Hokkaido Mathematical Journal Vol. 47 (2018) p. 487–500

The existence of Leray-Hopf weak solutions with linear strain

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(Received January 13, 2016; Revised August 31, 2017)

Abstract. This paper deals with the global existence of weak solutions to the initial value problem for the Navier-Stokes equations in $\mathbb{R}^n$ ($n \in \mathbb{Z}$, $n \geq 2$). Concerning initial data of the form $Ax + v(0)$, where $A \in M_n(\mathbb{R})$ and $v(0) \in L^2_\sigma(\mathbb{R}^n)$, the weak solutions are properly-defined with the aid of the alternativity of the trilinear from $(Ax \cdot \nabla)v \cdot \varphi$. Furthermore, we construct the Leray-Hopf weak solution which satisfies not only the Navier-Stokes equations but also the energy inequality via the Galerkin approximation. From the viewpoint of quadratic forms, the Gronwall-Bellman inequality admits the uniform boundedness of the approximate solution.

Key words: Navier-Stokes equations, Leray-Hopf weak solutions, Linear strain.

1. Introduction

Let $n \in \mathbb{Z}$, $n \geq 2$ and $T > 0$. Motion of incompressible viscous fluids with linear strain in $\mathbb{R}^n$ is described by the initial value problem for the Navier-Stokes equations as follows:

$$\begin{cases} \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \rho\{\partial_t + (u \cdot \nabla)\}u + \nabla p - \mu \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u|_{t=0} = Ax + v(0) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u = (u_1, \dots, u_n)^T$ is the fluid velocity, p is the pressure, ρ is the density, μ is the coefficient of viscosity, $A \in M_n(\mathbb{R})$, i.e., A is a real square matrix of order n and $\cdot^T$ is the transposition. These equations correspond to the laws of conservation of mass and momentum respectively. Moreover, it is required that ρ and μ are positive constants. See, for example, Lamb [5] and Serrin [10] on conservation laws of fluid motion and derivation of the above equations.

This paper is concerned with the fluid velocity perturbation $v := u - Ax$ from linear strain Ax. Note that Ax is characterized as an exact solution to

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2010 Mathematics Subject Classification : Primary 35Q30; Secondary 76D03, 76D05.

(1.1). Indeed,

$$u := Ax, \quad p := -\frac{\rho}{2}Ax \cdot A^T x$$

is a stationary solution to (1.1) provided that $\mathrm{tr}A = 0$ and $A^2 \in S_n(\mathbb{R})$, i.e., A^2 is a real symmetric matrix of order n. See, for example, Majda and Bertozzi [7] and Okamoto [9] on exact solutions to the Navier-Stokes equations and their fluid mechanical properties. Substituting

$$v := u - Ax, \quad q := p + \frac{\rho}{2}Ax \cdot A^T x$$

into (1.1), we consider a solution (v, q) to the following initial value problem:

$$\begin{cases} \operatorname{div} v = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \rho\{\partial_t v + (v \cdot \nabla)v + (Ax \cdot \nabla)v + Av\} + \nabla q - \mu \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ v|_{t=0} = v(0) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2}$$

In the case where A is a zero matrix, there are many results on the global existence and kinetic energy of weak solutions to (1.1) or equivalently, (1.2). For any $v(0) \in L^2_\sigma(\mathbb{R}^3)$ and $T > 0$, Leray [6] constructed a weak solution u to (1.1) such that the energy inequality

$$\frac{1}{2}\|u(t)\|^2 + \frac{\mu}{\rho}\int_0^t \|\nabla u(\tau)\|^2 d\tau \le \frac{1}{2}\|v(0)\|^2 \tag{1.3}$$

holds for any $0 < t < T$ via the heat kernel. Moreover, it follows from Hopf [4] that the Galerkin approximation works in domains rather than $\mathbb{R}^3$. On the other hand, Masuda [8] proved the global existence of weak solutions to (1.1) which seem to have a somewhat stronger property than Leray-Hopf weak solutions to (1.1). If A is a non-zero matrix, Campiti, Galdi and Hieber [1] recently obtained the global existence and uniqueness of strong solutions to (1.2) in the case of $n = 2$. However, it is unknown except for $n = 2$ whether (1.2) admits the global existence of weak solutions or not. Concerning the local existence and uniqueness of mild solutions to (1.2), Hieber and Sawada [2] defined the operator A_p in $L^p_\sigma(\mathbb{R}^n)$ $(1 < p < \infty)$ as

