_* preserves (or commutes with) filtered colimits.

Denote by $\mathbf{B}$ the full sub 2-category of $\mathbf{A}$ (locally presentable categories) whose objects are those categories $\mathscr{B}$ which are small generated and small presented by categories *with finite limits*. In other words, $\mathbf{B}$ denotes the 2-category of *locally finitely presentable categories*.

Equivalently, the objects of $\mathbf{B}$ are those categories $\mathscr{B}$ that appear as *coinverters* in $\mathbf{A}$ of the form

$$\begin{tikzcd} P(\mathbb{D}) \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[r, phantom, "\Downarrow\sigma"] & P\mathbb{C} \arrow[r, two heads] & \mathscr{B} \end{tikzcd}$$

where $\mathbb{C}$ and $\mathbb{D}$ are categories in $\mathscr{S}$ *with finite limits* and where

$$a : P\mathbb{C} \longrightarrow \mathscr{B}$$

as well as d_0, d_1 are $\mathscr{S}$-cocontinuous and *finite limits preserving*.

Denote by CohTop the 2-category whose objects are the coherent toposes, whose morphisms are the coherent morphisms, and whose 2-cells are those of $\mathrm{Top}_{\mathscr{S}}$. Let CohFrm be the opposite (for 1-cells) of CohTop. There is a forgetful 2-functor

$$\mathscr{U} : CohFrm \longrightarrow \mathbf{B}$$

which assigns to the frame of a coherent topos $\mathscr{E}$ its canonical presentation as a category of sheaves on a (finitary) site.

Let Lex denote the 2-category of small categories with finite limits, finite limit preserving functors, and all natural transformations. The *pretopos (co)completion* of a small category $\mathbb{C}$ with finite limits is a small pretopos $\mathbb{C}_\bullet$ together with a finite limits preserving functor

$$u : \mathbb{C} \longrightarrow \mathbb{C}_\bullet$$

with the property that every finite limits preserving functor from $\mathbb{C}$ to a pretopos $\mathbb{D}$ extends uniquely (up to isomorphism) to a pretopos morphism from $\mathbb{C}_\bullet$ to $\mathbb{D}$. One way to obtain it is to define, for $\mathbb{C}$ a small category with finite limits, the (small) category $\mathbb{C}_\bullet$ as the full subcategory of coherent objects of the (coherent) topos $P\mathbb{C}$.

The pretopos (co)completion of a small category with finite limits extends to a left 2-adjoint

$$(-)_{\bullet} : \mathrm{Lex} \longrightarrow \mathrm{Pretop}$$

to the forgetful 2-functor

$$U : \mathrm{Pretop} \longrightarrow \mathrm{Lex}.$$

The 2-adjoint pair $(-)_{\bullet} \dashv U$ induces a KZ-monad on Lex.

Theorem 7.1. The left 2-adjoint

$$\Theta : \mathbf{B} \longrightarrow \mathrm{CohFrm}$$

to the forgetful

$$\mathscr{U} : \mathrm{CohFrm} \longrightarrow \mathbf{B}$$

exists and induces a KZ-monad on $\mathbf{B}$.

Proof. Consider the following double coinverter diagram in $\mathbf{B}$

$$\begin{tikzcd}
P(\mathbb{D}) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[r, phantom, "\sigma\vee"] \arrow[d, "j_!"'] & P(\mathbb{C}) \arrow[r, two heads, "a"] \arrow[d, "i_!"] & \mathscr{B} \arrow[d, "\eta_{\mathscr{B}}"] \\
P(\mathbb{D}_{\bullet}) \arrow[r, shift left, "(d_1)_{\bullet}"] \arrow[r, shift right, "(d_1)_{\bullet}"'] \arrow[r, phantom, "(\sigma)_{\bullet}\vee"] & P(\mathbb{C}_{\bullet}) \arrow[r, two heads, "b"] & \Theta\mathscr{B} .
\end{tikzcd}$$

where $\mathscr{B}$ is an object of $\mathbf{B}$ with its given presentation as a coinverter in a diagram where all functors depicted are $\mathscr{S}$-cocontinuous and finite limit preserving, and where the parallel arrows $d_{0\bullet}$, $d_{1\bullet}$ and the 2-cell $\sigma_{\bullet}$ are induced via pretopos completions and left Kan extensions along the inclusions $i : \mathbb{C} \longrightarrow (\mathbb{C})_{\bullet}$ and $j : \mathbb{D} \longrightarrow (\mathbb{D})_{\bullet}$. Since i and j are finite limits preserving, so are the corresponding left Kan extensions $i_!$ and $j_!$. They are also $\mathscr{S}$-cocontinuous.

