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A Lê-Greuel type formula for the image Milnor number

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Abstract. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 finitely determined map germ. For a generic linear form $p : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ we denote by $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$ the transverse slice of f with respect to p. We prove that the sum of the image Milnor numbers $\mu_I(f) + \mu_I(g)$ is equal to the number of critical points of $p|_{X_s} : X_s \to \mathbb{C}$ on all the strata of X_s, where X_s is the disentanglement of f (i.e., the image of a stabilisation f_s of f).

Key words: Image Milnor number, Lê-Greuel formula, finite determinacy.

1. Introduction

The Lê-Greuel formula [4], [6] provides a recursive method to compute the Milnor number of an isolated complete intersection singularity (ICIS). We recall that if $(X, 0)$ is a d-dimensional ICIS defined as the zero locus of a map germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d}, 0)$, then the Milnor fibre $X_s = g^{-1}(s)$ (where s is a generic value in $\mathbb{C}^{n-d}$) has the homotopy type of a bouquet of d-spheres and the number of such spheres is called the Milnor number $\mu(X, 0)$. If $d > 0$, we can take $p : \mathbb{C}^n \to \mathbb{C}$ a generic linear projection with $H = p^{-1}(0)$ and such that $(X \cap H, 0)$ is a $(d-1)$-dimensional ICIS. Then,

$$\mu(X, 0) + \mu(X \cap H, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(g) + J(g, p)}, \tag{1}$$

where $\mathcal{O}_n$ is the ring of function germs from $(\mathbb{C}^n, 0)$ to $\mathbb{C}$, (g) is the ideal in $\mathcal{O}_n$ generated by the components of g and $J(g, p)$ is the Jacobian ideal of (g, p) (i.e., the ideal generated by the maximal minors of the Jacobian matrix). Note that X_s is smooth and if p is generic enough, then the re-

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striction $p|_{X_s} : X_s \to \mathbb{C}$ is a Morse function and the dimension appearing in the right hand side of (1) is equal to the number of critical points of $p|_{X_s}$.

The aim of this paper is to obtain a Lê-Greuel type formula for the image Milnor number of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. Mond showed in [11] that the disentanglement X_s (i.e., the image of a stabilisation f_s of f) has the homotopy type of a bouquet of n-spheres and the number of such spheres is called the image Milnor number $\mu_I(f, 0)$. The celebrated Mond's conjecture says that

$$\mathcal{A}_e\text{-codim}(f) \leq \mu_I(f),$$

with equality if f is weighted homogeneous. Mond's conjecture is known to be true for $n = 1, 2$ but it remains still open for $n \geq 3$ (see [11], [12]). We feel that our Lê-Greuel type formula can be useful to find a proof of the conjecture in the general case. In fact, it would be enough to prove that the module which controls the number of critical points of a generic linear function is Cohen-Macaulay and then, use an induction argument on the dimension n (see [1] for details about Mond's conjecture).

We assume that f has corank 1 and $n > 1$. Then given a generic linear form $p : \mathbb{C}^{n+1} \to \mathbb{C}$ we can see f as a 1-parameter unfolding of another map germ $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$ which is the transverse slice of f with respect to p. This means that g has image $(X \cap H, 0)$, where $(X, 0)$ is the image of f and $H = p^{-1}(0)$. The disentanglement X_s is not smooth but it has a natural Whitney stratification given by the stable types. If p is generic enough, the restriction $p|_{X_s} : X_s \to \mathbb{C}$ is a Morse function on each stratum. Our Lê-Greuel type formula is

$$\mu_I(f) + \mu_I(g) = \#\Sigma(p|_{X_s}), \tag{2}$$

where the right hand side of equation is the number of critical points of $p|_{X_s}$ on all the strata of X_s. The case $n = 1$ has to be considered separately, in this case we have

$$\mu_I(f) + m_0(f) - 1 = \#\Sigma(p|_{X_s}), \tag{3}$$

where $m_0(f)$ is the multiplicity of the curve parametrized by f. This makes sense, since $\mu(X, 0) = m_0(X, 0) - 1$ for a 0-dimensional ICIS $(X, 0)$.

