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Estimates for the first eigenvalue of the drifting Laplacian on embedded hypersurfaces

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Abstract. For an $(n-1)$-dimensional compact orientable smooth metric measure space $\big(M, g, e^{-f}dv_g\big)$ embedded in an n-dimensional compact orientable Riemannian manifold N, we successfully give a lower bound for the first nonzero eigenvalue of the *drifting Laplacian* on M, provided the Ricci curvature of N is bounded from below by a positive constant and the weighted function f on M satisfies two constraints.

Key words: Ricci curvature, eigenvalues, drifting Laplacian, smooth metric measure spaces.

1. Introduction

Suppose that N is an n-dimensional $(n \geqslant 2)$ compact orientable Riemannian manifold with Ricci curvature bounded from below by some positive constant k. For a compact orientable hypersurface M *minimally* embedded in N, Choi and Wang [1] proved $\lambda_1 \geqslant k/2$, where λ_1 is the first nonzero eigenvalue of the Laplacian $\widetilde{\Delta}$ on M.

A smooth metric measure space (also known as the weighted measure space) is actually a Riemannian manifold equipped with some measure which is conformal to the usual Riemannian measure. More precisely, for a given complete Riemannian manifold $(\mathscr{M}, g)$ with the metric g, the triple $(\mathscr{M}, g, e^{-f}dv_g)$ is called a smooth metric measure space, where f is a *smooth real-valued* function on $\mathscr{M}$ and dv_g is the Riemannian volume element associated with g (sometimes, we also call dv_g the volume density). On a smooth metric measure space $(\mathscr{M}, g, e^{-f}dv_g)$, we can define the so-called *drifting Laplacian* (also called *weighted Laplacian*) $\mathbb{L}_f$ as follows

$$\mathbb{L}_f := \widetilde{\Delta} - \langle \widetilde{\nabla} f, \widetilde{\nabla} \cdot \rangle_g,$$

where $\langle \cdot, \cdot \rangle_g$ is the inner product induced by the metric g, $\widetilde{\nabla}$ is the gradient operator on $\mathscr{M}$, and, as before, $\widetilde{\Delta}$ is the corresponding Laplace operator.

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Denote by Ric and Hess the Ricci tensor and the Hessian operator, respectively. On $(\mathscr{M}, g, e^{-f}dv_g)$, we can also define the so-called ∞-*Bakry-Émery Ricci tensor* Ric_f given by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\, f,$$

which is also called the *weighted Ricci curvature*. Clearly, the weighted Laplacian and the weighted Ricci curvature are natural generalizations of the classical Laplacian and Ricci curvature in the Riemannian geometry.

Maybe people would have an illusion that smooth metric measure spaces are not necessary to be studied since they are simply obtained from corresponding Riemannian manifolds by adding a conformal measure to the Riemannian measure. However, the truth is not like this, and they do have many differences. For instance, when Ric_f is bounded from below, the Myers' theorem, Bishop-Gromov's volume comparison, Cheeger-Gromoll's splitting theorem and Abresch-Gromoll excess estimate cannot hold as the Riemannian case. However, we know some generalizations. For example, Cheeger-Gromoll's splitting theorem was generalized by Lichnerowicz [4] and Fang-Li-Zhang [3] under $\mathrm{Ric}_f \geqslant 0$ and $\sup f < \infty$, and some volume comparison results are included in Wei-Wylie [8]. Here, for the purpose of comprehension, we would like to repeat an example given in [8, Example 2.1]. That is, for the smooth metric measure space $(\mathbb{R}^n, g_{\mathbb{R}^n}, e^{-f}dv_{g_{\mathbb{R}^n}})$, where $g_{\mathbb{R}^n}$ and $dv_{g_{\mathbb{R}^n}}$ are the usual Euclidean metric and the Euclidean volume density related to $g_{\mathbb{R}^n}$ respectively, if $f(x) = (\lambda/2)|x|^2$ for $x \in \mathbb{R}^n$, then we have $\mathrm{Hess} f = \lambda g_{\mathbb{R}^n}$ and $\mathrm{Ric}_f = \lambda g_{\mathbb{R}^n}$. Therefore, from this example, we know that unlike in the case of Ricci curvature bounded from below uniformly by some positive constant, a metric measure space is not necessarily compact provided $\mathrm{Ric}_f \geqslant \lambda$ and $\lambda > 0$. So, it is interesting to know whether a classical result in the Riemannian geometry could be extended to the weighted case or not. The first author here has been walking on this way and some interesting results have also been obtained (see, e.g., [2], [5], [6]).