$$A_p = -\Delta + (Ax \cdot \nabla) - A$$

with its domain $D(A_p) := \{u \in (W^{2,p}(\mathbb{R}^n))^n \cap L^p_\sigma(\mathbb{R}^n); (Ax \cdot \nabla)u \in (L^p(\mathbb{R}^n))^n\}$, and proved that $-A_p$ generates a C_0(but non-analytic)-semigroup $\{e^{-tA_p}\}_{t\geq 0}$ on $L^p_\sigma(\mathbb{R}^n)$. Applying L^p-L^q smoothing estimates for $\{e^{-tA_p}\}_{t\geq 0}$ to the successive approximation, the local existence and uniqueness result was given as follows: Let $n \leq p < \infty$, $p \leq q \leq \infty$ and $v(0) \in L^p_\sigma(\mathbb{R}^n)$. Then there exists $T_* > 0$ such that (1.2) uniquely has a mild solution v satisfying

$$t^{n/2(1/p-1/q)} v \in C([0, T_*); L^q_\sigma(\mathbb{R}^n)).$$

The aim of this paper is to establish the global existence of Leray-Hopf weak solutions to (1.2). More precisely, weak solutions to (1.2) are properly-defined with the aid of

$$\int_{\mathbb{R}^n} (Ax \cdot \nabla)v \cdot \varphi dx = -\int_{\mathbb{R}^n} v \cdot (Ax \cdot \nabla)\varphi dx$$

for any $v \in H^1_\sigma(\mathbb{R}^n)$ and $\varphi \in C^1_{0,\sigma}(\mathbb{R}^n)$. Furthermore, we construct a weak solution v to (1.2) such that the energy inequality

$$\frac{1}{2}\|v(t)\|^2 + \int_0^t (Av(\tau), v(\tau))d\tau + \frac{\mu}{\rho}\int_0^t \|\nabla v(\tau)\|^2 d\tau \leq \frac{1}{2}\|v(0)\|^2 \tag{1.4}$$

holds for any $0 < t < T$ via the Galerkin approximation. The crucial point is that the quadratic inequality

$$Ax \cdot x \geq a|x|^2 \tag{1.5}$$

holds for any $x \in \mathbb{R}^n$, where $a := \min\{\lambda; \lambda \in \sigma(S)\}$ and $S := (1/2)(A + A^T)$. Note that $\mathrm{tr}\, A = 0$ implies $a \leq 0$. Then it follows from (1.4), (1.5) and the Gronwall-Bellman inequality that a priori estimate

$$\frac{1}{2}\|v(t)\|^2 + \frac{\mu}{\rho}\int_0^t \|\nabla v(\tau)\|^2 d\tau \leq \frac{1}{2}\|v(0)\|^2 \exp(-2at) \tag{1.6}$$

holds for any $0 < t < T$. By virtue of the global existence result of Leray-Hopf weak solutions to (1.2), the fluid velocity perturbation $v = u - Ax$

is in the energy class $C_w([0,T];L^2_\sigma(\mathbb{R}^n)) \cap L^2((0,T);H^1_\sigma(\mathbb{R}^n))$. Moreover, our main result includes the case of the Navier-Stokes equations with the Coriolis force, e.g., Hieber and Shibata [3].

This paper is organized as follows: In Subsection 2.1, we define basic notation used in this paper. Subsection 2.2 provides the notion of weak solutions and Leray-Hopf weak solutions to (1.2) and our main result. In Subsection 3.1, we state some auxiliary lemmas. Finally, the global existence of Leray-Hopf weak solutions to (1.2) is established in Subsection 3.2.

2. Preliminaries and a main result

2.1. Function spaces

Function spaces and basic notation which we use throughout this paper are introduced as follows: $M_n(\mathbb{R})$ is the set of all real square matrices of order n. In particular, $S_n(\mathbb{R})$ is the set of all real symmetric matrix of order n. For any $A \in M_n(\mathbb{R})$, the spectrum of A is denoted by $\sigma(A)$.

