A CHARACTERIZATION OF MODULAR NORMS IN TERMS OF SIMILAR TRANSFORMATIONS

Dedicated to Professor Kinjiro Kunugi on his 60 th birthday

By

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1. Introduction. *A modulared semi-ordered linear space is a universally continuous semi-ordered linear space*[1] R with a non-negative functional m called a *modular* which satisfies the following conditions:

M. 1) $|x| \leqq |y|$, $x, y \in R$ *implies* $m(x) \leqq m(y)$;

M. 2) $m(\xi x) = 0$ *for each* $\xi > 0$ *implies* $x = 0$;

M. 3) $\lim_{\xi \to 0} m(\xi x) = 0$ *for each* $x \in R$;

M. 4) $m(\xi x)$ *is a convex function of* $\xi > 0$ *for each* $x \in R$;

M. 5) $x \perp y$[2] *implies* $m(x+y) = m(x) + m(y)$;

M. 6) $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ *implies* $\sup_{\lambda \in \Lambda} m(x_\lambda) = m(x)$.

On a modulared space $(R. m)$ a semi-continuous norm[3] $\|\cdot\|_m$ can be defined by

$$(1.1) \qquad \|x\|_m = \inf \left\{ \frac{1}{|\xi|} ; \quad m(\xi x) \leqq 1 \right\}^{4)} \qquad (x \in R),$$

that is, R is a normed semi-ordered linear space with the norm $\|\cdot\|_m$ at the same time. The converse of this, *Every normed semi-ordered linear space* $(R, \|\cdot\|)$ *has an equivalent norm* $\|\cdot\|_m$ *defined by an appropriate modular* m, is not true in general. Counter examples were constructed by the present author [7] and T. Andô [1].

$\boldsymbol{L}_p$-spaces ($p \geqq 1$) and Orlicz spaces $\boldsymbol{L}_\Phi$[5] on a σ-finite measure space (E, Ω, μ), with a countably additive non-negative measure μ defined on a σ-field Ω of E,

1) A semi-ordered linear space R is called universally continuous, if $0 \leq x_\lambda (\lambda \in \Lambda)$ implies $\bigcap_{\lambda \in \Lambda} x_\lambda \in R$, i.e. a conditionally complete vector lattice in Birkhoff's sense.

2) $x \perp y$ means that x and y are mutually orthogonal, i.e. $|x| \cap |y| = 0$.

3) A norm $\|\cdot\|$ is called semi-continuous, if $|x_\lambda| \uparrow_{\lambda \in \Lambda} |x|$ implies $\|x\| = \sup_{\lambda \in \Lambda} \|x_\lambda\|$.

4) $\|\cdot\|_m$ is termed the modular norm by m.

5) For the definition of an Orlicz space see [4].

are considered as modulared spaces with modulars $m_p(x)=\int_E |x(t)|^p d\mu(t)$ and $m_\Phi(x)=\int_E \Phi(|x(t)|)d\mu(t)$ respectively, where $x\leqq y$ means $x(t)\leqq y(t)$ a.e..

A modular m on R is called *finite* if $m(x)<+\infty$ for each $x\in R$, and is called *almost finite* if m is finite on a complete semi-normal manifold[6] M of R. It is evident that the modulars of $\boldsymbol{L}_p$-type ($1\leqq p<+\infty$) are finite and the modulars m_Φ of Orlicz spaces are almost finite. m_Φ is finite if and only if Φ satisfies the so-called Δ_2-condition.

An excellent axiomatic characterization of $\boldsymbol{L}_p$-spaces in terms of norms on semi-ordered linear spaces was established by F. Bohnenblust in [2]. Later on, H. Nakano characterized norms of $\boldsymbol{L}_p$-spaces as norms of unique indicatrix [5]. Since these chracterizations are based on the particular structure of $\boldsymbol{L}_p$-norms, it seems to be difficult to obtain similarly simple characterizations of general modular norms, even of modular norms of Orlicz spaces, as $\boldsymbol{L}_p$-norms.

