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On a certain invariant of differential equations associated with nilpotent graded Lie algebras

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Abstract. In this paper, we provide a new invariant for partial differential equations (PDEs) under contact transformations by using nilpotent graded Lie algebras. By virtue of this invariant, various geometric behavior of PDEs can be understood. As a typical class, we clarify geometric behavior of second-order PDEs in terms of our invariant.

Key words: Invariant of differential equations, (Linear) differential systems, Nilpotent graded Lie algebras.

1. Introduction

Historically, Sophus Lie initiated the geometric study of differential equations by considering a local equivalence problem of second-order ordinary differential equations, that is, a classification problem for differential equations under contact transformations. He also investigated the symmetries of differential equations via prolongations of vector fields (cf. [9], [1]). After the work of Lie, E. Cartan studied geometric structures associated with differential equations by his theory of differential systems and the method of equivalence in terms of Cartan connections and their curvatures [4]. On the other hand, Tanaka formulated the geometry of differential systems associated with simple graded Lie algebras over $\mathbb{R}$ or $\mathbb{C}$ and provided the theory of the construction of normal Cartan connections for such geometry [11]. This field is called parabolic geometries (cf. [3], [13]).

In this manner, many important relationships between differential equations, symmetries and geometry were provided. However, there are uncharted territories for the geometry of differential equations. Indeed many of obtained results are given under some regularity conditions. For instance, parabolic geometries and Tanaka theory are formulated under the regularity (i.e. stability) condition for nilpotent graded Lie algebras (symbol algebras). Hence it is important to study differential equations or differential systems

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which do not satisfy these regularity conditions. In particular, it is also natural to exploit useful tools for understanding of such differential equations. As a successful attempt for such a problem, Morimoto introduced a notion of the filtered manifold [6]. This notion is very effective to investigate such differential equations. In this paper, we apply this notion in order to define our invariant. We also give a paper [7] as one of the other trials. In [7], the authors provided a certain (local) invariant for second-order scalar PDEs by using the discriminant of the defining function of the equation. However, applicable PDEs are not so wide. Hence, it is important to give invariants which can be applicable to wider class of PDEs. For this purpose, in this present paper, we provide a new (local) invariant for PDEs by using nilpotent graded Lie algebras which are called symbol algebras. Here, it is known that the symbol algebras are invariants of filtered manifolds under contact transformations. Roughly speaking, we define our invariant by counting of path-connected components of symbol algebras defined for differential equations which can be realized as filtered manifolds. This invariant can be defined for higher-order PDEs of several unknown functions of several independent variables over $\mathbb{R}$. By the introduction of this invariant, we can obtain a method of quantification of geometric behavior of differential equations. Indeed, we can characterize differential equations which have regular (i.e. stable) symbol algebras by the simplest form of our invariant. Moreover, through the observation of numerical data of our invariant, we can also analyze complicated behavior of differential equations which have non-regular (i.e. unstable) symbol algebras in detail.

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2. Differential systems and Symbol algebras

2.1. Derived systems and Weak derived systems

In general, by a differential system (R, D), we mean a distribution D on a manifold R, that is, D is a subbundle of the tangent bundle TR of R. The sheaf of sections to D is denoted by $\mathcal{D} = \Gamma(D)$. The derived system ∂D of a differential system D is defined, in terms of sections, by $\partial\mathcal{D} := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$. In general, ∂D is obtained as a subsheaf of the tangent sheaf of R. Moreover, higher derived systems $\partial^k D$ are defined successively by $\partial^k\mathcal{D} := \partial(\partial^{k-1}\mathcal{D})$, where we set $\partial^0 D = D$ by convention. On the other hand, the k-th weak derived systems $\partial^{(k)}D$ of D are defined inductively by $\partial^{(k)}\mathcal{D} := \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}]$. These derived systems are also interpreted by using annihilators as follows; Let $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$ be a differential system on R. We denote by $D^\perp$ the annihilator subbundle of D in T^*R. Then the annihilator $(\partial D)^\perp$ of the first derived system of D is given by $(\partial D)^\perp = \{\varpi \in D^\perp \mid d\varpi \equiv 0 \pmod{D^\perp}\}$. The annihilator $(\partial^{(k+1)}D)^\perp$ of the $(k+1)$-th weak derived system of D is also given by

$$\begin{aligned}(\partial^{(k+1)}D)^\perp = \{\varpi \in (\partial^{(k)}D)^\perp \mid d\varpi \equiv 0 \ (\mathrm{mod}\ &(\partial^{(k)}D)^\perp,\\ &(\partial^{(p)}D)^\perp \wedge (\partial^{(q)}D)^\perp,\ 2 \le p,\ q \le k-1)\}.\end{aligned}$$

A differential system D is called *regular* (resp. *weakly regular*), if $\partial^k D$ (resp. $\partial^{(k)}D$) is a subbundle for each k. We set $D^{-1} := D$, $D^{-k} := \partial^{(k-1)}D$ $(k \ge 2)$, for a weakly regular differential system D. Then we have ([10, Proposition 1.1]);

(T1) There exists a unique positive integer μ such that

$$D^{-1} \subset D^{-2} \subset \cdots \subset D^{-k} \subset \cdots \subset D^{-(\mu-1)} \subset D^{-\mu} = D^{-(\mu+1)} = \cdots;$$

(T2) $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ for all $p,\ q < 0$.

2.2. Symbol algebra of regular differential system

Let (R, D) be a weakly regular differential system such that $TR = D^{-\mu} \supset D^{-(\mu-1)} \supset \cdots \supset D^{-1} =: D$. For all $x \in R$, we set $\mathfrak{g}_{-1}(x) := D^{-1}(x) = D(x)$, $\mathfrak{g}_p(x) := D^p(x)/D^{p+1}(x)$, $(p = -2, -3, \ldots, -\mu)$ and $\mathfrak{m}(x) := \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x)$. Then, $\dim \mathfrak{m}(x) = \dim R$ holds. We set $\mathfrak{g}_p(x) = \{0\}$ when $p \le -\mu - 1$. For $X \in \mathfrak{g}_p(x),\ Y \in \mathfrak{g}_q(x)$, the Lie bracket $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined as follows; Let ϖ_p be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$ and $\tilde{X} \in \mathcal{D}^p$, $\tilde{Y} \in \mathcal{D}^q$ be any extensions such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$. Then $[\tilde{X}, \tilde{Y}] \in \mathcal{D}^{p+q}$, and we define $[X, Y] := \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x) \in \mathfrak{g}_{p+q}(x)$. It does not depend on the choice of the extensions. Hence, $\mathfrak{m}(x)$ is a nilpotent graded Lie algebra. We call $(\mathfrak{m}(x),\ [\ ,\ ])$ the *symbol algebra* of (R, D) at x. Note that the symbol algebra $(\mathfrak{m}(x),\ [\ ,\ ])$ satisfies the generating conditions $[\mathfrak{g}_p, \mathfrak{g}_{-1}] = \mathfrak{g}_{p-1}$ $(p < 0)$. For two differential systems (R, D) and (R', D'), we define (local) contact transformations ϕ from R to R' by (local) diffeomorphisms $\phi : R \to R'$ satisfying $\phi_* D = D'$. The symbol algebra is an invariant of differential systems under contact transformations. Namely, if there exists a (local) contact transformation $\phi : R \to R'$, then we obtain the graded Lie algebra isomorphism $\mathfrak{m}(x) \cong \mathfrak{m}(\phi(x))$ at each point x [10].