$L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ and $H^k(\mathbb{R}^n)$ $(k \in \mathbb{Z},\ k \geq 0)$ are the Lebesgue and L^2-Sobolev spaces respectively. Moreover, the scalar product and the norm in $L^2(\mathbb{R}^n)$ is denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. Let us introduce solenoidal function spaces. $C^k(\mathbb{R}^n)$ $(k \in \mathbb{Z},\ k \geq 1)$ is the space of all functions in $\mathbb{R}^n$ which are continuously differentiable up to order k in $\mathbb{R}^n$. In particular, we denote by $C^k_0(\mathbb{R}^n)$ the space of all C^k-functions in $\mathbb{R}^n$ whose support are compact. Set $C^k_{0,\sigma}(\mathbb{R}^n) := \{u \in (C^k_0(\mathbb{R}^n))^n; \operatorname{div} u = 0\}$. $L^p_\sigma(\mathbb{R}^n)$ $(1 < p < \infty)$ and $H^1_\sigma(\mathbb{R}^n)$ are the completions of $C^1_{0,\sigma}(\mathbb{R}^n)$ in $(L^p(\mathbb{R}^n))^n$ and in $(H^1(\mathbb{R}^n))^n$ respectively.

Let I be a bounded open interval in $\mathbb{R}$, and X be a Banach space. $L^q(I;X)$ $(1 \leq q \leq \infty)$ and $H^k(I;X)$ $(k \in \mathbb{Z},\ k \geq 0)$ are the Lebesgue and L^2-Sobolev spaces of X-valued functions respectively.

Let I be a bounded closed interval in $\mathbb{R}$, and X be a Banach space. $C(I;X)$ is the Banach space of all X-valued functions which are continuous in I. In particular, $C^k(I;X)$ $(k \in \mathbb{Z},\ k \geq 1)$ is the Banach space of all X-valued functions which are continuously differentiable up to order k in I.

Let I be a bounded closed interval in $\mathbb{R}$, and X be a Hilbert space with the scalar product $(\cdot,\cdot)_X$. As for the Banach space of weak continuous functions, we set $C_w(I;X) := \{u : I \to X; \forall\varphi \in X,\ (u,\varphi)_X \in C(I;\mathbb{R})\}$.

2.2. Leray-Hopf weak solutions to (1.2) and a main result

This subsection provides the notion of weak solutions and Leray-Hopf weak solutions to (1.2) and our main result. First, we define weak solutions to (1.2) which satisfy the weak formulation of (1.2).

Definition 2.1 Let $A \in M_n(\mathbb{R})$, $v(0) \in L^2_\sigma(\mathbb{R}^n)$ and $T > 0$. Then $v \in C_w([0,T]; L^2_\sigma(\mathbb{R}^n)) \cap L^2((0,T); H^1_\sigma(\mathbb{R}^n))$ is called a weak solution to (1.2) if the functional equation

$$\begin{aligned}
&- \int_s^t (v(\tau), \partial_\tau \varphi(\tau))d\tau + \int_s^t ((v(\tau)\cdot\nabla)v(\tau), \varphi(\tau))d\tau \\
&\quad - \int_s^t (v(\tau), (Ax\cdot\nabla)\varphi(\tau))d\tau + \int_s^t (Av(\tau), \varphi(\tau))d\tau \\
&\quad + \frac{\mu}{\rho}\int_s^t (\nabla v(\tau), \nabla\varphi(\tau))d\tau = -(v(t), \varphi(t)) + (v(s), \varphi(s)) \qquad (2.1)
\end{aligned}$$

holds for any $\varphi \in H^1((s,t); C^1_{0,\sigma}(\mathbb{R}^n))$ and $0 \le s < t < T$.

Second, we introduce Leray-Hopf weak solutions to (1.2) which satisfy not only the weak formulation of (1.2) but also the energy inequality, i.e., (1.4).

Definition 2.2 Let $A \in M_n(\mathbb{R})$, $v(0) \in L^2_\sigma(\mathbb{R}^n)$ and $T > 0$. Then a weak solution $v \in C_w([0,T]; L^2_\sigma(\mathbb{R}^n)) \cap L^2((0,T); H^1_\sigma(\mathbb{R}^n))$ to (1.2) is called a Leray-Hopf weak solution to (1.2) if the energy inequality

$$\frac{1}{2}\|v(t)\|^2 + \int_0^t (Av(\tau), v(\tau))d\tau + \frac{\mu}{\rho}\int_0^t \|\nabla v(\tau)\|^2 d\tau \le \frac{1}{2}\|v(0)\|^2 \qquad (2.2)$$

holds for any $0 < t < T$.