2. Multiple point spaces and Marar's formula

In this section we recall Marar's formula for the Euler characteristic of the disentanglement of a corank 1 finitely determined map germ. We first recall the Marar-Mond [9] construction of the kth-multiple point spaces for corank 1 map germs, which is based on the iterated divided differences. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a corank 1 map germ. We can choose coordinates in the source and target such that f is written in the following form:

$$f(x, z) = (x, f_n(x, z), \dots, f_p(x, z)),\ x \in \mathbb{C}^{n-1},\ z \in \mathbb{C}.$$

This forces that if $f(x_1, z_1) = f(x_2, z_2)$ then necessarily $x_1 = x_2$. Thus, it makes sense to embed the double point space of f in $\mathbb{C}^{n-1} \times \mathbb{C}^2$ instead of $\mathbb{C}^n \times \mathbb{C}^n$. Analogously, we will consider the kth-multiple point space embedded in $\mathbb{C}^{n-1} \times \mathbb{C}^k$.

We construct an ideal $I_k(f) \subset \mathcal{O}_{n+k-1}$ defined as follows: $I_k(f)$ is generated by $(k-1)(p-n+1)$ functions $\Delta_i^{(j)} \in \mathcal{O}_{n+k-1}$, $1 \le i \le k-1$, $n \le j \le p$. Each $\Delta_i^{(j)}$ is a function only of the variables $x, z_1, \dots, z_{i+1}$ such that:

$$\Delta_1^{(j)}(x, z_1, z_2) = \frac{f_j(x, z_1) - f_j(x, z_2)}{z_1 - z_2},$$

and for $1 \le i \le k-2$,

$$\Delta_{i+1}^{(j)}(x, z_1, \dots, z_{i+2}) = \frac{\Delta_i^{(j)}(x, z_1, \dots, z_i, z_{i+1}) - \Delta_i^{(j)}(x, z_1, \dots, z_i, z_{i+2})}{z_{i+1} - z_{i+2}}.$$

Definition 2.1 The *kth-multiple point space* is $D^k(f) = V(I_k(f))$, the zero locus in $(\mathbb{C}^{n+k-1}, 0)$ of the ideal $I_k(f)$.

(We remark that the kth-multiple point space is denoted by $\widetilde{D}^k(f)$ instead of $D^k(f)$ in [9]).

If f is stable, then, set-theoretically, $D^k(f)$ is the Zariski closure of the set of points $(x, z_1, \dots, z_k) \in \mathbb{C}^{n+k-1}$ such that:

$$f(x, z_1) = \cdots = f(x, z_k), \quad z_i \neq z_j,\ \text{for } i \neq j,$$

(see [9], [13]). But, in general, this may be not true if f is not stable. For instance, consider the cusp $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ given by $f(z) = (z^2, z^3)$. Since f is one-to-one, the closure of the double point set is empty, but

$$D^2(f) = V(z_1 + z_2, z_1^2 + z_1 z_2 + z_2^2).$$

This example also shows that the kth-multiple point space may be non-reduced in general.

The main result of Marar-Mond in [9] is that the kth-multiple point spaces can be used to characterize the stability and the finite determinacy of f.

Theorem 2.2 ([9, 2.12]) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ $(n < p)$ be a finitely determined map germ of corank* 1. *Then*:

(1) *f is stable if and only if $D^k(f)$ is smooth of dimension $p - k(p - n)$, or empty, for $k \geq 2$.*
(2) *f is finitely determined if and only if for each k with $p - k(p - n) \geq 0$, $D^k(f)$ is either an ICIS of dimension $p - k(p - n)$ or empty, and if, for those k such that $p - k(p - n) < 0$, $D^k(f)$ consists at most of the point $\{0\}$.*

The following construction is also due to Marar-Mond [9] and gives a refinement of the types of multiple points.

Definition 2.3 Let $\mathcal{P} = (r_1, \ldots, r_m)$ be a partition of k (that is, $r_1 + \cdots + r_m = k$, with $r_1 \geq \cdots \geq r_m$). Let $I(\mathcal{P})$ be the ideal in $\mathcal{O}_{n-1+k}$ generated by the $k - m$ elements $z_i - z_{i+1}$ for $r_1 + \cdots + r_{j-1} + 1 \leq i \leq r_1 + \cdots + r_j$ for $j = 1, \ldots, m$. Define the ideal $I_k(f, \mathcal{P}) = I_k(f) + I(\mathcal{P})$ and the *k-multiple point space* of f with respect to the partition $\mathcal{P}$ as $D^k(f, \mathcal{P}) = V(I_k(f, \mathcal{P}))$.