2.3. Filtered manifolds and Symbol algebras

Morimoto introduced the notion of a filtered manifold as a generalization of weakly regular differential systems [6]. We define a filtered manifold (R, F) by a pair of a manifold R and a tangential filtration F. Here, a tangential filtration F on R is a sequence $\{F^p\}_{p<0}$ of subbundles of the tangent bundle TR and the following conditions are satisfied;

(M1) $TR = F^k = \cdots = F^{-\mu} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset F^0 = \{0\}$,
(M2) $[\mathcal{F}^p, \mathcal{F}^q] \subset \mathcal{F}^{p+q} \qquad$ for all $p,\ q < 0$,

where $\mathcal{F}^p = \Gamma(F^p)$ is the space of sections of F^p. Let (R, F) be a filtered manifold. For $x \in R$, we set $\mathfrak{f}_p(x) := F^p(x)/F^{p+1}(x)$ and $\mathfrak{f}(x) := \bigoplus_{p<0} \mathfrak{f}_p(x)$. For $X \in \mathfrak{f}_p(x)$, $Y \in \mathfrak{f}_q(x)$, the Lie bracket $[X, Y] \in \mathfrak{f}_{p+q}(x)$ is defined in the same way as before. The Lie algebra $\mathfrak{f}(x)$ is also a nilpotent graded Lie algebra. We call $(\mathfrak{f}(x),\ [\ ,\ ])$ the *symbol algebra* of (R, F) at x. In general it does not satisfy the generating conditions. Suppose (R, F) and (R', F') are filtered manifolds. Then, (local) contact transformations between (R, F) and (R', F') are defined by (local) diffeomorphisms $\phi : R \to R'$ such that $\phi_* F^p = F^{p\prime}$. This symbol algebras is also an invariant of filtered manifolds under contact transformations [6].

3. A certain invariant of differential equations

In this section, we provide a new invariant utilizing notions introduced in section 2.

Let $J^k(\mathbb{R}^n, \mathbb{R}^m)$ be the k-jet space for n independent and m dependent variables;

$$J^k(\mathbb{R}^n, \mathbb{R}^m) := \big\{(x_{i_1}, z^j, p^j_{i_1}, p^j_{i_1 i_2}, \ldots, p^j_{i_1 \cdots i_k}) \mid \\ 1 \leq i_1 \leq \cdots \leq i_k \leq n,\ 1 \leq j \leq m\big\}. \quad (1)$$

This space has the k-th order canonical differential system $C^k = \big\{\varpi^j = \varpi^j_{i_1} = \varpi^j_{i_1 i_2} = \cdots = \varpi^j_{i_1 \cdots i_{k-1}} = 0\big\}$ given by the 1-forms $\varpi^j := dz^j -$

$\sum_{i_1=1}^{n} p_{i_1}^j dx_{i_1}$, $\varpi_{i_1}^j := dp_{i_1}^j - \sum_{i_2=1}^{n} p_{i_1 i_2}^j dx_{i_2}$, $\ldots$, $\varpi_{i_1\cdots i_{k-1}}^j := dp_{i_1\cdots i_{k-1}}^j - \sum_{i_k=1}^{n} p_{i_1\cdots i_k}^j dx_{i_k}$, where $p_{i_1\cdots i_k}^j$ are the coordinate functions which are symmetric for the indices $i_1, \ldots, i_k$. On the jet space, we consider PDEs $F_1 = \cdots = F_r = 0$ by taking C^∞-functions $F_1, \ldots, F_r$. To study various behavior of PDEs, we introduce the notion of corresponding canonical differential systems. Let R be a zero subset $R = \{F_1 = \cdots = F_r = 0\}$ of $J^k(\mathbb{R}^n, \mathbb{R}^m)$ defined by $F_1, \ldots, F_r$ and $D := C^k|_R$ the restriction of C^k into R. In general R is not a submanifold of J^k and D is not a differential system. We impose on this equation R the following setting to define our invariant.

Definition 3.1 Let $R = \{F_1 = \cdots = F_r = 0\}$ be a subset of $J^k(\mathbb{R}^n, \mathbb{R}^m)$ defined by functions $F_1, \ldots, F_r$ and D be the restriction $D := C^k|_R$ of C^k into R. We impose on R the following two conditions.

(1) A pair (R, D) is a differential system. Namely, R is a regular submanifold and D is a subbundle of TR.
(2) On TR, a filtration structure $F := \{F^p\}_{p<0}$ associated with D is specified. Namely, F is a certain filtration such that $F^{-1} = D$.

Then, we call (R, F) *a filtered equation.*

We remark that our invariant depends on the choice of the filtration. Thus we need to take filtrations which give invariants having many information of differential equations.

Now, we explain the detailed formulation of our invariant for differential equations. For this purpose, we first fix a class Λ consisting of filtered equations (submanifolds of codimension r) $R = \{F_1 = \cdots = F_r = 0\} \subset J^k(\mathbb{R}^n, \mathbb{R}^m)$ with a specified filtration $\{F^p\}_{p<0}$. Here, according to the manner of Section 2.3, for two filtered equations $R = \{F_1 = \cdots = F_r = 0\}$ and $R' = \{F_1{}' = \cdots = F_r{}' = 0\}$, *contact transformations* $\phi : R \to R'$ are defined by (local) diffeomorphisms $\phi : R \to R'$ such that $\phi_* F^p = F^{p\prime}$. Let $\Lambda_{\mathfrak{g}}$ be the set of the isomorphic class of the symbol algebra at each point v on each equation manifold R, that is, the moduli space of the symbol algebras for filtered equations belonging to Λ, i.e. $\Lambda_{\mathfrak{g}} := \{[\mathfrak{g}(v)] \mid v \text{ is any point on each equation } R \in \Lambda\}$. Now, we take an equation $R \in \Lambda$ and a base point $w \in R$. Then a neighborhood $B(w)$ of w is decomposed as follows;

$$B(w) = \bigcup_{i \in \Lambda_{\mathfrak{g}}} B^i(w) \quad \text{(disjoint union)}, \tag{2}$$

where each component $B^i(w)$ consists of points which have isomorphic symbol algebras. For each component $B^i(w)$, an equivalence relation $w_1 \sim w_2$ $(w_1, w_2 \in B^i(w))$ is defined as follows; There exists a continuous curve,

$$c : [0,1] \to B^i(w) \quad \text{such that} \quad c(0) = w_1,\ c(1) = w_2. \tag{3}$$

Let $\#(B^i(w)/\sim)$ be the number of elements of the quotient space $B^i(w)/\sim$ consisting of path-connected components.

We next fix a diffeomorphism $\phi : U_w \to \phi(U_w)$ defined on an open neighborhood U_w of w in $J^k(\mathbb{R}^n, \mathbb{R}^m)$, where $\phi(U_w) \subset \mathbb{R}^K$ and $K = \dim J^k(\mathbb{R}^n, \mathbb{R}^m)$. Then, the standard Euclidean distance on $\mathbb{R}^K$ induces a distance on U_w by this diffeomorphism. Namely, we can define the following induced distance d on U_w;

$$d(p, q) := \|\phi(p) - \phi(q)\|, \quad \text{for} \quad p, q \in U_w, \tag{4}$$

where $\|\cdot\|$ is the Euclidean norm. Utilizing this distance, we choose the following ε-neighborhood of w in the filtered equation R;

$$B_\varepsilon(w) := \{v \in R \mid d(v, w) < \varepsilon\}, \tag{5}$$

where d is the restriction of the distance on U_w into R. Then we have the decomposition;

$$B_\varepsilon(w) = \bigcup_{i \in \Lambda_{\mathfrak{g}}} B^i_\varepsilon(w) \quad \text{(disjoint union)}, \tag{6}$$

similar to (2). Here we consider the numbers;

$$N^i_d(w) := \lim_{\varepsilon \to 0} \#(B^i_\varepsilon(w)/\sim) \quad \text{for } i \in \Lambda_{\mathfrak{g}}. \tag{7}$$

We remark that if a limit does not exist, then we set the value ∞. These numbers do not depend on the contact isomorphisms of (R, F), but there is the possibility that they depend on the local distance on $J^k(\mathbb{R}^n, \mathbb{R}^m)$ under the local identification $J^k(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^K$. Hence, we define the invariant of (R, F) at w as follows;

Definition 3.2 We set $N^i(w) := \min_{J^k \cong \mathbb{R}^K} \left(\lim_{\varepsilon \to 0} \#(B^i_\varepsilon(w)/\ \sim)\right)$, where the minimum is taken over all local diffeomorphisms $\phi : J^k \to \mathbb{R}^K$. We also set $N(w) := (N^i(w))_{i \in \Lambda_{\mathfrak{g}}}$.