Finally, we state our main result of this paper, i.e., the global existence result of Leray-Hopf weak solutions to (1.2) is established. The following theorem means that for any $T > 0$, a Leray-Hopf weak solution to (1.2) exists in $\mathbb{R}^n \times (0,T)$.

Theorem 2.1 *Let $A \in M_n(\mathbb{R})$, $\mathrm{tr}A = 0$, $v(0) \in L^2_\sigma(\mathbb{R}^n)$ and $T > 0$. Then (1.2) has a Leray-Hopf weak solution $v \in C_w([0,T]; L^2_\sigma(\mathbb{R}^n)) \cap L^2((0,T); H^1_\sigma(\mathbb{R}^n))$ satisfying*

$$\lim_{t\to+0}\|v(t)-v(0)\|=0. \tag{2.3}$$

Remark 2.1 In the case where A is a zero matrix, Theorems 2.1 is [8, Theorem 1].

3. Proof of a main result

3.1. Auxiliary Lemmas

Some auxiliary lemmas are given in this subsection. First, we have the following lemma on the approximation of functions in $L^2((s,t);X)$ and in $H^1((s,t);X)$:

Lemma 3.1 *Let X be a Banach space with the norm $\|\cdot\|_X$, Y be a dense subset of X and $0\le s<t$, and set*

$$F([s,t];Y):=\left\{\sum_{\text{finite}} a_k\psi_k;\, a_k\in C^1([s,t];\mathbb{R}),\ \psi_k\in Y\right\}.$$

Then

(1) *For any $\varphi\in L^2((s,t);X)$, there exists a sequence $\{\varphi_m; m\in\mathbb{N}\}$ in $F([s,t];Y)$ satisfying*

$$\lim_{m\to\infty}\varphi_m=\varphi \text{ in } L^2((s,t);X).$$

(2) *For any $\varphi\in H^1((s,t);X)$, there exists a sequence $\{\varphi_m; m\in\mathbb{N}\}$ in $F([s,t];Y)$ satisfying*

$$\lim_{m\to\infty}\varphi_m=\varphi \text{ in } H^1((s,t);X).$$

Proof. See [8, Lemma 2.2]. □

Second, we proceed to the alternativity of the trilinear form $((u\cdot\nabla)v,w)$. This property is established as follows:

Lemma 3.2 *Let $u\in H^1_\sigma(\mathbb{R}^n)\cap L^n_\sigma(\mathbb{R}^n)$. Then*

$$((u\cdot\nabla)v,w)=-((u\cdot\nabla)w,v) \tag{3.1}$$

holds for any $v,w\in H^1_\sigma(\mathbb{R}^n)$.

Proof. See [8, Lemma 2.3]. □

Finally, the Friedrichs inequality for the trilinear form $((u \cdot \nabla)v, w)$ is given. The following lemma is used for the equicontinuity.

Lemma 3.3 *Let $T > 0$ and $w \in C([0,T]; L^n_\sigma(\mathbb{R}^n))$. Then for any $\varepsilon > 0$, there exists $C(T,w,\varepsilon) > 0$, $N(T,w,\varepsilon) \in \mathbb{N}$ and a sequence $\{\varphi_k; k \in \{1,\dots,N(T,w,\varepsilon)\}\}$ in $L^2_\sigma(\mathbb{R}^n)$ depending only on n, T, w and ε such that*

$$\begin{aligned}\int_s^t &|((u(\tau)\cdot\nabla)v(\tau), w(\tau))|d\tau \\ &\le \varepsilon \int_s^t (\|\nabla u(\tau)\|^2 + \|u(\tau)\|\|\nabla v(\tau)\| + \|\nabla v(\tau)\|^2)d\tau \\ &\quad + C(T,w,\varepsilon)\sum_{k=1}^{N(T,w,\varepsilon)} \int_s^t |(u(\tau),\varphi_k)|^2 d\tau \end{aligned} \tag{3.2}$$

holds for any $u, v \in L^2((s,t); H^1_\sigma(\mathbb{R}^n))$ and $0 \le s < t \le T$.