Definition 2.4 We define a *generic point* of $D^k(f, \mathcal{P})$ as a point

$$(x, z_1, \ldots, z_1, \ldots, z_m, \ldots, z_m),$$

(z_i iterated r_i times, and $z_i \neq z_j$ if $i \neq j$) such that the local algebra of f at (x, z_i) is isomorphic to $\mathbb{C}[t]/(t^{r_i})$, and such that

$$f(x, z_1) = \cdots = f(x, z_m).$$

If f is stable, then $D^k(f, \mathcal{P})$ is equal to the Zariski closure of its generic points (see [9]). Moreover, we have the following corollary, which extends Theorem 2.2 to the multiple point spaces with respect to the partitions.

Corollary 2.5 ([9, 2.15]) *If f is finitely determined (resp. stable), then for each partition $\mathcal{P} = (r_1, \dots, r_m)$ of k satisfying $p - k(p-n+1) + m \geq 0$, the germ of $D^k(f,\mathcal{P})$ at $\{0\}$ is either an ICIS (resp. smooth) of dimension $p-k(p-n+1)+m$, or empty. Moreover, those $D^k(f,\mathcal{P})$ for $\mathcal{P}$ not satisfying the inequality consist at most of the single point $\{0\}$.*

Let $f : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ be a finitely determined map germ of corank 1 and let $f_s : U_s \to X_s$ be a stabilization of f. For a partition $\mathcal{P}$ of k, we denote by $\rho_{\mathcal{P}}$ the mapping given as the composition of the inclusion $D^k(f_s,\mathcal{P}) \hookrightarrow D^k(f_s)$, the projection $D^k(f_s) \to U_s$ and f_s. The following two results will be useful in the next section.

Remark 2.6 ([8]) Let $\mathcal{P} = (a_1, \dots, a_h)$ be a partition of k, with $a_i \geq a_{i+1}$. If y is a generic point of $D^k(f_s,\mathcal{P}')$, where $\mathcal{P}' = (b_1, \dots, b_q)$, with $b_i \geq b_{i+1}$ and $\mathcal{P} < \mathcal{P}'$ then $\#\rho_{\mathcal{P}}^{-1}(\rho_{\mathcal{P}'}(y))$ is the coefficient of the monomial $x_1^{b_1} x_2^{b_2} \dots x_q^{b_q}$ in the polynomial $\prod_{i\geq 1}(x_1^{a_i} + x_2^{a_i} + \cdots x_q^{a_i})$.

Lemma 2.7 ([7]) *Let h_k be the k-th complete symmetric function in variables $x_1, \dots, x_q$, i.e., h_k is the sum of all monomials of degree k in the variables $x_1, \dots, x_q$. Then*

$$h_k = \sum_{\mathcal{P}} \frac{1}{\prod_{i\geq 1} \alpha_i! i^{\alpha_i}} \prod_{i\geq 1} (x_1^i + \cdots + x_q^i)^{\alpha_i},$$

where $\mathcal{P}$ runs through the set of all ordered partitions of k.

The next step is to observe that the kth-multiple point space $D^k(f)$ is invariant under the action of the kth symmetric group S_k.

Definition 2.8 Let M be a $\mathbb{Q}$-vector space upon which S_k acts. Then the *alternating part* of M, denoted by $\mathrm{Alt}_k M$, is defined to be

$$\mathrm{Alt}_k M := \{m \in M : \sigma(m) = \mathrm{sign}(\sigma) m, \text{ for all } \sigma \in S_k\}.$$

Given a topological space X on which S_k acts, the *alternating Euler characteristic* is

$$\chi^{alt}(X) := \sum_i (-1)^i \dim_{\mathbb{Q}} \mathrm{Alt}_k(H_i(X, \mathbb{Q})).$$

The following theorem of Goryunov-Mond in [3] allows us to compute the image Milnor number of f by means of a spectral sequence associated to the multiple point spaces.