The value $N(w)$ is well-defined. Thus it is an (local) invariant of (R, F) under contact transformations. This invariant describes behavior of filtered equations (R, F). We also denote the notion of our invariant by N. For the invariance of this invariant, we remark that if the isomorphisms of D are in bijective correspondence with the isomorphisms of F, then the symbol algebra and our invariant for (R, F) are also invariants of (R, D). As such a typical case, in Section 4, we will investigate second-order scalar PDEs (R, D). On the other hand, for second-order regular overdetermined systems in section 5, we need to define an appropriate filtration F associated with D to give an invariant of (R, F).

As a direct consequence from the definition of our invariant, we obtain a simple characterization for filtered equations which have regular symbol. Here, a filtered equation (R, F) which have *regular symbol* is defined by the condition that the symbol algebra is isomorphic at each point of R. We also define a filtered equation (R, F) which have *regular symbol around a base point* $w \in R$ by the condition; There exists an open neighborhood U around w such that symbol algebras at all points on U are isomorphic. Now we can state the clear description of equations which have (locally) regular symbol by our invariant N.

Theorem 3.3 *Let (R, F) be a filtered equation. Then, (R, F) has regular symbol around a base point $w \in R$ if and only if the value of one component of $N(w)$ is* 1 *and the values of the other components of $N(w)$ are all* 0.

In Section 4 and 5, we have the corresponding statements for second-order PDEs. On the other hand, there exist the classifications of the structure equations for other categories of equations. (cf. [2], [5]). Thus, we can apply these results to our invariant.

4. Second-order scalar PDEs

In this section, we treat scalar PDEs of second order with two independent and one dependent variables. For this class, an invariant was given in [7]. We have the agreement between our invariant and the invariant in [7]. For this purpose, we recall the invariant in [7]. Let $J^2(\mathbb{R}^2, \mathbb{R}) :=$

$\{(x,y,z,p,q,r,s,t)\}$ be the two-jet space with two independent and one dependent variables. The canonical system $C^2 = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ is given by the one forms $\varpi_0 := dz - pdx - qdy$, $\varpi_1 := dp - rdx - sdy$ and $\varpi_2 := dq - sdx - tdy$. Let $R = \{F = 0\} \subset J^2(\mathbb{R}^2, \mathbb{R})$ be a smooth hypersurface satisfying $dF \neq 0$, where F is a C^∞-function on J^2. In this section, we call a second order PDE $R = \{F = 0\}$ with $dF \neq 0$ a hypersurface. We recall the notion of the discriminant. Let $R := \{F = 0\}$ be a hypersurface of $J^2(\mathbb{R}^2, \mathbb{R})$. Then, for the discriminant $\Delta := F_r F_t - F_s^2/4$ of F, a point $w \in R$ is said to be *hyperbolic* or *elliptic* if $\Delta(w) < 0$ or $\Delta(w) > 0$ respectively. Moreover, a point $w \in R$ is said to be *parabolic* if $(F_r, F_s, F_t)_w \neq (0,0,0)$ and $\Delta(w) = 0$. For any $w \in R$ and an neighborhood U of w in R, U is decomposed as $U = U_H \cup U_E \cup U_P \cup U_{Sing}$, where each component is given by $U_H := \{v \in U \mid v \text{ is hyperbolic}\}$, $U_E := \{v \in U \mid v \text{ is elliptic}\}$, $U_P := \{v \in U \mid v \text{ is parabolic}\}$ and $U_{Sing} := U \backslash (U_H \cup U_E \cup U_P)$. We put $K_U := U_H$, U_E, U_P or U_{Sing}. An equivalence relation $w_1 \sim w_2$ $(w_1, w_2 \in K_U)$ is defined in the same way as (3). Let $\#(K_U/\sim)$ be the number of elements of the quotient space $K_U/\sim$ consisting of path-connected components. We fix a local diffeomorphism ϕ between $J^2(\mathbb{R}^2, \mathbb{R})$ and $\mathbb{R}^8$. Then the standard metric on $\mathbb{R}^8$ induces a local distance d on $J^2(\mathbb{R}^2, \mathbb{R})$ in the same way as (4). We take the neighborhood $U = B_\varepsilon(w)$ given by (5). We have the decomposition $U = B_\varepsilon^H(w) \cup B_\varepsilon^E(w) \cup B_\varepsilon^P(w) \cup B_\varepsilon^{Sing}(w)$ as above. Here we consider four numbers $H_d(w) := \lim_{\varepsilon \to 0} \#(B_\varepsilon^H(w)/\sim)$, $E_d(w) := \lim_{\varepsilon \to 0} \#(B_\varepsilon^E(w)/\sim)$, $P_d(w) := \lim_{\varepsilon \to 0} \#(B_\varepsilon^P(w)/\sim)$ and $S_d(w) := \lim_{\varepsilon \to 0} \#(B_\varepsilon^{Sing}(w)/\sim)$, where if a limit is not determined, we set a value ∞. Numbers do not depend on the contact isomorphisms of the differential system $(R, D := C^2|_R)$, but there is the possibility that they depend on the above local distance on $J^2(\mathbb{R}^2, \mathbb{R})$. Hence we define as follows;

Definition 4.1 We define as follows; $H(w) := \min_{J^2 \cong \mathbb{R}^8} \big(\lim_{\varepsilon \to 0} \# \cdot (B_\varepsilon^H(w)/\sim)\big)$, $E(w) := \min_{J^2 \cong \mathbb{R}^8} \big(\lim_{\varepsilon \to 0} \#(B_\varepsilon^E(w)/\sim)\big)$, $P(w) := \min_{J^2 \cong \mathbb{R}^8} \cdot \big(\lim_{\varepsilon \to 0} \#(B_\varepsilon^P(w)/\sim)\big)$, $S(w) := \min_{J^2 \cong \mathbb{R}^8} \big(\lim_{\varepsilon \to 0} \#(B_\varepsilon^{Sing}(w)/\sim)\big)$, where min is that of all local diffeomorphisms $\phi : J^2 \to \mathbb{R}^8$. We also define $(H,E,P,S)_w := (H(w), E(w), P(w), S(w))$.

Since $(H,E,P,S)_w$ is well-defined, it is an (local) invariant of (R,D) at w.