In this paper we shall present a necessary and sufficient condition in order that a norm $\|\cdot\|$ on R be the modular norm by a finite (almost finite) modular, in terms of the existence of a similar transformation T acting from R onto itself with the following property: *for any* $x, y\in R$ *with* $\|x\|=1$ *and* $x\perp y$, $\|T(x+y)\|=1$ *holds if and only if* $\|y\|=1$ *does* (*Theorems* 1, 2). According to the representation theory, this gives also an axiomatic characterization of modulared function spaces $\boldsymbol{L}_{M(\xi,t)}$.[7] In **5** we shall state some supplementary remarks with concrete explanations of these results in Banach function spaces.

2. Notations and the theorems. In what follows, let $(R, \|\cdot\|)$ be a non-atomic[8] universally continuous semi-ordered linear space with a semi-continuous norm $\|\cdot\|$. A norm $\|\cdot\|$ is called *continuous* if $x_\nu\downarrow_{\nu=1}^\infty 0$ implies $\|x_\nu\|\downarrow_{\nu=1}^\infty 0$ always. If there exists a complete semi-normal manifold M such that $\|\cdot\|$ is continuous on M, $\|\cdot\|$ is called *almost continuous*. The modular norm $\|\cdot\|_m$ is continuous if and only if m is finite. We denote by V the unit ball and by S its surface respectively, i.e. $V=\{x: \|x\|\leqq 1\}$ and $S=\{x: \|x\|=1\}$. We write $z=x\oplus y$, if $z=x+y$ with $x\perp y$ holds.

A one to one transformation T from R onto R is called *similar*, if it satisfies

(2.1) $T([p]x)=[p](Tx)$ *for each* $x\in R$ *and projector* $[p]$;

(2.2) $Tx\leqq Ty$ *if and only if* $x\leqq y$;

(2.3) $T(-x)=-Tx$ *for each* $x\in R$.

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6) A linear lattice manifold M is called semi-normal if $|y|\leqq|x|$, $x\in M$ implies $y\in M$. A semi-normal manifold M is complete, if $M^\perp=\{0\}$.

7) For the definition of $L_{M(\xi,t)}$ see [3 or 6].

8) R is termed non-atomic, if each $0\neq x\in R$ can be decomposed into $x=y+z$ with $y, z\neq 0$ and $y\perp z$.

We see easily from the definition that for a similar transformation T, T^{-1} is also such a one, and that T is order-continuous, i.e. $x_\nu\uparrow_{\nu=1}^{\infty}a$ (or $x_\nu\downarrow_{\nu=1}^{\infty}b$) implies $Tx_\nu\uparrow_{\nu=1}^{\infty}Ta$ (resp. $Tx_\nu\downarrow_{\nu=1}^{\infty}Tb$).

Here we consider the following condition which establishes a relation between a similar transformation T and the norm on R:

(T.C.) *For any x, y with $x\in S$ and $x\perp y$, $T(x+y)\in S$ holds if and only if $y\in S$.*

Now we can prove

Theorem 2.1. *In order that a given continuous norm $\|\cdot\|$ on R be the modular norm $\|\cdot\|_m$ by a modular m, it is necessary and sufficient that there exists a similar transformation T on R satisfying the condition* (T.C.).

If a modular m is almost finite the modular norm is almost continuous. For an almost continuous norm $\|\cdot\|$ we denote by R_C the *continuous manifold of R* with respect to $\|\cdot\|$, i.e., the totality of all continuous elements[9] of R. Evidently R_C is a complete semi-normal manifold on which $\|\cdot\|$ is continuous. Here we put $V_C=V\cap R_C$ and $S_C=S\cap R_C$. Then, for almost continuous norms we obtain

Theorem 2.2. *In order that a given almost continuous norm $\|\cdot\|$ on R be the modular norm $\|\cdot\|_m$ by a modular m, it is necessary and sufficient that there exists a similar transformation T on R_C onto R_C which satisfies the following condition:*

(T.C′.) *For any $x, y\in R_C$ with $x\in S_C$ and $x\perp y$, $T(x+y)\in S_C$ holds if and only if $y\in S_C$.*

To the proofs of these theorems the succeeding sections **3** and **4** shall be devoted.