Theorem 2.9 ([3, 2.6]) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 map germ and f_s a stabilisation of f, for $s \neq 0$ and X_s the image of f_s. Then,*

$$H_n(X_s, \mathbb{Q}) \cong \bigoplus_{k=2}^{n+1} \mathrm{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})).$$

Note that since X_s has the homotopy type of a wedge of n-spheres, the image Milnor number of f is the rank of $H_n(X_s, \mathbb{Q})$. If we consider $H_n(X_s, \mathbb{Q})$ as a $\mathbb{Q}$-vector space,

$$\mu_I(f) = \dim_{\mathbb{Q}} H_n(X_s, \mathbb{Q}).$$

So, by Theorem 2.9, the image Milnor number is

$$\mu_I(f) = \sum_{k=2}^{n+1} \dim_{\mathbb{Q}} \mathrm{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})).$$

By [5, Corollary 2.8], we can compute the alternating Euler characteristic of $D^k(f_s)$ as follows: for each partition $\mathcal{P} = (r_1, \ldots, r_s)$, we set

$$\beta(\mathcal{P}) = \frac{\mathrm{sign}(\mathcal{P})}{\prod_i \alpha_i! i^{\alpha_i}},$$

where $\alpha_i := \#\{j : \ r_j = i\}$ and $\mathrm{sign}(\mathcal{P})$ is the number $(-1)^{k-\sum_i \alpha_i}$. Then,

$$\chi^{alt}(D^k(f_s)) = \sum_{|\mathcal{P}|=k} \beta(\mathcal{P}) \chi(D^k(f_s, \mathcal{P})).$$

Moreover, by Theorem 2.2 and Corollary 2.5, $D^k(f_s)$ (resp. $D^k(f_s, \mathcal{P})$) is a Milnor fibre of the ICIS $D^k(f)$ (resp. $D^k(f, \mathcal{P})$), and hence it has the homotopy type of a wedge of spheres of real dimension $\dim D^k(f) = n-k+1$ (resp. $\dim D^k(f, \mathcal{P})$). Thus,

$$\dim_{\mathbb{Q}} \operatorname{Alt}_k(H_{n-k+1}(D^k(f_s),\mathbb{Q})) = (-1)^{n-k+1}\chi^{alt}(D^k(f_s)),$$

and

$$\chi(D^k(f_s,\mathcal{P})) = 1 + (-1)^{\dim D^k(f,\mathcal{P})}\mu(D^k(f,\mathcal{P})).$$

This gives the following version of Marar's formula [8] in terms of the Milnor numbers of the multiple point spaces:

$$\mu_I(f) = \sum_{k=2}^{n+1}(-1)^{n-k+1}\sum_{|\mathcal{P}|=k}\beta(\mathcal{P})\big(1+(-1)^{\dim D^k(f,\mathcal{P})}\mu(D^k(f,\mathcal{P}))\big), \quad (4)$$

where the coefficients $\beta(\mathcal{P}) = 0$ when the sets $D^k(f,\mathcal{P})$ are empty, for $k = 2,\dots,n+1$.

3. Lê-Greuel type formula

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 finitely determined map germ. Let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear projection such that $H = p^{-1}(0)$ is a generic hyperplane through the origin in $\mathbb{C}^{n+1}$. We can choose linear coordinates in $\mathbb{C}^{n+1}$ such that $p(y_1,\dots,y_{n+1}) = y_1$. Then, we choose the coordinates in $\mathbb{C}^n$ in such a way that f is written in the form

$$f(x_1,\dots,x_{n-1},z) = (x_1,\dots,x_{n-1},h_1(x_1,\dots,x_{n-1},z),h_2(x_1,\dots,x_{n-1},z)),$$

for some holomorphic functions h_1, h_2. We see f as a 1-parameter unfolding of the map germ $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$ given by

$$\begin{aligned} &g(x_2,\dots,x_{n-1},z)\\ &\quad = (x_2,\dots,x_{n-1},h_1(0,x_2,\dots,x_{n-1},z),h_2(0,x_2,\dots,x_{n-1},z)). \end{aligned}$$

We say that g is the transverse slice of f with respect to the generic hyperplane H. If f has image $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$, then the image of g in $(\mathbb{C}^n, 0)$ is isomorphic to $(X \cap H, 0)$.

We take f_s a stabilisation of f and denote by X_s the image of f_s (see [11] for the definition of stabilisation). Since f has corank 1, X_s has a natural Whitney stratification given by the stable types of f_s. In fact, the strata are the submanifolds

$$M^k(f_s,\mathcal{P}) := \epsilon^k(D^k(f_s,\mathcal{P})^0) \setminus \epsilon^{k+1}(D^{k+1}(f_s)),$$

where $D^k(f_s,\mathcal{P})^0$ is the set of generic points of $D^k(f_s,\mathcal{P})$, $\epsilon^k : \mathbb{C}^{n+k-1} \to \mathbb{C}^{n+1}$ is the map $(x, z_1, \dots, z_k) \mapsto f_s(x, z_1)$ and $\mathcal{P}$ runs through all the partitions of k with $k = 2, \dots, n+1$. We can choose the generic linear projection $p : \mathbb{C}^{n+1} \to \mathbb{C}$ in such a way that the restriction to each stratum $M^k(f_s,\mathcal{P})$ is a Morse function. In other words, such that the restriction $p|_{X_s} : X_s \to \mathbb{C}$ is a Morse function on each stratum (this is one of the condition of be a stratifed Morse function in the sense of [2]). We will denote by $\#\Sigma(p|_{X_s})$ the number of critical points on all the strata of X_s. Our first result in this section is for the case of a plane curve.