Proof. See [8, Lemma 2.5]. □

3.2. Proof of Theorem 2.1

In this subsection, we will prove Theorem 2.1. Since $H^1_\sigma(\mathbb{R}^n) \cap L^n_\sigma(\mathbb{R}^n)$ is a separable Banach space and $C^1_{0,\sigma}(\mathbb{R}^n)$ is dense in $H^1_\sigma(\mathbb{R}^n) \cap L^n_\sigma(\mathbb{R}^n)$, see [8, Proposition 1 and Lemma 3.1], there exists a linearly independent total sequence $\{\psi_k; k \in \mathbb{N}\}$ in $H^1_\sigma(\mathbb{R}^n) \cap L^n_\sigma(\mathbb{R}^n)$ which admits $\psi_k \in C^1_{0,\sigma}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$. Note that $C^1_{0,\sigma}(\mathbb{R}^n) \subseteq H^1_\sigma(\mathbb{R}^n) \cap L^n_\sigma(\mathbb{R}^n) \subseteq L^2_\sigma(\mathbb{R}^n)$ and $C^1_{0,\sigma}(\mathbb{R}^n)$ is dense in $L^2_\sigma(\mathbb{R}^n)$. Without loss of generality, we may assume that $\{\psi_k; k \in \mathbb{N}\}$ is an orthonormal basis for $L^2_\sigma(\mathbb{R}^n)$. The approximate solution v_m to (1.2) of the form

$$v_m(x,t) := \sum_{k=1}^m b_k(t)\psi_k(x) \quad (x \in \mathbb{R}^n,\ 0 \le t \le T)$$

is constructed by the sequence $\{b_k; k \in \{1,\dots,m\}\}$ in $C^1([0,T];\mathbb{R})$ which is a solution to the following initial value problem for the system of m ordinary differential equations:

$$b_k'(t) + \sum_{(i,j)=(1,1)}^{(m,m)} ((\psi_i \cdot \nabla)\psi_j, \psi_k) b_i(t) b_j(t) - \sum_{i=1}^{m} (\psi_i, (Ax \cdot \nabla)\psi_k) b_i(t)$$
$$+ \sum_{i=1}^{m} (A\psi_i, \psi_k) b_i(t) + \frac{\mu}{\rho} \sum_{i=1}^{m} (\nabla\psi_i, \nabla\psi_k) b_i(t) = 0$$
$$(k \in \{1, \dots, m\}), \qquad (3.3)$$
$$b_k(0) = (v(0), \psi_k) \quad (k \in \{1, \dots, m\}). \qquad (3.4)$$

Note that (3.3), (3.4) uniquely has a solution $\{b_k; k \in \{1, \dots, m\}\}$ in $C^1([0,T];\mathbb{R})$. Furthermore, (3.3) is rewritten to the system of m ordinary differential equations

$$(\partial_t v_m(t), \psi_k) + ((v_m(t) \cdot \nabla) v_m(t), \psi_k) - (v_m(t), (Ax \cdot \nabla)\psi_k)$$
$$+ (A v_m(t), \psi_k) + \frac{\mu}{\rho} (\nabla v_m(t), \nabla\psi_k) = 0 \quad (k \in \{1, \dots, m\}). \qquad (3.5)$$

First, for any $k \in \mathbb{N}$, the (uniform) boundedness of $\{(v_m, \psi_k); m \in \mathbb{N}\}$ in $C([0,T];\mathbb{R})$ is derived from the following lemma on the energy equality for v_m:

Lemma 3.4 *The energy equality*

$$\frac{1}{2}\|v_m(t)\|^2 + \int_s^t (A v_m(\tau), v_m(\tau)) d\tau + \frac{\mu}{\rho} \int_s^t \|\nabla v_m(\tau)\|^2 d\tau = \frac{1}{2}\|v_m(s)\|^2 \qquad (3.6)$$

holds for any $0 \le s < t \le T$.

Proof. After multiplying (3.5) by b_k, we integrate it with respect to time over $[s,t]$. This integration yields the system of m integral equations

$$\int_s^t (\partial_\tau v_m(\tau), b_k(\tau)\psi_k) d\tau + \int_s^t ((v_m(\tau) \cdot \nabla) v_m(\tau), b_k(\tau)\psi_k) d\tau$$
$$- \int_s^t (v_m(\tau), (Ax \cdot \nabla)(b_k(\tau)\psi_k)) d\tau + \int_s^t (A v_m(\tau), b_k(\tau)\psi_k) d\tau$$
$$+ \frac{\mu}{\rho} \int_s^t (\nabla v_m(\tau), \nabla(b_k(\tau)\psi_k)) d\tau = 0 \quad (k \in \{1, \dots, m\}). \qquad (3.7)$$