In this paper, we also work along this direction, and successfully prove the following main theorem.

Theorem 1.1 *Assume that $(M, g, e^{-f}dv_g)$ is an $(n-1)$-dimensional compact orientable smooth metric measure space embedded in an n-dimensional $(n \geqslant 2)$ compact orientable Riemannian manifold N, and the weight function f satisfies the following conditions*:

- *the norm of the gradient of f is bounded by some nonnegative constant C_1, that is, $|\widetilde{\nabla} f| \leqslant C_1$, where $\widetilde{\nabla}$ is the gradient operator on M;*
- *the absolute value of each element of the Hessian matrix of f is bounded by some nonnegative constant C_2, that is, for any $x \in M$ and a local orthonormal frame $\{\widetilde{e}_1, \ldots, \widetilde{e}_{n-1}\}$ at x, $|(\widetilde{D}^2 f)(\widetilde{e}_i, \widetilde{e}_j)| = |\widetilde{e}_i(\widetilde{e}_j f) - (\widetilde{\nabla}_{\widetilde{e}_i} \widetilde{e}_j) f| = |f_{ij}| \leqslant C_2$ for any $i, j = 1, 2, \ldots, n-1$, where $\widetilde{D}^2$ is the Hessian tensor on M, and $\widetilde{\nabla}_{\widetilde{e}_i} \widetilde{e}_j$ is the covariant derivative associated with the Levi-Civita connection on M.*

If the Ricci curvature of N is bounded from below by a positive constant k, then

(1) $2\lambda_{1,f} \geqslant k - 2C_3$ *provided M is a minimal hypersurface, where $C_3 := \max\{\sqrt{n-1} \cdot C_2, C_1\}$ and $\lambda_{1,f}$ is the first nonzero eigenvalue of the drifting Laplacian $\mathbb{L}_f$ on M;*
(2) $2\lambda_{1,f} \geqslant k - (n-1)\max_M |H|$ *provided M is not a minimal hypersurface, where H is the mean curvature of M.*

Especially, if furthermore f is a constant function on M, then equalities in the above two eigenvalue inequalities cannot hold.

Remark 1.2 If f is a constant function on M, then the drifting Laplacian $\mathbb{L}_f$ degenerates into the normal Laplacian $\widetilde{\Delta}$ on M and $C_1 = C_2 = 0$, which leads to the facts that $C_3 = 0$ and $\lambda_{1,f} = \lambda_1$ with, as before, λ_1 the first nonzero eigenvalue of $\widetilde{\Delta}$. Correspondingly, in this situation, our eigenvalue lower bounds become $2\lambda_1 > k - (n-1)\max_M |H|$. Clearly, if furthermore M is minimally embedded in N, then $2\lambda_1 > k$, which is sharper than Choi-Wang's lower bound $\lambda_1 \geqslant k/2$ shown in [1].

Yau's conjecture [9] asserts that the first nonzero eigenvalue of the Laplacian on a compact embedded minimal hypersurface of $\mathbb{S}^n$ must be $n-1$. Choi and Wang [1] have shown that this first eigenvalue is bounded from below by $(n-1)/2$. Therefore, it is meaningful to see what we can get if $N \equiv \mathbb{S}^n$ in Theorem 1.1. In fact, we can obtain the following.

Corollary 1.3 *If furthermore the ambient Riemannian manifold N in the assumption of Theorem 1.1 satisfies $N \equiv \mathbb{S}^n$, then $\lambda_{1,f} \geqslant (n-1)/2 - C_3$ provided M is a minimal hypersurface of $\mathbb{S}^n$, and $\lambda_{1,f} \geqslant (1/2)(n-1)\big(1 - \max_M |H|\big)$ provided M is not a minimal hypersurface of $\mathbb{S}^n$. Especially, if furthermore f is a constant function on M, then equalities in these two* *eigenvalue inequalities cannot be attained.*

2. Proof of main theorem

Proof of Theorem 1.1. Since the Ricci curvature of N is strictly positive, the first Betti number of N must be zero. Furthermore, together with the fact that M and N are orientable and considering the exact sequences of homology groups, one can get the conclusion that M divides N into two components, say Ω_1 and Ω_2, such that $\partial\Omega_1 = \partial\Omega_2 = M$.