We next give the description of our invariant N for a hypersurface R. Let

$D := C^2|_R$ be a differential system on R. Then, from [7, Proposition 3.2], we obtain that D is weakly-regular. Hence the natural filtration associated to the weak-derived system defines the symbol algebras in the sense of Tanaka. For a base point $w \in R$, w is either submersion point or nonsubmersion point for the restricted projection $\pi^2_1|_R : R \to J^1(\mathbb{R}^2, \mathbb{R})$, where $\pi^2_1 : J^2(\mathbb{R}^2, \mathbb{R}) \to J^1(\mathbb{R}^2, \mathbb{R})$. We first consider when a base point $w \in R$ is a submersion point. Then, it is well-known that the symbol algebra $\mathfrak{s}(w)$ is one of the following three types consisting of hyperbolic, parabolic and elliptic ([14, the case of codim $\mathfrak{f} = 1$ of Case $n = 2$ in p. 346]). The symbol algebra $\mathfrak{s}(w)$ is represented as $\mathfrak{s}(w) = \mathfrak{s}_{-3}(w) \oplus \mathfrak{s}_{-2}(w) \oplus \mathfrak{s}_{-1}(w)$, where each component is given by $\mathfrak{s}_{-3}(w) \cong \mathbb{R}$, $\mathfrak{s}_{-2}(w) \cong V^*$ (V is a 2-dimensional integral subspace of $D(w)$), $\mathfrak{s}_{-1}(w) = V \oplus \mathfrak{f}(w)$ for $\mathfrak{f}(w) \subset S^2(V^*)$, and $\mathfrak{f}(w)$ has the following structure corresponding to each type at w:

$$\mathfrak{f}(w) = \begin{cases} \langle e_1^* \circledcirc e_1^*,\ e_2^* \circledcirc e_2^* \rangle & \text{in the case of hyperbolic type,} \\ \langle e_1^* \circledcirc e_2^*,\ e_2^* \circledcirc e_2^* \rangle & \text{in the case of parabolic type,} \\ \langle e_1^* \circledcirc e_2^*,\ e_1^* \circledcirc e_1^* - e_2^* \circledcirc e_2^* \rangle & \text{in the case of elliptic type,} \end{cases} \quad (8)$$

where $\langle e_1, e_2 \rangle$ is the basis of V and the notation $\circledcirc$ means that the symmetric tensor, so we have $S^2(V^*) = \langle e_1^* \circledcirc e_1^*, 2e_1^* \circledcirc e_2^*, e_2^* \circledcirc e_2^* \rangle$. We denote by $\mathfrak{s}^{hyp}(w)$, $\mathfrak{s}^{par}(w)$, $\mathfrak{s}^{ell}(w)$ the symbol algebras corresponding to each case. We next consider when a base point $w \in R$ is a nonsubmersion point. In this case, we can give the uniqueness of the symbol algebra $\mathfrak{s}^{nonsub}(w)$ at this point w in [7, Theorem 3.2]. We omit the brackets of this algebra $\mathfrak{s}^{nonsub}(w)$. Under the classification of four symbol algebras consisting of $\mathfrak{s}^{hyp}(w)$, $\mathfrak{s}^{par}(w)$, $\mathfrak{s}^{ell}(w)$ and $\mathfrak{s}^{nonsub}(w)$, we have the decomposition $B_\varepsilon(w) = B_\varepsilon^{hyp}(w) \cup B_\varepsilon^{ell}(w) \cup B_\varepsilon^{par}(w) \cup B_\varepsilon^{nonsub}(w)$ of a ε-neighborhood on R. Hence, by the definition of our invariant, we obtain the expression $N(w) = (N^{hyp}(w), N^{ell}(w), N^{par}(w), N^{nonsub}(w))$ of the invariant N at $w \in R$. From the above discussion, it is also clear that $N(w) = (H, E, P, S)_w$ is satisfied.

Now we state the following simple characterization as a corollary of Theorem 3.3.

Corollary 4.2 *Let R be a regular scalar PDE of second order. Then, we obtain the following characterization under our invariant $N(w)$ at each point $w \in R$.*

(1) $(1,0,0,0)_w \iff R$ is locally hyperbolic around w.

(2) $(0,1,0,0)_w \iff R$ is locally elliptic around w.

(3) $(0,0,1,0)_w \iff R$ is locally parabolic around w.

Here "regular" means the submersion condition for the projection $\pi^2_1|_R : R \to J^1$.

We also have a result for scalar PDEs which have non-regular symbols.

Theorem 4.3 *Let* $R = \{F = 0\} \subset J^2$ *be a hypersurface given by* $F := f-(a_1x+a_2y+a_3z+a_4p+a_5q+a_6)$, *where* $(a_1, \ldots, a_5) \neq (0, \ldots, 0)$ *and* f *is a monomial which is one of the three forms consisting of* (I) $f = r^{k_1}s^{k_2}$, (II) $f = r^{k_1}t^{k_2}$, (III) $f = s^{k_1}t^{k_2}$, *for* $k_i \geq 0$. *Then the values of the invariants* $(H, E, P, S)_w$ (*or* $N(w)$) *are as follows*;

(1) $k_1 = k_2 = 0$,
In this case, for any form of (I), (II) *and* (III), *we have* $(0, 0, 0, 1)$ *for any* $w \in R$.

(2) $k_i = 1$, $k_j = 0$ *for* $i \neq j$,
In this case, for each class of (I), (II) *and* (III), *we have*;

(2-I-1) *If* $k_1 = 1$, $k_2 = 0$, *then we have* $(0, 0, 1, 0)_w$ *for any* $w \in R$.
(2-I-2) *If* $k_1 = 0$, $k_2 = 1$, *then we have* $(1, 0, 0, 0)_w$ *for any* $w \in R$.
(2-II-1) *If* $k_1 = 1$, $k_2 = 0$, *then we have* $(0, 0, 1, 0)_w$ *for any* $w \in R$.
(2-II-2) *If* $k_1 = 0$, $k_2 = 1$, *then we have* $(0, 0, 1, 0)_w$ *for any* $w \in R$.
(2-III-1) *If* $k_1 = 1$, $k_2 = 0$, *then we have* $(1, 0, 0, 0)_w$ *for any* $w \in R$.
(2-III-2) *If* $k_1 = 0$, $k_2 = 1$, *then we have* $(0, 0, 1, 0)_w$ *for any* $w \in R$.

(3) *The other cases* (*general index*),
In this case, R *has a nonsubmersion point. Let* w *be a nonsubmersion point such that* $D = C^2|_R$ *is a differential system around* w. *Then we have the following for each case.*

(3-I-1) *If* $k_1 \geq 2$, $k_2 = 0$, *then we have* $(0, 0, 2, 1)_w$ *at* $w \in \{r = 0\} \subset R$.
(3-I-2) *If* $k_1 = 0$, $k_2 \geq 2$, *then we have* $(2, 0, 0, 1)_w$ *at* $w \in \{s = 0\} \subset R$.
(3-I-3) *If* $k_1 = 1$, $k_2 = 1$, *then we have* $(2, 0, 2, 1)_w$ *at* $w \in \{r = s = 0\} \subset R$.
(3-I-4) *If* $k_1 \geq 2$, $k_2 = 1$, *then we have* $(2, 0, 0, 1)_w$ *at* $w \in \{r = 0\} \subset R$.
(3-I-5) *If* $k_1 = 1$, $k_2 \geq 2$, *then we have* $(4, 0, 2, 1)_w$ *at* $w \in \{r = s = 0\}$

$\subset R$ *and* $(2,0,0,1)_w$ *at* $w \in \{r \neq 0, s = 0\} \subset R$.

(3-I-6) *If* $k_1 \geq 2$, $k_2 \geq 2$, *then we have* $(4,0,0,1)_w$ *at* $w \in \{r = s = 0\} \subset R$ *and* $(2,0,0,1)_w$ *at* $w \in \{r \neq 0, s = 0\} \cup \{r = 0, s \neq 0\} \subset R$.

(3-II-1) *If* $k_1 \geq 2$, $k_2 = 0$, *then we have* $(0,0,2,1)_w$ *at* $w \in \{r = 0\} \subset R$. (*this case is equal to the case of* (3-I-1) *from the definition*).

(3-II-2) *If* $k_1 = 0$, $k_2 \geq 2$, *then we have* $(0,0,2,1)_w$ *at* $w \in \{t = 0\} \subset R$.

(3-II-3) *If* $k_1 = 1$, $k_2 = 1$, *then we have* $(2,2,4,1)_w$ *at* $w \in \{r = t = 0\} \subset R$.