Since $v_m \in C^1([0,T]; C^1_{0,\sigma}(\mathbb{R}^n))$,

$$((v_m \cdot \nabla)v_m, v_m) = 0 = (v_m, (Ax \cdot \nabla)v_m)$$

follows from Lemma 3.2. Consequently, the sum of (3.7) with respect to $k \in \{1, \dots, m\}$ yields

$$\int_s^t (\partial_\tau v_m(\tau), v_m(\tau))d\tau + \int_s^t (Av_m(\tau), v_m(\tau))d\tau + \frac{\mu}{\rho}\int_s^t \|\nabla v_m(\tau)\|^2 d\tau = 0. \tag{3.8}$$

It is easy to see (3.6) from (3.8) and the fundamental theorem of calculus, which completes the proof of Lemma 3.4. □

The following lemma yields not only the (uniform) boundedness of $\{(v_m, \psi_k); m \in \mathbb{N}\}$ in $C([0,T];\mathbb{R})$ but also the energy inequality for weak solutions to (1.2).

Lemma 3.5 *The energy inequality*

$$\frac{1}{2}\|v_m(t)\|^2 + \int_0^t (Av_m(\tau), v_m(\tau))d\tau + \frac{\mu}{\rho}\int_0^t \|\nabla v_m(\tau)\|^2 d\tau \le \frac{1}{2}\|v(0)\|^2 \tag{3.9}$$

holds for any $0 < t \le T$.

Proof. Substituting $s = 0$ into Lemma 3.4, we have

$$\frac{1}{2}\|v_m(t)\|^2 + \int_0^t (Av_m(\tau), v_m(\tau))d\tau + \frac{\mu}{\rho}\int_0^t \|\nabla v_m(\tau)\|^2 d\tau = \frac{1}{2}\|v_m(0)\|^2.$$

Moreover, $\|v_m(0)\| \le \|v(0)\|$ follows from the Bessel inequality. This completes the proof of Lemma 3.5. □

By the quadratic inequality,

$$\int_0^t (Av_m(\tau), v_m(\tau))d\tau \ge a\int_0^t \|v_m(\tau)\|^2 d\tau \tag{3.10}$$

holds for any $0 < t \le T$, where $a = \min\{\lambda; \lambda \in \sigma(S)\}$ and $S = (1/2)(A + A^T)$. Note that $\mathrm{tr}A = 0$ implies $a \le 0$. Then it follows from Lemma 3.5 and (3.10) that

$$\frac{1}{2}\|v_m(t)\|^2 + \frac{\mu}{\rho}\int_0^t \|\nabla v_m(\tau)\|^2 d\tau \leq -a\int_0^t \|v_m(\tau)\|^2 d\tau + \frac{1}{2}\|v(0)\|^2 \quad (3.11)$$

holds for any $0 < t \leq T$. Applying the Gronwall-Bellman inequality to (3.11),

$$\frac{1}{2}\|v_m(t)\|^2 + \frac{\mu}{\rho}\int_0^t \|\nabla v_m(\tau)\|^2 d\tau \leq \frac{1}{2}\|v(0)\|^2 \exp(-2at) \quad (3.12)$$

holds for any $0 < t \leq T$. Therefore, for any $k \in \mathbb{N}$, the Schwarz inequality, (3.12) and $\|\psi_k\| = 1$ admit that $\{(v_m, \psi_k); m \in \mathbb{N}\}$ is (uniformly) bounded in $C([0,T];\mathbb{R})$.

Second, for any $k \in \mathbb{N}$, we proceed to the equicontinuity of $\{(v_m, \psi_k); m \in \mathbb{N}\}$ on $[0,T]$. It is easy to see from (3.5) and the fundamental theorem of calculus that

$$\begin{aligned}(v_m(t), \psi_k) - (v_m(s), \psi_k) &= \int_s^t (\partial_\tau v_m(\tau), \psi_k) d\tau \\ &= -\int_s^t ((v_m(\tau)\cdot\nabla)v_m(\tau), \psi_k) d\tau \\ &\quad + \int_s^t (v_m(\tau), (Ax\cdot\nabla)\psi_k) d\tau \\ &\quad - \int_s^t (Av_m(\tau), \psi_k) d\tau - \frac{\mu}{\rho}\int_s^t (\nabla v_m(\tau), \nabla\psi_k) d\tau \\ &=: I_1(s,t) + I_2(s,t) + I_3(s,t) + I_4(s,t) \qquad (3.13)\end{aligned}$$

holds for any $0 \leq s < t \leq T$. Concerning continuity properties of the above integrals in (3.13) with respect to time, the following two lemmas are established.