Theorem 3.1 *Let $f : (\mathbb{C},0) \to (\mathbb{C}^2,0)$ be an injective map germ. Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a generic linear projection, then*

$$\#\Sigma(p|_{X_s}) = \mu_I(f) + m_0(f) - 1,$$

where $m_0(f)$ is the multiplicity of f.

Proof. After a change of coordinates, we can assume that

$$f(t) = (t^k, c_m t^m + c_{m+1} t^{m+1} + \cdots),$$

where $k = m_0(f)$, $m > k$ and $c_m \neq 0$. The stabilisation f_s is an immersion with only transverse double points. So, its image X_s has only two strata: $M^2(f_s,(1,1))$ is a 0-dimensional stratum composed by the transverse double points and $M^1(f_s,(1))$ is a 1-dimensional stratum given by the smooth points of X_s. Note that the number of double points of f_s is the delta invariant of the plane curve, $\delta(X,0)$, which is equal to $\mu_I(f)$ by [12, Theorem 2.3].

Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a generic linear projection such that $p|_{X_s}$ is a Morse function on each stratum. Then:

$$\#\Sigma(p|_{X_s}) = \#M^2(f_s,(1,1)) + \#\Sigma(p|_{M^1(f_s,(1))}) = \mu_I(f) + \#\Sigma(p|_{M^1(f_s,(1))}).$$

Since f_s is a local diffeomorphism on the stratum $M^1(f_s,(1))$, the number of critical points of $p|_{M^1(f_s,(1))}$ is equal to the number of critical points of $p \circ f_s$ (here the points of $M^2(f_s,(1,1))$ can be excluded by the genericity of p). Assume that $p(x,y) = Ax + By$ with $A \neq 0$. Then $p \circ f_s$ is a Morsification of the function

$$p \circ f(t) = At^k + B(c_m t^m + c_{m+1} t^{m+1} + \cdots)$$

The number of critical points of $p \circ f_s$ is equal to $\mu(p \circ f) = k - 1 = m_0(f) - 1$, which proves our formula. □

Next, we state and prove the formula for the case $n > 1$.

Theorem 3.2 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 finitely determined map germ with $n > 1$. Let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear projection which defines a transverse slice $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$. Then,*

$$\#\Sigma(p|_{X_s}) = \mu_I(f) + \mu_I(g).$$

Proof. By Marar's formula (4):

$$\mu_I(f) + \mu_I(g) = \sum_{k=2}^{n+1} (-1)^{n-k+1} \sum_{|\mathcal{P}|=k} \beta(\mathcal{P})(1 + (-1)^{\dim D^k(f,\mathcal{P})} \mu(D^k(f, \mathcal{P})))$$
$$+ \sum_{k=2}^{n} (-1)^{n-k} \sum_{|\mathcal{P}|=k} \beta(\mathcal{P})(1 + (-1)^{\dim D^k(g,\mathcal{P})} \mu(D^k(g, \mathcal{P})))$$

Note that if $\dim D^k(f, \mathcal{P}) > 0$, then $\dim D^k(f, \mathcal{P}) = 1 + \dim D^k(g, \mathcal{P})$. Moreover, if $\dim D^k(f, \mathcal{P}) = 0$, then $D^k(g, \mathcal{P}) = \emptyset$. So, we can separate the formula into two parts, the first one for partitions with $\dim D^k(f, \mathcal{P}) = 0$, the second one for partitions with $\dim D^k(f, \mathcal{P}) > 0$. Thus,

$$\mu_I(f) + \mu_I(g) = \sum_{k=2}^{n+1} (-1)^{n+k-1} \sum_{\substack{|\mathcal{P}|=k \\ \dim D^k(f,\mathcal{P})=0}} \beta(\mathcal{P})(1 + \mu(D^k(f, \mathcal{P})))$$
$$+ \sum_{k=2}^{n} (-1)^{n+k-1} \sum_{\substack{|\mathcal{P}|=k \\ \dim D^k(f,\mathcal{P})>0}}$$
$$\times \beta(\mathcal{P})(-1)^{\dim D^k(f,\mathcal{P})} \big(\mu(D^k(f, \mathcal{P})) + \mu(D^k(g, \mathcal{P}))\big)$$