(3-II-4) *If* $k_1 \geq 2$, $k_2 = 1$, *then we have* $(2,2,2,1)_w$ *at* $w \in \{r = t = 0\} \subset R$ *and* $(1,1,0,1)_w$ *at* $w \in \{r = 0, t \neq 0\} \subset R$.

(3-II-5) *If* $k_1 = 1$, $k_2 \geq 2$, *then we have* $(2,2,2,1)_w$ *at* $w \in \{r = t = 0\} \subset R$ *and* $(1,1,0,1)_w$ *at* $w \in \{r \neq 0, t = 0\} \subset R$.

(3-II-6) *If* $k_1 \geq 2$, $k_2 \geq 2$, *then we have* $(2,2,0,1)_w$ *at* $w \in \{r = t = 0\} \subset R$ *and* $(1,1,0,1)_w$ *at* $w \in \{r \neq 0, t = 0\} \cup \{r = 0, t \neq 0\} \subset R$.

(3-III-1) *If* $k_1 \geq 2$, $k_2 = 0$, *then we have* $(2,0,0,1)_w$ *at* $w \in \{s = 0\} \subset R$.

(3-III-2) *If* $k_1 = 0$, $k_2 \geq 2$, *then we have* $(0,0,2,1)_w$ *at* $w \in \{t = 0\} \subset R$.

(3-III-3) *If* $k_1 = 1$, $k_2 = 1$, *then we have* $(2,0,2,1)_w$ *at* $w \in \{s = t = 0\} \subset R$.

(3-III-4) *If* $k_1 \geq 2$, $k_2 = 1$, *then we have* $(4,0,2,1)_w$ *at* $w \in \{s = t = 0\} \subset R$ *and* $(2,0,0,1)_w$ *at* $w \in \{s = 0, t \neq 0\} \subset R$.

(3-III-5) *If* $k_1 = 1$, $k_2 \geq 2$, *then we have* $(2,0,0,1)_w$ *at* $w \in \{t = 0\} \subset R$.

(3-III-6) *If* $k_1 \geq 2$, $k_2 \geq 2$, *then we have* $(4,0,0,1)_w$ *at* $w \in \{s = t = 0\} \subset R$ *and* $(2,0,0,1)_w$ *at* $w \in \{s \neq 0, t = 0\} \cup \{s = 0, t \neq 0\} \subset R$.

Proof. We first consider the case of (1). Since R is a first-order PDE, we have the assertion. We next consider the case of (2). In this case, R is a regular PDE of second order. Therefore we have $S(w) = 0$ at any point $w \in R$. For the cases of (2-I-1), (2-II-1), (2-II-2) and (2-III-2), the discriminant Δ always vanishes. Hence, we have $(H, E, P, S)_w = (0,0,1,0)$ for any $w \in R$. On the other hand, for each case of (2-I-2) and (2-III-1), by the property $\Delta < 0$, we have $(H, E, P, S)_w = (1,0,0,0)$ for any $w \in R$. Finally, we prove the case of (3). For this purpose, we prepare the following proposition given in [7];

Proposition 4.4 ([7, Proposition 3.1]) *Let* $R = \{F = 0\} \subset J^2(\mathbb{R}^2, \mathbb{R})$ *be a second-order scalar PDE. Then, for a nonsubmersion point* w_0, $D = C^2|_R$ *is a differential system around* w_0 *if and only if* $dF/dx(w_0) \neq 0$, *or* $dF/dy(w_0) \neq 0$, *where* $d/dx := \partial/\partial x + p\partial/\partial z + r\partial/\partial p + s\partial/\partial q$, $d/dy :=$

$\partial/\partial y + q\partial/\partial z + s\partial/\partial p + t\partial/\partial q$.

We consider the case of (3-I-1). From $(a_1, \ldots, a_5) \neq 0$, R is a smooth hypersurface. By $(F_r, F_s, F_t) = (k_1 r^{k_1-1}, 0, 0)$, any point $w \in \{r = 0\} \subset R$ is a nonsubmersion point. From the assumption of the above proposition, we fix a base nonsubmersion point w satisfying $dF/dx(w) \neq 0$ or $dF/dy(w) \neq 0$. Namely D is a differential system around w (Hereafter, for the other cases, we have the same argument). Since $\Delta = 0$ on R, parabolic points appear in $\{r \neq 0\}$ which has two connected components and we have $(H, E, P, S)_w = (0, 0, 2, 1)$. For the case of (3-I-2), by $(F_r, F_s, F_t) = (0, k_2 s^{k_2-1}, 0)$ and $\Delta = -(k_2)^2 s^{2(k_2-1)}/4$, we have $(H, E, P, S)_w = (2, 0, 0, 1)$ at a nonsubmersion point $w \in \{s = 0\} \subset R$. For the case of (3-I-3), points on $\{r = s = 0\}$ are nonsubmersion points by $(F_r, F_s, F_t) = (s, r, 0)$. Since $\Delta = -r^2/4$, there are no elliptic points. Points on $\{r = 0, s \neq 0\}$ are parabolic. Moreover, hyperbolic points appear in $\{r \neq 0\}$ which has two connected components. Thus we have $(H, E, P, S)_w = (2, 0, 2, 1)$ at a nonsubmersion point w. We consider the case of (3-I-4). Since $(F_r, F_s, F_t) = (k_1 r^{k_1-1} s, r^{k_1}, 0)$ and $\Delta = -r^{2k_1}/4$, we have $(H, E, P, S)_w = (2, 0, 0, 1)$ at a nonsubmersion point $w \in \{r = 0\} \subset R$. For the case of (3-I-5), points on $\{s = 0\} \subset R$ are nonsubmersion points from $(F_r, F_s, F_t) = (s^{k_2}, k_2 r s^{k_2-1}, 0)$. Since $\Delta = -(k_2)^2 r^2 s^{2(k_2-1)}/4$, an elliptic point does not exist and points on $\{r = 0, s > 0\} \cup \{r = 0, s < 0\}$ are of parabolic type. Moreover, points in $\{r \neq 0, s \neq 0\}$ are of hyperbolic type. Thus we have $(H, E, P, S)_w = (4, 0, 2, 1)$ at a nonsubmersion point $w \in \{r = s = 0\}$ and $(H, E, P, S)_w = (2, 0, 0, 1)$ at a nonsubmersion point $w \in \{r \neq 0, s = 0\}$. For the case of (3-I-6), from $(F_r, F_s, F_t) = (k_1 r^{k_1-1} s^{k_2}, k_2 r^{k_1} s^{k_2-1}, 0)$ and $\Delta = -(k_2)^2 r^{2k_1} s^{2(k_2-1)}/4$, we have $(H, E, P, S)_w = (4, 0, 0, 1)$ at a nonsubmersion point $w \in \{r = s = 0\} \subset R$ and $(H, E, P, S)_w = (2, 0, 0, 1)$ at a nonsubmersion point $w \in \{r \neq 0, s = 0\} \cup \{r = 0, s \neq 0\} \subset R$.