Lemma 3.6 *Let $I_1(s,t)$ be taken as in (3.13). Then for any $\varepsilon > 0$, there exists $C(k,\varepsilon) > 0$ depending only on n, k and ε such that*

$$|I_1(s,t)| \leq \left\{\frac{\rho}{4\mu}\varepsilon + C(k,\varepsilon)(t-s)\right\}\|v(0)\|^2 \exp(-2aT) \quad (3.14)$$

holds for any $0 \leq s < t \leq T$.

Proof. By Lemma 3.3 and the Schwarz inequality, for any $\varepsilon > 0$, there exists $C(k, \varepsilon) > 0$ depending only on n, k and ε such that

$$|I_1(s,t)| \leq \frac{\varepsilon}{2} \int_s^t \|\nabla v_m(\tau)\|^2 d\tau + C(k,\varepsilon) \int_s^t \|v_m(\tau)\|^2 d\tau$$

holds for any $0 \leq s < t \leq T$. Consequently, (3.14) follows from (3.12). □

Lemma 3.7 *Let $I_2(s,t)$, $I_3(s,t)$ and $I_4(s,t)$ be taken as in* (3.13)*. Then*

$$|I_2(s,t)| \leq (t-s)\|(Ax \cdot \nabla)\psi_k\| \|v(0)\| \exp(-aT) \tag{3.15}$$

holds for any $0 \leq s < t \leq T$,

$$|I_3(s,t)| \leq (t-s)|A| \|v(0)\| \exp(-aT) \tag{3.16}$$

holds for any $0 \leq s < t \leq T$, *and*

$$|I_4(s,t)| \leq \left\{ \frac{\mu}{2\rho}(t-s) \right\}^{1/2} \|\nabla \psi_k\| \|v(0)\| \exp(-aT) \tag{3.17}$$

holds for any $0 \leq s < t \leq T$.

Proof. Analogously to the proof of Lemma 3.6, (3.15), (3.16) and (3.17) are derived from the Schwarz inequality, (3.12) and $\|\psi_k\| = 1$. □

Combining (3.13) with Lemmas 3.6 and 3.7, for any $\varepsilon > 0$, there exists $\delta(T, k, \varepsilon) > 0$ depending only on n, ρ, μ, a, $v(0)$, T, k and ε such that

$$|(v_m(t), \psi_k) - (v_m(s), \psi_k)| < \varepsilon$$

holds for any $m \in \mathbb{N}$ and $0 \leq s, t \leq T$ satisfying $|t-s| < \delta(T, k, \varepsilon)$. Therefore, for any $k \in \mathbb{N}$, $\{(v_m, \psi_k); m \in \mathbb{N}\}$ is equicontinuous on $[0, T]$.

The proof of Theorem 2.1 is based on the following lemma on the convergence of the approximate solution v_m to (1.2). Hereafter, a subsequence of $\{v_m; m \in \mathbb{N}\}$ is denoted by $\{v_m; m \in \mathbb{N}\}$ itself for the sake of simplicity of the notation.

Lemma 3.8 *There exists* $v \in C_w([0,T]; L^2_\sigma(\mathbb{R}^n)) \cap L^2((0,T); H^1_\sigma(\mathbb{R}^n))$ *satisfying*

$$\lim_{m\to\infty} v_m = v \textit{ weakly in } L^2((0,T);H^1_\sigma(\mathbb{R}^n))$$

and

$$\lim_{m\to\infty} v_m = v \text{ in } C_w([0,T];L^2_\sigma(\mathbb{R}^n)).$$

Proof. Since $\{v_m; m \in \mathbb{N}\}$ is (uniformly) bounded in $C([0,T];L^2_\sigma(\mathbb{R}^n))$ and in $L^2((0,T);H^1_\sigma(\mathbb{R}^n))$, which follows from (3.12), there exists $v \in L^2((0,T);H^1_\sigma(\mathbb{R}^n))$ satisfying

$$\lim_{m\to\infty} v_m = v \text{ weakly in } L^2((0,T);H^1_\sigma(\mathbb{R}^n)).$$