By construction, $\sigma_{\bullet}$ is the unique 2-cell for which

$$\sigma_{\bullet} \cdot j_! = i_! \cdot \sigma$$

so that

$$b \cdot i_! \cdot \sigma = b \cdot \sigma_{\bullet} \cdot j_!.$$

Therefore $b \cdot i_! \cdot \sigma$ is invertible since $b \cdot \sigma_{\bullet}$ is invertible. From the universal property of the top coinverter, it follows now that there exists a unique arrow $\eta_{\mathscr{B}}$ such that

$$\eta_{\mathscr{B}} \cdot a = b \cdot i_!.$$

In the diagram above, the $(-)_{\bullet}$ construction (which is part of a KZ-monad on Lex is lifted inside the presentation of $\mathscr{B}$, hence giving rise (as argued more generally in [4]) to a KZ-monad Θ on $\mathbf{B}$.

We claim that $\Theta\mathscr{B}$ is a *coherent (topos) frame* and so that b is the inverse image part of a coherent morphism between coherent toposes. In [4] (§3) we recalled (quoting [16] (Theorem 2.3)) that finite 2-colimits in Frm (exist and) are finite 2-colimits in $\mathbf{A}$ (locally presentable categories) so that, in particular, coinverters in Frm (exist and) are coinverters in **A**. This explicit argument given in [16] (Theorem 2.3) can easily be adapted to deduce that coinverters in CohFrm (exist and) are coinverters in **B** (locally finitely presentable categories). .

For any frame of a coherent topos $\mathscr{E}$, and any object $\mathscr{B}$ of **B**, there are equivalences of categories

$$\mathrm{CohFrm}(\Theta\mathscr{B},\mathscr{E}) \cong \mathrm{CohFrm}_{\sigma_\bullet}(P(\mathbb{C}_\bullet),\mathscr{E}) \cong \mathbf{B}_\sigma(P\mathbb{C},U\mathscr{E}) \cong \mathbf{B}(\mathscr{B},U\mathscr{E})$$

where the subindexes $\sigma_\bullet$ and σ are to be interpreted as restrictions of the functors in the Hom categories to those that take the 2-cell in question into an iso. These equivalences show that $\eta_{\mathscr{B}} : \mathscr{B} \to \mathscr{U}\Theta\mathscr{B}$ has the universal property that translates into the adjointness $\Theta \dashv \mathscr{U}$. In particular, the construction is independent of the presentation. □

As in [5], we now turn to the "geometric point of view", which amounts to regarding

$$\mathscr{U}^{\mathrm{op}} \dashv \Theta^{\mathrm{op}}$$

as inducing a co-KZ-monad $\mathscr{C}$ on CohTop. Let us call this monad $\mathscr{C}$ the *coherent monad*.

Proposition 7.2. 1. For a coherent topos $\mathscr{E}$, the topos $\mathscr{C}\mathscr{E}$ classifies pretopos morphisms $\mathscr{E}_{\mathrm{coh}} \to \mathscr{S}$.

2. Any coherent morphism between coherent toposes is a $\mathscr{C}$-map.

Proof.

1. For a coherent topos $\mathscr{E}$, $\mathscr{C}\mathscr{E}$ is the coherent topos of sheaves on the site consisting of the small category $\mathscr{E}_{\mathrm{coh}}$ with finite epimorphic coverings. The conclusion follows from an identification of this site in terms of any given finitary site of presentation for $\mathscr{E}$.

2. For the second statement, suppose that $f : \mathscr{F} \to \mathscr{E}$ is a coherent morphism between coherent toposes. Then,

$$\mathscr{C}f : \mathscr{C}\mathscr{F} \to \mathscr{C}\mathscr{E}$$

is 'composition with $f^\star$ restricted to the full subcategories of coherent objects, from which it follows that it always has a left adjoint, to wit

$$\lambda_f \dashv \mathscr{C}f,$$

with λ_f being 'composition with $f_\star$ (similarly restricted). In other words, *any* coherent morphism between coherent toposes is a $\mathscr{C}$-map.

□

Theorem 7.3. The co-KZ-monad $\mathscr{C}$ on the 2-category $\mathscr{K} = \mathrm{CohTop}$ is a Pitts co-KZ-monad.

Proof. The required condition about bicomma objects is shown to be satisfied in [14] (Theorem 2). In addition, the unit of the monad is locally fully faithful, as was already shown. □

Theorem 7.4. [21, 1, 15, 14] Coherent surjections between coherent toposes are of lax effective descent.