We consider each case of (3-II). It is clear that the equations (3-II-1) are equal to the equations (3-I-1) and the equations (3-II-2) are reduced to the equations (3-I-1) by contact transformations. In the case of (3-II-3), points on $\{r = t = 0\} \subset R$ are nonsubmersion points by $(F_r, F_s, F_t) = (t, 0, r)$. Since $\Delta = rt$, points on $\{r = 0, t \neq 0\} \cup \{r \neq 0, t = 0\}$ are of parabolic type. Two hyperbolic and two elliptic parts appear in connected components divided by $\{r = 0\}$ and $\{t = 0\}$. Thus we have $(H, E, P, S)_w = (2, 2, 4, 1)$ at a nonsubmersion point w. For the case of

(3-II-4), from $(F_r, F_s, F_t) = (k_1 r^{k_1-1} t, 0, r^{k_1})$, points on $\{r = 0\} \subset R$ are nonsubmersion points. Since $\Delta = k_1 r^{2k_1-1} t$, points on $\{r > 0, t = 0\} \cup \{r < 0, t = 0\}$ are of parabolic type. Moreover two hyperbolic and two elliptic parts appear in connected components divided by $\{r = 0\}$ and $\{t = 0\}$. Thus we have $(H, E, P, S)_w = (2, 2, 2, 1)$ at a nonsubmersion point $w \in \{r = t = 0\}$ and $(H, E, P, S)_w = (1, 1, 0, 1)$ at a nonsubmersion point $w \in \{r = 0, t \neq 0\}$. In the case of (3-II-5), these equations are reduced to the equations of (3-II-4) by contact transformations. For the case of (3-II-6), from the condition $(F_r, F_s, F_t) = (k_1 r^{k_1-1} t^{k_2}, 0, k_2 r^{k_1} t^{k_2-1})$, points on $\{r = 0\} \cup \{t = 0\} \subset R$ are nonsubmersion points. By the discriminant $\Delta = k_1 k_2 r^{2k_1-1} t^{2k_2-1}$, we have $(H, E, P, S)_w = (2, 2, 0, 1)$ at a nonsubmersion point $w \in \{r = t = 0\}$ and $(H, E, P, S)_w = (1, 1, 0, 1)$ at a nonsubmersion point $w \in \{r = 0, t \neq 0\} \cup \{r \neq 0, t = 0\}$.

We study each case of (3-III). Equations of (3-III-1) and (3-III-2) are equal to the equations of (3-I-2) and (3-II-2) respectively. Equations of (3-III-3), (3-III-4), (3-III-5) and (3-III-6) are reduced to equations of (3-I-3), (3-I-5), (3-I-4) and (3-I-6) respectively by contact transformations for fixed indices k_1, k_2. Now the proof is complete. $\square$

5. Second-order regular overdetermined systems

In this section, we apply our invariant to the following second-order regular overdetermined systems with two independent and one dependent variables.

For two smooth functions F and G on $J^2(\mathbb{R}^2, \mathbb{R})$, an overdetermined system $R = \{F = G = 0\}$ is *regular* if R is a submanifold of codimension two in J^2 and the restricted projection $\pi_1^2|_R : R \to J^1(\mathbb{R}^2, \mathbb{R})$ is a submersion. Let R be a regular overdetermined system. From the definition, the restricted 1-forms $\varpi_i|_R$ on R are linearly independent. Hence we have the differential system $D := C^2|_R$ of rank three on R. For brevity, we denote by ϖ_i each restricted generator 1-form $\varpi_i|_R$ of D in the following. These differential systems (R, D) are not weakly regular, in general. Hence, we need to take an appropriate filtration on R to discuss contact geometry of overdetermined systems in terms of symbol algebras. We take the filtration $F = \{F^p\}_{p<0}$ on R given by $F^{-3} = TR$, $F^{-2} = (\pi_1^2|_R)_*^{-1} C^1$, $F^{-1} = D$, where C^1 is the canonical contact system on 1-jet space $J^1(\mathbb{R}^2, \mathbb{R})$. Now, it is known that if R does not have torsion (obstruction-free to the existence of solutions), then the symbol algebra $\mathfrak{s}(w)$ at each point $w \in R$ is one of the following three types ([14, the case of codim $\mathfrak{f} = 2$ of Case $n = 2$ in p. 346–347]). The symbol algebra $\mathfrak{s}(w)$ is described as $\mathfrak{s}(w) = \mathfrak{s}_{-3}(w) \oplus \mathfrak{s}_{-2}(w) \oplus \mathfrak{s}_{-1}(w)$, where each component is given by $\mathfrak{s}_{-3}(w) \cong \mathbb{R}$, $\mathfrak{s}_{-2}(w) \cong V^*$, $\mathfrak{s}_{-1}(w) = V \oplus \mathfrak{f}(w)$ for $\mathfrak{f}(w) \subset S^2(V^*)$, and $\mathfrak{f}(w)$ has the following structure corresponding to each type at w:

$$\mathfrak{f}(w) = \begin{cases} \langle e_2^* \circledcirc e_2^* \rangle & \text{in the case of involutive type (i),} \\ \langle e_1^* \circledcirc e_2^* \rangle & \text{in the case of finite type (ii),} \\ \langle e_1^* \circledcirc e_1^* + e_2^* \circledcirc e_2^* \rangle & \text{in the case of finite type (iii),} \end{cases} \tag{9}$$

where $\langle e_1, e_2 \rangle$ is the basis of V. Among these symbol algebras, type (i) does not satisfy the generating condition (i.e. $[\mathfrak{s}_{-1}, \mathfrak{s}_{-1}] \neq \mathfrak{s}_{-2}$). Here, we denote by $\mathfrak{s}^{inv}(w)$, $\mathfrak{s}^{\mathfrak{o}(3,3)}(w)$, $\mathfrak{s}^{\mathfrak{o}(4,2)}(w)$ the symbol algebras corresponding to the three types (i), (ii), (iii) respectively. The reason of this notation is as follows. Among these three types of the symbol algebras, the first type $\mathfrak{s}^{inv}(w)$ represents an involutive symbol (see [2] and [12]). On the other hand, two symbol algebras $\mathfrak{s}^{\mathfrak{o}(3,3)}(w)$ and $\mathfrak{s}^{\mathfrak{o}(4,2)}(w)$ are of finite type. Moreover, in fact, these can be regarded as the symbol algebras of two parabolic geometries (cf. [13], [15]). Here parabolic geometries are geometries modeled after the homogeneous space G/P (where G is a simple Lie group and P is a parabolic subgroup of G), associated with simple graded Lie algebras $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$ over $\mathbb{R}$ or $\mathbb{C}$, where $\mathfrak{m} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1}$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{\mu}$ (parabolic subalgebra of $\mathfrak{g}$) for some $\mu \in \mathbb{N}$. For precise formulation of parabolic geometries, see Yamaguchi [13]. Let us return to the discussion of the two symbol algebras $\mathfrak{s}^{\mathfrak{o}(3,3)}(w)$, $\mathfrak{s}^{\mathfrak{o}(4,2)}(w)$. We can show that these symbol algebras correspond to the symbol algebras which formulate parabolic geometries associated with two gradations of $\mathfrak{o}(3,3)$ and $\mathfrak{o}(4,2)$ respectively (cf. [13], [15]). For this purpose, we describe explicitly the structures of these symbol algebras by using the matrix representation of Lie algebras $\mathfrak{o}(3,3)$ and $\mathfrak{o}(4,2)$. For these descriptions, we refer the reader to [15]. We introduce the following notation of these two Lie algebras.

$$\mathfrak{o}(I) := \left\{ X \in \mathfrak{g}l(6, \mathbb{R}) \mid {}^t X I + I X = 0 \right\}, \tag{10}$$

where $I = \left(\begin{smallmatrix} 0 & 0 & K \\ 0 & S & 0 \\ K & 0 & 0 \end{smallmatrix} \right)$ for $K = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$, $S = K$ or E_2 (identity matrix). We obtain $\mathfrak{o}(I) \cong \mathfrak{o}(3,3)$ or $\mathfrak{o}(4,2)$ for $S = K$ or E_2 respectively. Here, we consider the gradation;

$$\mathfrak{o}(I) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3, \tag{11}$$

where the non-negative component $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ is a parabolic (Borel) subalgebra of $\mathfrak{o}(I)$, the negative component $\mathfrak{m} := \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the symbol algebra for each parabolic geometry and each component of $\mathfrak{m}$ is given by,

$$\mathfrak{g}_{-3} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A & 0 & 0 \end{pmatrix} \middle|\ A \in \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\},$$