Let φ be the eigenfunction of the first nonzero eigenvalue $\lambda_{1,f}$ of the drifting Laplacian $\mathbb{L}_f$ on M, that is,

$$\mathbb{L}_f\varphi + \lambda_{1,f}\varphi = 0, \qquad \text{in } M. \tag{2.1}$$

Let u be the solution to the following Cauchy problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega_1, \\ u = \varphi, & \text{on } \partial\Omega_1 = M, \end{cases} \tag{2.2}$$

where Δ is the Laplace operator with respect to the Riemannian metric on Ω_1. Clearly, u is a function defined on Ω_1 smooth up to $\partial\Omega_1$.

For any $x \in \Omega_1$, one can always set up a local orthonormal frame $\{e_1, e_2, \ldots, e_{n-1}, e_n\}$, and moreover, if furthermore $x \in \partial\Omega_1$, one can require that at x, $\{e_1, e_2, \ldots, e_{n-1}\}$ forms an orthonormal basis of the tangent space $T_x(\partial\Omega_1)$ and e_n is the outward normal vector. Define the second fundamental form h as $h(v, w) = \langle \nabla_v e_n, w\rangle$, where v, w are vectors tangent to $\partial\Omega_1 = M$, and ∇ is the gradient operator with respect to the Riemannian metric on Ω_1. Then the mean curvature H of $\partial\Omega_1 = M$ is given by $H = \sum_{i=1}^{n-1}(h(e_i, e_i)/(n-1))$. To avoid confusion, as before, denote by $\widetilde{\Delta}$ and $\widetilde{\nabla}$ the Laplace and the gradient operators with respect to the induced Riemannian metric on $\partial\Omega_1 = M$ respectively. For $x \in \Omega_1$ and $X, Y \in T_x\Omega_1$, define the Hessian tensor $(D^2u)(X, Y) = X(Yu) - (\nabla_X Y)u$, where $\nabla_X Y$ is the covariant derivative of the Levi-Civita connection on Ω_1.

Now, we do computations in the above setting. As we know, for any $x \in \Omega_1$, one has

$$\Delta u = \sum_{i=1}^{n} D^2u(e_i, e_i) = \sum_{i=1}^{n} u_{ii}, \tag{2.3}$$

where $u_{ij} = D^2u(e_i, e_j)$ for any $i, j = 1, 2, \ldots, n$. On the other hand, when $x \in \partial\Omega_1$ and $i \neq n$, we have $\nabla_{e_i} e_i = \widetilde{\nabla}_{e_i} e_i + h_{ii} e_i$, where $h_{ii} = h(e_i, e_i)$. Therefore, combing (2.2) with (2.3), and together with the definition of the Hessian tensor, we can obtain

$$\begin{aligned}\Delta u &= u_{nn} + \widetilde{\Delta}\varphi + \sum_{i=1}^{n-1} h_{ii} e_n(u) \\ &= u_{nn} + \mathbb{L}_f \varphi + \langle \widetilde{\nabla} f, \widetilde{\nabla}\varphi \rangle_g + (n-1)He_n(u) \end{aligned} \tag{2.4}$$

for any $x \in \partial\Omega_1 = M$. Furthermore, by (2.1) and (2.4), we have

$$u_{nn} = \lambda_{1,f}\varphi - (n-1)He_n(u) - \langle \widetilde{\nabla} f, \widetilde{\nabla}\varphi \rangle_g \tag{2.5}$$

for $x \in \partial\Omega_1 = M$.