As is proved above, for any $k \in \mathbb{N}$, $\{(v_m, \psi_k); m \in \mathbb{N}\}$ is (uniformly) bounded in $C([0,T];\mathbb{R})$ and equicontinuous on $[0,T]$. Recall that $\{\psi_k; k \in \mathbb{N}\}$ is an orthonormal basis for $L^2_\sigma(\mathbb{R}^n)$. Then, by the Arzelà-Ascoli theorem and Cantor's diagonal argument, v is also in $C_w([0,T];L^2_\sigma(\mathbb{R}^n))$, and

$$\lim_{m\to\infty} v_m = v \text{ in } C_w([0,T];L^2_\sigma(\mathbb{R}^n)).$$

This completes the proof of Lemma 3.8. □

Finally, we will prove that v is a Leray-Hopf weak solution to (1.2). Let $0 \le s < t < T$, and set

$$F([s,t];\operatorname{span}\{\psi_k; k \in \mathbb{N}\}) := \left\{ \sum_{\text{finite}} a_k \psi_k; a_k \in C^1([s,t];\mathbb{R}) \right\}.$$

Then the same argument as in Lemma 3.4 shows that

$$\begin{aligned} &-\int_s^t (v_m(\tau), \partial_\tau \varphi(\tau))d\tau + \int_s^t ((v_m(\tau)\cdot\nabla)v_m(\tau), \varphi(\tau))d\tau \\ &\quad - \int_s^t (v_m(\tau), (Ax\cdot\nabla)\varphi(\tau))d\tau + \int_s^t (Av_m(\tau), \varphi(\tau))d\tau \\ &\quad + \frac{\mu}{\rho}\int_s^t (\nabla v_m(\tau), \nabla\varphi(\tau))d\tau = -(v_m(t), \varphi(t)) + (v_m(s), \varphi(s)) \end{aligned} \tag{3.18}$$

holds for any $\varphi \in F([s,t];\operatorname{span}\{\psi_k; k \in \mathbb{N}\})$. For the purpose of the conclusion, we have the following lemma on the convergence of the triliner form $((v_m(\tau)\cdot\nabla)v_m(\tau),\varphi(\tau))$:

Lemma 3.9 *Let* $v \in C_w([0,T];L^2_\sigma(\mathbb{R}^n)) \cap L^2((0,T);H^1_\sigma(\mathbb{R}^n))$ *be taken as in Lemma* 3.8. *Then*

$$\lim_{m\to\infty}\int_s^t ((v_m(\tau)\cdot\nabla)v_m(\tau),\varphi(\tau))d\tau = \int_s^t ((v(\tau)\cdot\nabla)v(\tau),\varphi(\tau))d\tau \quad (3.19)$$

holds for any $\varphi \in F([s,t];\operatorname{span}\{\psi_k;k\in\mathbb{N}\})$.

Proof. See [8, (3.12)]. □

Let $m\to\infty$ in (3.18). Then, by Lemmas 3.8 and 3.9, we obtain

$$\begin{aligned}
&-\int_s^t (v(\tau),\partial_\tau\varphi(\tau))d\tau + \int_s^t ((v(\tau)\cdot\nabla)v(\tau),\varphi(\tau))d\tau \\
&\quad -\int_s^t (v(\tau),(Ax\cdot\nabla)\varphi(\tau))d\tau + \int_s^t (Av(\tau),\varphi(\tau))d\tau \\
&\quad +\frac{\mu}{\rho}\int_s^t (\nabla v(\tau),\nabla\varphi(\tau))d\tau = -(v(t),\varphi(t)) + (v(s),\varphi(s)) \qquad (3.20)
\end{aligned}$$

for any $\varphi \in F([s,t];\operatorname{span}\{\psi_k;k\in\mathbb{N}\})$. Thus, it follows from (3.20) and Lemma 3.1 with $X = H^1_\sigma(\mathbb{R}^n)\cap L^n_\sigma(\mathbb{R}^n)$ and $Y = C^1_{0,\sigma}(\mathbb{R}^n)$ that v is a weak solution to (1.2). Moreover, Lemma 3.5 and 3.8 admit the energy inequality for v, so v is also a Leray-Hopf weak solution to (1.2). This completes the proof of Theorem 2.1.

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