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & \hat{B} & 0 \end{pmatrix} \middle|\ \begin{array}{l} \xi \in \mathbb{R}^2, \\ B = (\xi, 0), \end{array} \hat{B} = \begin{pmatrix} 0 \\ -{}^t\xi S \end{pmatrix} \right\},$$

$$\mathfrak{g}_{-1} = V \oplus \mathfrak{f},$$

$$V = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ 0 & \hat{C} & 0 \end{pmatrix} \middle|\ \begin{array}{l} x \in \mathbb{R}^2, \\ C = (0, x), \end{array} \hat{C} = \begin{pmatrix} -{}^t x S \\ 0 \end{pmatrix} \right\},$$

$$\mathfrak{f} = \left\{ \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -D \end{pmatrix} \middle|\ \alpha \in \mathbb{R}, D = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \right\}.$$

We can show easily that the negative components $\mathfrak{m}$ for Lie algebras $\mathfrak{o}(3,3)$ and $\mathfrak{o}(4,2)$ are isomorphic to the symbol algebras $\mathfrak{s}^{\mathfrak{o}(3,3)}(w)$, $\mathfrak{s}^{\mathfrak{o}(4,2)}(w)$ in (9) respectively. Hence, according the manner in [13], two geometries consisting of differential systems which have each regular symbol algebra $\mathfrak{s}^{\mathfrak{o}(3,3)}$, $\mathfrak{s}^{\mathfrak{o}(4,2)}$ can be regraded as parabolic geometries associated with simple graded Lie algebras $\mathfrak{o}(3,3)$ or $\mathfrak{o}(4,2)$ respectively.

Now, we remark that the above discussion was done under the non-existence of the torsion. Hence we next consider when the torsion exists. In this case, we can prove the uniqueness of the symbol algebra $\mathfrak{s}^{tor}(w)$ of torsion type utilizing the technique of the proof of [8, Theorem 3.3]. We omit the brackets of this algebra $\mathfrak{s}^{tor}(w)$.

In this way, regular overdetermined systems have the symbol algebras of four-types consisting of involutive type $\mathfrak{s}^{inv}(w)$, two finite types $\mathfrak{s}^{\mathfrak{o}(3,3)}(w)$, $\mathfrak{s}^{\mathfrak{o}(4,2)}(w)$, and torsion type $\mathfrak{s}^{tor}(w)$, in general. Since these symbol algebras are invariants of (R, F) under contact transformations, we can form our invariant N for regular overdetermined systems. We fix a base point $w \in R$. Then we have the decomposition $B_\varepsilon(w) = B_\varepsilon^{inv}(w) \cup B_\varepsilon^{\mathfrak{o}(3,3)}(w) \cup B_\varepsilon^{\mathfrak{o}(4,2)}(w) \cup B_\varepsilon^{tor}(w)$ of a ε-neighborhood on R. Hence, from the discussion similar to the previous two sections, we obtain the expression of our invariant at $w \in R$;

$$N(w) = \big(N^{inv}(w), N^{\mathfrak{o}(3,3)}(w), N^{\mathfrak{o}(4,2)}(w), N^{tor}(w)\big). \tag{12}$$

We first state the following simple characterization as a corollary of Theorem 3.3.

Corollary 5.1 *Let (R, F) be a regular overdetermined system. Then, for a base point $w \in R$, we have*;

(1) $(1,0,0,0)_w \iff (R, F)$ is involutive around w.

(2) $(0,1,0,0)_w \iff (R, F)$ has the structure of parabolic geometry associated to the simple graded Lie algebra $\mathfrak{o}(3,3)$ given by (11) around w.

(3) $(0,0,1,0)_w \iff (R, F)$ has the structure of parabolic geometry associated to the simple graded Lie algebra $\mathfrak{o}(4,2)$ given by (11) around w.

(4) $(0,0,0,1)_w \iff (R, F)$ *is of torsion type around* w.

We next have a result for regular overdetermined systems with non-regular symbols.

Theorem 5.2 *For the regular overdetermined systems given by $r = at^m$, $s = bt^n$ for constants $a > 0$, $b > 0$ and $m, n \in \mathbb{N}$, we have the value of the invariant $N(w) = (N^{inv}(w), N^{\mathfrak{o}(3,3)}(w), N^{\mathfrak{o}(4,2)}(w), N^{tor}(w))$ at each point $w \in R$ as follows*:

(1) $m - 1 = 2(n-1)$.

(1-1) $m = 1, n = 1$.

(1-1-1) *If $a - b^2 = 0$, then we have $(1,0,0,0)_w$ for any $w \in R$.*

(1-1-2) *If* $a - b^2 > 0$, *then we have* $(0,0,1,0)_w$ *for any* $w \in R$.
(1-1-3) *If* $a - b^2 < 0$, *then we have* $(0,1,0,0)_w$ *for any* $w \in R$.

(1-2) $m \geq 3$, $n \geq 2$.
(1-2-1) *If* $am - b^2n^2 = 0$, *then we have* $(1,0,0,0)_w$ *for any* $w \in R$.
(1-2-2) *If* $am - b^2n^2 > 0$, *then we have* $(1,0,2,0)_w$ *for any* $w \in \{t = 0\} \subset R$.
(1-2-3) *If* $am - b^2n^2 < 0$, *then we have* $(1,2,0,0)_w$ *for any* $w \in \{t = 0\} \subset R$.

(2) $m - 1 > 2(n-1)$.

(2-1) $n = 1$.
(2-1-1) *If* $m-1$ *is odd, then we have* $(1,1,1,0)_w$ *for any* $w \in \big\{t = (b^2/am)^{1/(m-1)}\big\} \subset R$.
(2-1-2) *If* $m-1$ *is even, then we have* $(1,1,1,0)_w$ *for any* $w \in \big\{t = (b^2/am)^{1/(m-1)}\big\} \cup \big\{t = -(b^2/am)^{1/(m-1)}\big\} \subset R$.

(2-2) $n \geq 2$.
(2-2-1) *If* $m - 2n + 1$ *is odd, then we have* $(1,2,0,0)_w$ *for any* $w \in \{t = 0\} \subset R$ *and* $(1,1,1,0)_w$ *for any* $w \in \big\{t = (b^2n^2/am)^{1/(m-2n+1)}\big\} \subset R$.
(2-2-2) *If* $m - 2n + 1$ *is even, then we have* $(1,1,1,0)_w$ *for any* $w \in \big\{t = (b^2n^2/am)^{1/(m-2n+1)}\big\} \cup \big\{t = -(b^2n^2/am)^{1/(m-2n+1)}\big\} \subset R$, *and* $(1,2,0,0)_w$ *for any* $w \in \{t = 0\} \subset R$.

(3) $m - 1 < 2(n-1)$.

(3-1) $m = 1$.
We have $(1,1,1,0)_w$ *for any* $w \in \big\{t = (a/b^2n^2)^{1/2(n-1)}\big\} \cup \big\{t = -(a/b^2n^2)^{1/2(n-1)}\big\} \subset R$.

(3-2) $m \geq 2$.
(3-2-1) *If* $2n - m - 1$ *is odd, then we have* $(1,1,1,0)_w$ *for any* $w \in \{t = 0\} \cup \big\{t = (am/b^2n^2)^{1/(2n-m-1)}\big\} \subset R$.
(3-2-2) *If* $2n - m - 1$ *is even, then we have* $(1,1,1,0)_w$ *for any* $w \in \big\{t = (am/b^2n^2)^{1/(2n-m-1)}\big\} \cup \big\{t = -(am/b^2n^2)^{1/(2n-m-1)}\big\} \subset R$ *and* $(1,0,2,0)_w$ *for any* $w \in \{t = 0\} \subset R$.