By the fact that $\Delta u = 0$ and Bochner's formula, for $x \in \Omega_1$, we have

$$\Delta|\nabla u|^2 = 2\sum_{ij}^{n} u_{ij}^2 + 2\sum_{ij}^{n} R_{ij} u_i u_j,$$

where $R_{ij} = \mathrm{Ric}(e_i, e_j)$. Together with the fact that the Ricci curvature of N is bounded from below by k, it follows that $\Delta|\nabla u|^2 \geqslant 2|D^2u|^2 + 2k|\nabla u|^2$, which implies that

$$\int_{\Omega_1} \Delta|\nabla u|^2 \geqslant 2\int_{\Omega_1} |D^2u|^2 + 2k\int_{\Omega_1} |\nabla u|^2. \tag{2.6}$$

Here we drop the volume element in all integrations in (2.6), and for convenience, we make an agreement that *the volume element will be dropped in every integration below.*

On the other hand, when $i \neq n$, $u_{in} = D^2u(e_i, e_n) = e_i(e_n u) - (\nabla_{e_i} e_n)u = e_i(u_n) - \sum_{j=1}^{n-1} h_{ij} u_j$. Furthermore, by using Green's formula, (2.1), (2.2) and (2.5), one can obtain

$$\int_{\Omega_1} \Delta|\nabla u|^2 = 2\int_{\partial\Omega_1} \sum_{i=1}^{n-1} u_i u_{in} + 2\int_{\partial\Omega_1} u_n u_{nn}$$

$$
\begin{aligned}
&= 2\int_{\partial\Omega_1} \widetilde{\nabla}\varphi \cdot \widetilde{\nabla} u_n - 2\int_{\partial\Omega_1} \sum_{i,j=1}^{n-1} h_{ij} u_i u_j + 2\int_{\partial\Omega_1} u_n u_{nn} \\
&= -2\int_{\partial\Omega_1} u_n \widetilde{\Delta}\varphi - 2\int_{\partial\Omega_1} h(\nabla u, \nabla u) + 2\int_{\partial\Omega_1} u_n u_{nn} \\
&= 4\lambda_{1,f}\int_{\partial\Omega_1} u_n\varphi - 4\int_{\partial\Omega_1} u_n \langle \widetilde{\nabla} f, \widetilde{\nabla}\varphi\rangle_g - 2\int_{\partial\Omega_1} h(\nabla u, \nabla u) \\
&\quad - 2(n-1)\int_{\partial\Omega_1} H u_n^2 \\
&= 4\lambda_{1,f}\int_{\partial\Omega_1} u_n\varphi - 4\int_{\partial\Omega_1} \sum_{i=1}^{n-1} u_n f_i \varphi_i - 2\int_{\partial\Omega_1} h(\nabla u, \nabla u) \\
&\quad - 2(n-1)\int_{\partial\Omega_1} H u_n^2. \qquad (2.7)
\end{aligned}
$$

For the first term of the right hand side (RHS for short) of (2.7), by Green's formula and (2.2), we have

$$
4\lambda_{1,f}\int_{\Omega_1} |\nabla u|^2 = 4\lambda_{1,f}\Big(-\int_{\Omega_1} u\Delta u + \int_{\partial\Omega_1} u u_n\Big) = 4\lambda_{1,f}\int_{\partial\Omega_1} u_n\varphi. \quad (2.8)
$$

For the last term of the RHS of (2.7), by (2.2), Green's formula, and Hölder's inequality, we can obtain

$$
\begin{aligned}
&- 2(n-1)\int_{\partial\Omega_1} H u_n^2 \\
&\quad \leqslant 2(n-1)\max_M |H| \int_{\partial\Omega_1} u_n^2 \\
&\quad = 2(n-1)\max_M |H| \Big(\int_{\Omega_1} u_n \Delta u + \int_{\Omega_1} \nabla u_n \cdot \nabla u\Big) \\
&\quad \leqslant 2(n-1)\max_M |H| \Big(\int_{\Omega_1} |\nabla u_n|^2\Big)^{1/2} \Big(\int_{\Omega_1} |\nabla u|^2\Big)^{1/2}. \qquad (2.9)
\end{aligned}
$$