If $\dim D^k(f, \mathcal{P}) = 0$, the Milnor number of $D^k(f, \mathcal{P})$ is

$$\mu(D^k(f,\mathcal{P})) = deg(D^k(f,\mathcal{P})) - 1,$$

where deg is the degree of the map germ that defines the 0-dimensional ICIS $D^k(f,\mathcal{P})$. Note that we can see $deg(D^k(f,\mathcal{P}))$ as the number of critical points of $\tilde{p}|_{D^k(f_s,\mathcal{P})}$.

We choose the coordinates such that $p(y_1,\dots,y_{n+1}) = y_1$. We denote by $\tilde{p}:\mathbb{C}^{n+k-1}\to\mathbb{C}$ the projection onto the first coordinate. Then:

$$D^k(g,\mathcal{P}) = D^k(f,\mathcal{P}) \cap \tilde{p}^{-1}(0).$$

By the Lê-Greuel formula for ICIS [4], [6],

$$\mu(D^k(f,\mathcal{P})) + \mu(D^k(g,\mathcal{P})) = \#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P})}).$$

It is easy to check that $(-1)^{\dim D^k(f)}\mathrm{sign}(\mathcal{P})(-1)^{\dim D^k(f,\mathcal{P})} = 1$ for any partition $\mathcal{P}$. Thus, we get:

$$\mu_I(f) + \mu_I(g) = \sum_{k=2}^{n+1}\sum_{|\mathcal{P}|=k}\frac{\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P})})}{\gamma(\mathcal{P})},$$

where $\gamma(\mathcal{P}) = \prod_i \alpha_i! i^{\alpha_i}$.

Let $\mathcal{P}$ be a partition of k, if $|\mathcal{P}'| = k$ and $\mathcal{P}' \geq \mathcal{P}$ then any critical point of $\tilde{p}|_{D^k(f_s,\mathcal{P}')}$ is a critical point of $\tilde{p}|_{D^k(f_s,\mathcal{P})}$. This implies

$$\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P})}) = \sum_{\substack{|\mathcal{P}'|=k\\ \mathcal{P}'\geq\mathcal{P}}}\alpha(\mathcal{P},\mathcal{P}')\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P}')^0}),$$

where $\alpha(\mathcal{P},\mathcal{P}')$ is defined by

$$\alpha(\mathcal{P},\mathcal{P}') := \frac{\#\rho_{\mathcal{P}}^{-1}(\rho_{\mathcal{P}'}(y))}{\#\rho_{\mathcal{P}'}^{-1}(\rho_{\mathcal{P}'}(y))}$$

for a generic point y in $D^k(f_s,\mathcal{P}')$. We can see $\alpha(\mathcal{P},\mathcal{P}')$ as the number of times that a generic point of $D^k(f_s,\mathcal{P}')$ appears repeated in $D^k(f_s,\mathcal{P})$. By Remark 2.6 and Lemma 2.7,

$$
\begin{aligned}
\mu_I(f)+\mu_I(g) &= \sum_{k=2}^{n+1}\sum_{|\mathcal{P}|=k}\frac{\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P})})}{\gamma(\mathcal{P})}\\
&= \sum_{k=2}^{n+1}\sum_{|\mathcal{P}|=k}\sum_{\substack{|\mathcal{P}'|=k\\ \mathcal{P}'\geq\mathcal{P}}}\frac{\alpha(\mathcal{P},\mathcal{P}')}{\gamma(\mathcal{P})}\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P}')^0})\\
&= \sum_{k=2}^{n+1}\sum_{|\mathcal{P}'|=k}\left(\sum_{\substack{|\mathcal{P}|=k\\ \mathcal{P}\leq\mathcal{P}'}}\frac{\#\rho_{\mathcal{P}}^{-1}(\rho_{\mathcal{P}'}(y))}{\gamma(\mathcal{P})}\right)\frac{\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P}')^0})}{\#\rho_{\mathcal{P}'}^{-1}(\rho_{\mathcal{P}'}(y))}\\
&= \sum_{k=2}^{n+1}\sum_{|\mathcal{P}'|=k}\frac{\#\Sigma(\tilde{p}|_{D^k(f_s,\mathcal{P}')^0})}{\#\rho_{\mathcal{P}'}^{-1}(\rho_{\mathcal{P}'}(y))}\\
&= \sum_{k=2}^{n+1}\sum_{|\mathcal{P}'|=k}\#\Sigma(p|_{M^k(f_s,\mathcal{P}')}),
\end{aligned}
$$