Proof. The differential system $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ on R is given by $\varpi_0 = dz - pdx - qdy$, $\varpi_1 = dp - at^m dx - bt^n dy$ and $\varpi_2 = dq - bt^n dx - tdy$. The structure equation of (R, D) is written as;

$$d\varpi_0 = \begin{pmatrix} dx & dy \end{pmatrix} \wedge \begin{pmatrix} dp \\ dq \end{pmatrix}, \quad \begin{pmatrix} d\varpi_1 \\ d\varpi_2 \end{pmatrix} = -\begin{pmatrix} amt^{m-1} & nbt^{n-1} \\ nbt^{n-1} & 1 \end{pmatrix} dt \wedge \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Here we observe the $(2,2)$-type real symmetric matrix $A := \left(\begin{smallmatrix} amt^{m-1} & nbt^{n-1} \\ nbt^{n-1} & 1 \end{smallmatrix}\right)$. Since $\det(\lambda E - A) = \lambda^2 - \lambda\{amt^{m-1}+1\} + amt^{m-1} - b^2n^2t^{2(n-1)}$, if we set $\alpha := amt^{m-1}+1$, $\beta := amt^{m-1} - b^2n^2t^{2(n-1)}$, then we have the eigenvalues $\lambda = \left(\alpha \pm \sqrt{\alpha^2 - 4\beta}\right)/2$ of A. Now, by the form of the defining equation or the structure equation of D, R does not have torsion. Hence, for any point $w \in R$, the symbol algebra $\mathfrak{s}(w)$ is one of the three types $\mathfrak{s}^i(w)$ ($i = inv$, $\mathfrak{o}(3,3)$, $\mathfrak{o}(4,2)$). By the proof of Theorem 3.3 in [8], the classification of these symbol algebras can be obtained in terms of the eigenvalues λ of A as follows;

(I) If the eigenvalues are degenerated type (i.e. either of the eigenvalues is zero) at w, the symbol algebra $\mathfrak{s}(w)$ is isomorphic to $\mathfrak{s}^{inv}(w)$.
(II) If the eigenvalues are $(+,-)$ type (i.e. two eigenvalues have the distinct signatures), the symbol algebra $\mathfrak{s}(w)$ is isomorphic to $\mathfrak{s}^{\mathfrak{o}(3,3)}(w)$.
(III) If the eigenvalues are $(+,+)$ or $(-,-)$ type (i.e. two eigenvalues have the same signatures), the symbol algebra $\mathfrak{s}(w)$ is isomorphic to $\mathfrak{s}^{\mathfrak{o}(4,2)}(w)$.

Hence, it is sufficient to clarify the signatures of the eigenvalues of A to prove our assertion. From the expression of the eigenvalues λ of A, we have the following characterization of each case; (I) $\beta(w) = 0$, (II) $\beta(w) < 0$ and (III) $\beta(w) > 0$. We prove each case.

First, we consider each case of (1) $m-1 = 2(n-1)$. For the cases of (1-1) (i.e. $m = n = 1$), from $\beta = a - b^2$, we have $N(w) = (1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ at any point $w \in R$ for each case $\beta = 0, < 0, > 0$ respectively. For the cases of (1-2) (i.e. $m \geq 3$, $n \geq 2$), we can write $\beta = t^{2(n-1)}(am - b^2n^2)$. Hence, if $am - b^2n^2 = 0$, then we have the statement of (1-2-1). Moreover, if $am - b^2n^2 > 0$ (resp. < 0), then β vanishes on the hypersurface $\{t = 0\} \subset R$ and we have $\beta > 0$ (resp. $\beta < 0$) on two connected components given by $\{t \neq 0\} \subset R$. Thus we have the statements of (1-2-2) and (1-2-3).

Next, we consider each case of (2) (i.e. $m-1 > 2(n-1)$). For the cases of (2-1) $m \geq 2$, $n = 1$, from $\beta = amt^{m-1} - b^2$, if $m-1$ is odd (resp. even), we have $\beta = 0$ on the hypersurface $\left\{t = (b^2/am)^{1/(m-1)}\right\} \subset R$, (resp. $\left\{t = \pm(b^2/am)^{1/(m-1)}\right\} \subset R$). If $m-1$ is odd, then we have $\beta < 0$ on $\left\{t < (b^2/am)^{1/(m-1)}\right\} \subset R$ and $\beta > 0$ on $\left\{t > (b^2/am)^{1/(m-1)}\right\} \subset R$.

Hence we have the result of (2-1-1). On the other hand, if $m-1$ is even, then $\beta < 0$ is satisfied on $\{|t| < (b^2/am)^{1/(m-1)}\} \subset R$ and $\beta > 0$ is satisfied on $\{|t| > (b^2/am)^{1/(m-1)}\} \subset R$. Thus we have the result of (2-1-2). In the cases of (2-2), from $\beta = t^{2(n-1)}(amt^{m-2n+1} - b^2n^2)$, if $m-2n+1$ is odd or even, we have $\beta = 0$ on the hypersurfaces $\{t = 0\} \cup \{t = (b^2n^2/am)^{1/(m-2n+1)}\} \subset R$, or $\{t = 0\} \cup \{t = \pm(b^2n^2/am)^{1/(m-2n+1)}\} \subset R$ respectively. If $m-2n+1$ is odd, then we have $\beta < 0$ on $\{t < 0\} \cup \{0 < t < (b^2n^2/am)^{1/(m-2n+1)}\} \subset R$ and $\beta > 0$ on $\{t > (b^2n^2/am)^{1/(m-2n+1)}\} \subset R$. Thus we have the result of (2-2-1). On the other hand, if $m-2n+1$ is even, then we have $\beta > 0$ on $\{|t| > (b^2n^2/am)^{1/(m-2n+1)}\} \subset R$ and $\beta < 0$ on the hypersurfaces $\{0 < |t| < (b^2n^2/am)^{1/(m-2n+1)}\} \subset R$. Thus we have the result of (2-2-2).

Finally, we study each case of (3) (i.e. $m-1 < 2(n-1)$). For the case of (3-1) given by $m = 1$, $n \geq 2$, since $\beta = a - b^2n^2t^{2(n-1)}$, we have $\beta = 0$ on the hypersurfaces $\{t = \pm(a/b^2n^2)^{1/2(n-1)}\} \subset R$. We also have $\beta < 0$ on $\{|t| > (a/b^2n^2)^{1/2(n-1)}\} \subset R$ and $\beta > 0$ on $\{|t| < (a/b^2n^2)^{1/2(n-1)}\} \subset R$. Hence we have the result of (3-1). We prove the cases of (3-2) (i.e. $m \geq 2$). Since $\beta = t^{m-1}(am - b^2n^2t^{2n-m-1})$, if $2n-m-1$ is odd or even, then we have $\beta = 0$ on $\{t = 0\} \cup \{t = (am/b^2n^2)^{1/(2n-m-1)}\} \subset R$ or $\{t = 0\} \cup \{t = \pm(am/b^2n^2)^{1/(2n-m-1)}\} \subset R$ respectively. Moreover, if $2n-m-1$ is odd, then we have $\beta < 0$ on $\{t < 0\} \cup \{t > (am/b^2n^2)^{1/(2n-m-1)}\} \subset R$ and $\beta > 0$ on $\{0 < t < (am/b^2n^2)^{1/(2n-m-1)}\} \subset R$. Hence we have the result of (3-2-1). On the other hand, if $2n-m-1$ is even , then we have $\beta < 0$ on $\{|t| > (am/b^2n^2)^{1/(2n-m-1)}\} \subset R$ and $\beta > 0$ on $\{0 < |t| < (am/b^2n^2)^{1/(2n-m-1)}\} \subset R$. Thus we have the result of (3-2-2). $\square$

In this theorem, the equation given by $a = 1/3$, $b = 1/2$, $m = 3$ and $n = 2$ is Cartan's overdetermined system which has the infinitesimal automorphism G_2 (cf. [4]).

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