Consider the second term of the RHS of (2.7), by two constraints for the weight function f in the assumption of Theorem 1.1, (2.2), Green's formula, and Hölder's inequality, one can get

$$
\begin{aligned}
&-4\int_{\partial\Omega_1}\sum_{i=1}^{n-1}u_nf_i\varphi_i\\
&=-4\int_{\partial\Omega_1}\sum_{i=1}^{n-1}u_nf_iu_i\\
&\leqslant 4\Big|\int_{\Omega_1}\Big(\sum_{i=1}^{n-1}f_iu_i\Big)\Delta u+\int_{\Omega_1}\nabla\Big(\sum_{i=1}^{n-1}f_iu_i\Big)\nabla u\Big|\\
&=4\Big|\sum_{i=1}^{n-1}\Big[\int_{\Omega_1}\Big(\nabla f_i\cdot\nabla u\Big)u_i+\int_{\Omega_1}\Big(\nabla u_i\cdot\nabla u\Big)f_i\Big]\Big|\\
&\leqslant 4\sum_{i=1}^{n-1}\Big[\Big(\int_{\Omega_1}|\nabla f_i|^2u_i^2\Big)^{1/2}\Big(\int_{\Omega_1}|\nabla u|^2\Big)^{1/2}\\
&\qquad\qquad+\Big(\int_{\Omega_1}|\nabla u_i|^2|f_i|^2\Big)^{1/2}\Big(\int_{\Omega_1}|\nabla u|^2\Big)^{1/2}\Big]\\
&\leqslant 4\sum_{i=1}^{n-1}\Big[\sqrt{n-1}\cdot C_2\Big(\int_{\Omega_1}u_i^2\Big)^{1/2}+C_1\Big(\int_{\Omega_1}|\nabla u_i|^2\Big)^{1/2}\Big]\\
&\quad\times\Big(\int_{\Omega_1}|\nabla u|^2\Big)^{1/2}\\
&\leqslant 4C_3\sum_{i=1}^{n-1}\Big[\Big(\int_{\Omega_1}u_i^2\Big)^{1/2}+\Big(\int_{\Omega_1}|\nabla u_i|^2\Big)^{1/2}\Big]\Big(\int_{\Omega_1}|\nabla u|^2\Big)^{1/2},
\end{aligned}\tag{2.10}
$$

where, as defined before, $C_3=\max\{\sqrt{n-1}\cdot C_2, C_1\}$. Since $\int_{\partial\Omega_1}h(\nabla u,\nabla u)$ $=\int_M h(\nabla\varphi,\nabla\varphi)$ and the outward normal vector of $\partial\Omega_2$ is $-e_n$, we have $\int_{\partial\Omega_1}h(\nabla u,\nabla u)=-\int_{\partial\Omega_2}h(\nabla u,\nabla u)$. So, without loss of generality, we can assume $\int_{\partial\Omega_1}h(\nabla u,\nabla u)\geqslant 0$, otherwise, we can work on Ω_2 rather than Ω_1. Putting (2.8), (2.9), (2.10) into (2.7), and together with the fact that $\int_{\partial\Omega_1}h(\nabla u,\nabla u)\geqslant 0$, it follows that

$$
\begin{aligned}
&\int_{\Omega_1}\Delta|\nabla u|^2\\
&\quad\leqslant 4\lambda_{1,f}\int_{\Omega_1}|\nabla u|^2+2(n-1)\max_M|H|\Big(\int_{\Omega_1}|\nabla u_n|^2\Big)^{1/2}\Big(\int_{\Omega_1}|\nabla u|^2\Big)^{1/2}
\end{aligned}
$$

$$+ 4C_3 \sum_{i=1}^{n-1} \left[\left(\int_{\Omega_1} u_i^2 \right)^{1/2} + \left(\int_{\Omega_1} |\nabla u_i|^2 \right)^{1/2} \right] \left(\int_{\Omega_1} |\nabla u|^2 \right)^{1/2}. \quad (2.11)$$

Combining (2.6) and (2.11), we have

$$\begin{aligned} &(2\lambda_{1,f} - k) \int_{\Omega_1} |\nabla u|^2 \\ &\quad \geqslant \int_{\Omega_1} |D^2 u|^2 - (n-1) \max_M |H| \left(\int_{\Omega_1} |\nabla u_n|^2 \right)^{1/2} \left(\int_{\Omega_1} |\nabla u|^2 \right)^{1/2} \\ &\qquad - 2C_3 \sum_{i=1}^{n-1} \left[\left(\int_{\Omega_1} u_i^2 \right)^{1/2} + \left(\int_{\Omega_1} |\nabla u_i|^2 \right)^{1/2} \right] \left(\int_{\Omega_1} |\nabla u|^2 \right)^{1/2}. \end{aligned} \quad (2.12)$$

Now, we divide the rest of the proof into two steps.