which is nothing but the number of critical points of $p|_{X_s}$. □

4. Examples

In this section, we give some examples to illustrate the formulas of theorems 3.1 and 3.2.

Example 4.1 (The singular plane curve E_6) Let $f(z)=(z^3,z^4)$ be the singular plane curve E_6, let $f_s(z)=(z^3+sz, z^4+(5/4)sz^2)$ be a stabilisation of f, for $s\neq 0$.

Let $M^2(f_s,(1,1))$ be the 0-dimensional stratum of X_s. It is composed by three points, they correspond to three double transversal points. Let $M^1(f_s,(1))$ be the 1-dimensional stratum. If we compose f_s with $p(z,u)=z$ there are two critical points in a neighbourhood of the origin, so $\#\sum p_{|X_s} = 5$.

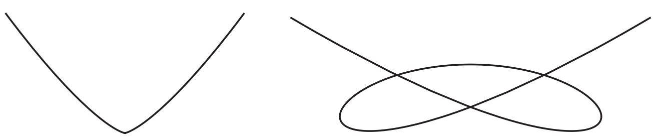

Figure 1. The curve E_6 and its stabilisation for $s<0$.

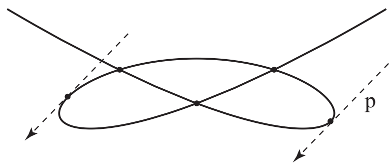

Figure 2. Critical points in X_s.

Now, since the multiplicity of f, $m_0(f) = 3$ and the image Milnor number of f is $\mu_I(f) = 3$, $\mu_I(f) + m_0(f) - 1 = 5$ as predicted by the formula.

When $n > 1$, we proceed in the following way: Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 finitely determined map germ written as

$$f(x, z) = (x, h_1(x, z), h_2(x, z)),\ x \in \mathbb{C}^{n-1},\ z \in \mathbb{C}.$$

Let f_s be a stabilisation of f. The image of f_s is denoted by X_s. First, we calculate the number of critical points of the restriction of p to X_s, for the generic linear projection $p(y_1, \dots, y_{n+1}) = y_1$. We separate the image set X_s in strata of different dimensions given by stable types, which correspond to the sets $M^k(f_s, \mathcal{P})$. The n-dimensional stratum, $M^1(f_s, (1))$, is composed of the regular part of f_s. So, the restriction $p|_{M^1(f_s)}$ has not critical points.

The $(n-1)$-dimensional stratum is composed of $M^2(f_s, (1, 1))$. To calculate the critical points, we will work with the inverse image by ϵ^2, that is, $D^2(f_s, (1, 1)) = D^2(f_s)$. The double point space $D^2(f_s)$ is a subset of $\mathbb{C}^{n+1}$, but we take a projection of $D^2(f_s)$ in the first n variables. So, we denote by $D(f_s)$ the projection of double point space in $\mathbb{C}^n$. The double point space $D(f_s)$ is a hypersurface in $\mathbb{C}^n$ given by the resultant of P_s and Q_s with respect to z_2, where $P_s = (h_{1,s}(x, z_2) - h_{1,s}(x, z_1))/(z_2 - z_1)$ and $Q_s = (h_{2,s}(x, z_2) - h_{2,s}(x, z_1))/(z_2 - z_1)$. This gives the defining equation of $D(f_s)$, denoted by $\lambda_s(x, z) = 0$.

To calculate the critical points of the set $D(f_s)$ we take the linear projection $\tilde{p}(x_1, \dots, x_{n-1}, z) = x_1$. Note that the hypersuface $D(f_s)$ also contains the critical points of the other k-dimensional strata, with $k < n - 1$. Then, it will be sufficient to compute critical points here, in order to have all the critical points. We have that $(x_1, \dots, x_{n-1}, z)$ is a critical point of $\tilde{p}|_{D(f_s)}$ if $\lambda_s(x, z) = 0$ and $J(\lambda_s, \tilde{p})(x, z) = 0$, where $J(\lambda_s, \tilde{p})$ is the Jacobian determinant of λ and $\tilde{p}$.