Proof. A coherent surjection between coherent toposes is a surjective $\mathscr{C}$-map. The objects classifier $\mathscr{S}[U] = \mathscr{S}^{\mathscr{S}_{\mathrm{fin}}}$ is a coherent topos and it is trivially a $\mathscr{C}$-algebra. Apply Theorem 3.13. $\square$

Remark 7.5. *As remarked in [15, 1], the lax descent property of a coherent surjection $f : \mathscr{F} \twoheadrightarrow \mathscr{E}$ restricts to an equivalence of pretoposes, from the category $\mathscr{E}_{\mathrm{coh}}$ to the category of objects in $\mathscr{F}_{\mathrm{coh}}$ with lax descent data. The latter statement was proved originally in [21] and interpreted as a general definability theorem for coherent theories. It was precisely this connection with 'geometric logic' that motivated A. M. Pitts [17] in his work on lax descent, including the theorem about essential surjections, proved therein as part of a possible route towards a proof of his conjecture.*

Remark 7.6. *In [15] it is shown that relatively tidy surjections of toposes are of lax effective descent and that tidy surjections of toposes are of effective descent. An open question is that of the existence of a Pitts monad on* $\mathrm{Top}_{\mathscr{S}}$ *that accounts for these facts.*

8 Ackowledgements

I record here my indebtedness to Marco Grandis for providing me with a stimulating environment during my sabbatical year 1994-95 at the DIMA of the Universitá di Genova and for his invitation last summer to give them a seminar on the contents of his paper. Thanks are due to the anonymous referee for expert advice. Diagrams were typeset with Michael Barr's diagxy package for xy-pic. In addition, I am grateful to Michael Barr for his help with various special diagrams that I needed, most notably the cube in the proof of Theorem 3.7.

References

[1] D. Ballard and W. Boshuck, *Definability and Descent*, Journal of Symbolic Logic **63, 2** (1998) 372–378.

[2] M. Barr and R. Paré, *Molecular toposes*, J. Pure and Apl. Algebra **17** (1980) 127–152.

[3] J. Bénabou and J. Roubaud, *Monades et descente*, C.R.A.S. **270** (1970) 96–98.

[4] M. Bunge and A. Carboni, *The symmetric topos*, J. Pure and Appl. Alg. **105** (1995) 233–249.

[5] M. Bunge and J. Funk, *Singular Coverings of Toposes,* Lecture Notes in Mathematics **1890**, Springer, 2006.

[6] M. H. Escardó, *Properly injective spaces and function spaces*, Topology and Its Applications **89, 1-2** (1998) 75–120.

[7] P. T. Johnstone, *Topos Theory,* London Mathematical Society Monographs **10**, Academic Press, 1977.

[8] P. T. Johnstone, *Sketches on an Elephant. A Topos Theory Compendium, Volumes 1 and 2* Oxford Logic Guides **43**, Oxford University Press, 2002.

[9] A. Joyal and M. Tierney, *An extension of the Galois theory of Grothendieck*, Memoirs of the American Mathematical Society **309**, 1984.

[10] A. Kock, *Monads for which structure is adjoint to units*, J. Pure and Appl. Alg. **104** (1995) 41–59.

[11] M. Korostenski and C. C. A. Labuschagne, *Lax proper maps of locales*, J. Pure and Appl. Alg. **208** (2007) 655– 664.

[12] F. W. Lawvere, *Integration on presheaf toposes*, Proceedings of the Oberwolfach Meeting on Category Theory, 1966.

[13] S. Mac Lane, *Categories for the Working Mathematician*, Second Edition, Graduate Texts in Mathematics **5**, Springer, 1998.

[14] I. Moerdijk and J. J. C. Vermeulen, *Proof of a conjecture of A. Pitts*, J. Pure Appl. Algebra **143** (1999) 329–338.

[15] I. Moerdijk and J. J. C. Vermeulen, *Proper Maps of Toposes*, Memoirs of the American Mathematical Society **148, 705**, 2000.

[16] A. M. Pitts, *On product and change of base for toposes*, Cahiers de Top. et Géom. Diff. Catégoriques **26-1** (1985) 43–61.

[17] A. M. Pitts, *Lax descent for essential surjections*, Slides for a talk at the Category Theory Conference, Cambridge, July 21, 1986.

[18] J. J. C. Vermeulen, *Proper maps of locales*, J. Pure Appl. Alg, **92** (1994) 79–107.

[19] S. J. Vickers. *Locales are not pointless*, In C. Hankin, I. Mackie, and R. Nagarajan, editors, Theory and Formal Methods 1994. IC Press, 1995.

[20] S. J. Vickers, *Cosheaves and connectedness in formal topology*, Annals of Pure and Applied Logic **63** (2012) 157–174.

[21] M. W. Zawadowski, *Descent and duality*, Ann. Pure Appl. Logic **71** (1995) 131–188.