Step 1. If furthermore M is a minimal hypersurface, then $H \equiv 0$, which implies that the second term of the RHS of (2.12) vanishes identically. So, in this setting, we have

$$\begin{aligned} &(2\lambda_{1,f} - k) \int_{\Omega_1} |\nabla u|^2 \\ &\quad \geqslant \int_{\Omega_1} |D^2 u|^2 - 2C_3 \sum_{i=1}^{n-1} \left[\left(\int_{\Omega_1} u_i^2 \right)^{1/2} + \left(\int_{\Omega_1} |\nabla u_i|^2 \right)^{1/2} \right] \\ &\qquad \times \left(\int_{\Omega_1} |\nabla u|^2 \right)^{1/2}. \end{aligned} \quad (2.13)$$

In the following, for convenience, set $\sharp := \sum_{i=1}^{n-1} \left[\left(\int_{\Omega_1} u_i^2 \right)^{1/2} + \left(\int_{\Omega_1} |\nabla u_i|^2 \right)^{1/2} \right]$. Consider (2.13) into the following two cases:

Case (A). If $\sharp \leqslant \left(\int_{\Omega_1} |\nabla u|^2 \right)^{1/2}$, then from (2.13), one can get

$$(2\lambda_{1,f} - k) \int_{\Omega_1} |\nabla u|^2 \geqslant \int_{\Omega_1} |D^2 u|^2 - 2C_3 \int_{\Omega_1} |\nabla u|^2 \geqslant -2C_3 \int_{\Omega_1} |\nabla u|^2,$$

which implies $2\lambda_{1,f} \geqslant k - 2C_3$. This is because the harmonic function u cannot be constant in Ω_1 (otherwise it contradicts the boundary condition $u = \varphi$ on $\partial\Omega$) and so $\int_{\Omega_1} |\nabla u|^2 > 0$.

Case (B). If $\sharp \geqslant \left(\int_{\Omega_1} |\nabla u|^2\right)^{1/2}$, then we have

$$-2C_3\sharp^2 \leqslant -2C_3 \cdot \sharp \cdot \left(\int_{\Omega_1} |\nabla u|^2\right)^{1/2}.$$

Together with (2.13), one can obtain

$$(2\lambda_{1,f} - k)\int_{\Omega_1} |\nabla u|^2 + 2C_3\sharp^2 \geqslant \int_{\Omega_1} |D^2 u|^2 \geqslant 0.$$

Therefore, we have

$$(2\lambda_{1,f} - k + 2C_3)\sharp^2 \geqslant (2\lambda_{1,f} - k)\int_{\Omega_1} |\nabla u|^2 + 2C_3\sharp^2 \geqslant \int_{\Omega_1} |D^2 u|^2 \geqslant 0,$$

which implies that $2\lambda_{1,f} - k + 2C_3 \geqslant 0$. This fact holds since in this case $\sharp^2 \geqslant \int_{\Omega_1} |\nabla u|^2 > 0$.

Step 2. If M is not a minimal hypersurface, then there must exist some point $x_0 \in M$ such that $H(x_0) \neq 0$, which implies that $\max_M |H| > 0$. So, in this setting, by (2.12) we have

$$\begin{aligned}
&(2\lambda_{1,f} - k)\int_{\Omega_1} |\nabla u|^2 \\
&\quad \geqslant -(n-1)\max_M |H| \left(\int_{\Omega_1} |\nabla u_n|^2\right)^{1/2} \left(\int_{\Omega_1} |\nabla u|^2\right)^{1/2} \\
&\qquad - 2C_3\sharp \left(\int_{\Omega_1} |\nabla u|^2\right)^{1/2} + \int_{\Omega_1} |D^2 u|^2 \\
&\quad = -(n-1)\max_M |H| \left[\left(\int_{\Omega_1} |\nabla u_n|^2\right)^{1/2} + \frac{2C_3\sharp}{(n-1)\max_M |H|}\right] \left(\int_{\Omega_1} |\nabla u|^2\right)^{1/2} \\
&\qquad + \int_{\Omega_1} |D^2 u|^2. \qquad (2.14)
\end{aligned}$$