If a critical point of $\tilde{p}_{|_{D(f_s)}}$ corresponds to a m-multiple point, then we will have m critical points in $D(f_s)$ for one in the image of f_s. Thus, once the critical points of each type are obtained, we have to divide by the multiplicity of the point. In this way, we obtain the number of critical points of p in the image of f_s.

On the other hand, we compute separately the image Milnor numbers of f and g in order to check the formulas.

Example 4.2 (The germ F_4 in $\mathbb{C}^3$) Let $f(x,z) = (x, z^2, z^5 + x^3 z)$ be the germ F_4. Let $f_s(x,z) = (x, z^2, z^5 + xsz^3 + (x^3 - 5xs - s)z)$ be a stabilisation of f, for $s \neq 0$. By [10], f is a 1-parameter unfolding of the plane curve A_4, $g(z) = (z^2, z^5)$ and in fact, g is the transverse slice of f.

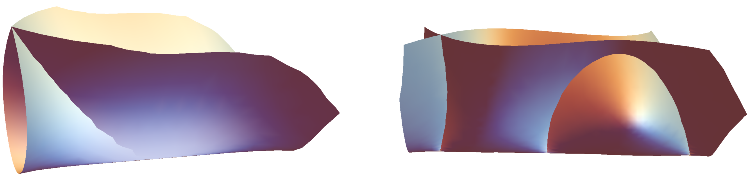

Figure 3. The germ F_4 and its stabilisation for $s > 0$.

Let $M^3(f_s,(1,1,1)) \cup M^2(f_s,(2))$ be the 0-dimensional strata of X_s. In our case, there are not triple points and there are three cross caps in $M^2(f_s,(2))$.

Let $M^2(f_s,(1,1))$ be the 1-dimensional stratum of X_s. As we said, let $D^2(f_s)$ be the double point curve in $\mathbb{C}^3$ and by projecting in the first two coordinates, we have the double point curve in $\mathbb{C}^2$, denoted by $D(f_s)$.

We compute the resultant of P_s and Q_s respect to z_2, where P_s and Q_s are the divided differences. The double point curve of f_s in $\mathbb{C}^2$ is the plane curve

$$\lambda_s(x,z) = -s - 5sx + x^3 + sxz^2 + z^4.$$

The critical points of the restriction $p|_D(f_s)$ are given by $\lambda_s(x_0,z_0) = 0$ and $J(\lambda_s,\tilde{p})(x_0,z_0) = 0$, where $\tilde{p}(x,z) = x$.

Nine critical points are obtained. Three of these points are cusps in $g_{x,s}$ which correspond to the three cross caps of f_s. Then, the other six critical points in $\tilde{p}_{|_{\lambda_s(x_0,z_0)=0}}$ correspond to three tacnodes in $g_{x,s}$ which are represented in the double point curve when a vertical line is tangent at two

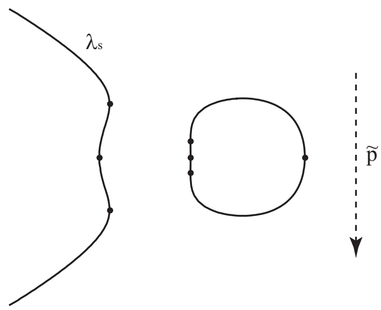

Figure 4. Cusps and tacnodes in the double point curve.

points of $D(f_s)$. So, each two of these critical points in λ_s correspond to one tacnode of $g_{x,s}$ in $M^2(f_s, (1,1))$. Note that in the Fig. 4 there are only two tacnodes, that is because the other is a complex tacnode.

Finally, in the 2-dimensional stratum $M^1(f_s, (1))$ there are not critical points. So, the number of critical points in X_s is $\#\Sigma p|_{X_s} = 6$, three cusps, three tacnodes and zero triple points. Then, $\#\Sigma p|_{X_s} = C + J + T$ where C, J, T are the numbers of cusps, tacnodes and triple points respectively of $g_{x,s}$. By [10], $\mu_I(f) = C + J + T - \delta(g)$. Since g is a plane curve, we have that $\mu_I(g) = \delta(g)$ (see [12]). So,

$$\#\Sigma p|_{X_s} = C + J + T = \mu_I(f) + \mu_I(g).$$

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