For convenience, set $\flat := \left(\int_{\Omega_1} |\nabla u_n|^2\right)^{1/2} + 2C_3\sharp/((n-1)\max_M |H|)$. As in Step 1, we consider (2.14) into the following two cases:

Case (C). If $\flat \leqslant \left(\int_{\Omega_1} |\nabla u|^2\right)^{1/2}$, then from (2.14) and the fact that

$\int_{\Omega_1} |\nabla u|^2 > 0$, we have

$$(2\lambda_{1,f} - k)\int_{\Omega_1} |\nabla u|^2 \geqslant \int_{\Omega_1} |D^2u|^2 - (n-1)\max_M |H| \int_{\Omega_1} |\nabla u|^2$$
$$\geqslant -(n-1)\max_M |H| \int_{\Omega_1} |\nabla u|^2,$$

which implies $2\lambda_{1,f} \geqslant k - (n-1)\max_M |H|$.

Case (D). If $\flat \geqslant \big(\int_{\Omega_1} |\nabla u|^2\big)^{1/2}$, then

$$-(n-1)\max_M |H| \cdot \flat^2 \leqslant -(n-1)\max_M |H| \cdot \flat \bigg(\int_{\Omega_1} |\nabla u|^2\bigg)^{1/2}.$$

Together with (2.14), we can obtain

$$(2\lambda_{1,f} - k)\int_{\Omega_1} |\nabla u|^2 + (n-1)\max_M |H| \cdot \flat^2 \geqslant \int_{\Omega_1} |D^2u|^2 \geqslant 0.$$

Hence, one has

$$\Big(2\lambda_{1,f} - k + (n-1)\max_M |H|\Big) \cdot \flat^2$$
$$\geqslant (2\lambda_{1,f} - k)\int_{\Omega_1} |\nabla u|^2 + (n-1)\max_M |H| \cdot \flat^2 \geqslant \int_{\Omega_1} |D^2u|^2 \geqslant 0,$$

which implies that $2\lambda_{1,f} - k + (n-1)\max_M |H| \geqslant 0$. This inequality holds since in this case $\flat^2 \geqslant \int_{\Omega_1} |\nabla u|^2 > 0$.

Through the above two steps, we have proved two eigenvalue inequalities involved $\lambda_{1,f}$ in Theorem 1.1. At the end, we would like to give the proof of the last assertion of Theorem 1.1. In fact, by the argument in Steps 1 and 2, if equalities in the two eigenvalue inequalities in Theorem 1.1 holds, we must have $\int_{\Omega_1} |D^2u|^2 = 0$, which implies that $u_{ij} = 0$ on Ω_1 for all $1 \leqslant i, j \leqslant n$. Furthermore, we have $\varphi_{ij} = 0$ on M for all $1 \leqslant i, j \leqslant n-1$, since u is smooth up to $\partial\Omega_1 = M$ and $u = \varphi$ on Ω_1. It follows that $\widetilde{\Delta}\varphi = 0$ in M. This is equivalent to say that $\mathbb{L}_f\varphi = 0$ in M when f is a constant function. However, this is a contradiction since φ is the first eigenfunction of $\mathbb{L}_f$. Therefore, when f is a constant function, equalities in the two eigenvalue inequalities in Theorem 1.1 cannot be achieved.

The proof of Theorem 1.1 is finished. □

Remark 2.1 For $\Phi \in C^{\infty}(N)$, the so-called *weighted mean curvature* H_{Φ} (see, e.g., [7] for this notion about a generalization of the mean curvature) can be well-defined on the ambient manifold N. *Maybe it is possible to* get analogous estimates for the first nonzero eigenvalue λ_1 of $\widetilde{\Delta}$ on M under $\mathrm{Ric}_{\Phi} \geqslant k$ on N and some conditions on H_{Φ}.

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