3. Construction of orthogonal additive functional ρ. In this section, let $\|\cdot\|$ be continuous on R and T be a similar transformation satisfying the condition (T.C.) From (2.1)$\sim$(2.3) it follows that

$$T(x\oplus y)=Tx\oplus Ty \quad \text{and} \quad |Tx|=T(|x|) \qquad \text{for } x, y\in R. \tag{3.1}$$

First we shall prove several auxiliary lemmas easily derived from the assumption.

Lemma 1. *We have $T(V)\subset V-S$.*

Proof. Suppose $y\in S$ with $Ty\in S$. Then we have $T(y+0)=Ty\in S$, which implies $0\in S$ by (T.C.), a contradiction. On account of (2.2) and the semi-continuity of $\|\cdot\|$, it is now clear that $T(V)\subset V-S$ holds. Q.E.D.

9) If $\|x_\nu\|\downarrow 0$ for each $x_\nu\downarrow 0$ with $|x_\nu|\leqq|a|(1\leqq\nu)$, $a\in R$ is termed a continuous element of R with respect to $\|\cdot\|$.

In the sequel, we use the following notations:

(3. 2) $$S_0 = S \quad \text{and} \quad S_n = TS_{n-1} \qquad (n=1, 2, \cdots).$$

Now we have

Lemma 2. *$S_i \cap S_j = \phi$ holds for $i \neq j$ $(i, j = 1, 2, \cdots)$.*

Proof. If $z \in S_i \cap S_j$ for some i, j with $i<j$, *i.e.*, $z = T^i x = T^j y$ for some $x, y \in S$ we get $x = T^{j-i}y$. Putting $c = T^{j-i-1}y$, we obtain $x = Tc$ and $c \in V$, which is inconsistent with Lemma 1.

Lemma 3. *For each $x \in S_n$ $(n=0, 1, 2, \cdots)$ x can be decomposed into $x = x_1 \oplus x_2$ in such a way that $x_i \in S_{n+1}$ $(i=1, 2)$ holds.*

Proof. $x \in S_n$ implies $T^{-n}x \in S$, whence $\|T^{-(n+1)}x\| > 1$. Now we put $a = T^{-(n+1)}x$. Since R contains no atomic element and $\|\cdot\|$ is continuous, we can find an element p such that $[p]a \in S$ holds. Because of $[p]a$, $Ta \in S$, it follows from (T.C.) that $(1-[p])a \in S$ holds. Hence $x = T^{n+-}a = T^{n+1}[p]a + T^{n+1}((1-[p])a)$ with $T^{n+1}[p]a$, $T^{n+1}((1-[p])a) \in S_{n+1}$ simultaneously.

It is obvious from Lemma 3 that $x \in S$ if and only if x is represented as, for any fixed n,

(3. 3) $$x = T^n(\overset{2^n}{\underset{i=1}{\oplus}} x_i),$$

where $x_i \in S$ $(i=1, 2, \cdots, 2^n)$.

Lemma 4. *Let a, $b \in S$* and $a \perp b$. Then

(3. 4) $$\|T^n a \oplus b\| > 1$$

stands for each $n \geqq 1$.

Proof. We shall prove this lemma by inducotin. In case of $n=1$, $\|Ta \oplus b\| = 1$ implies $\|T(a \oplus T^{-1}b)\| = 1$, whence $T^{-1}b \in S$, contradicting Lemma 1. Thus (3. 4) is valid for $n=1$. Now suppose that (3. 4) holds for each $n \leqq k$ and $\|T^{k+1}a \oplus b\| = 1$ for some $a, b \in S$ with $a \perp b$. Then $\|T(T^k a \oplus T^{-1}b)\| = 1$ holds and $T^{-1}b$ can be represented as $T^{-1}b = b_1 \oplus b_2$, $b_i \in S$ $(i=1, 2)$. From this and $\|T\{(T^k a \oplus b_1) + b_2\}\| = 1$, it follows that $\|T^k a \oplus b_1\| = 1$ holds on account of (T.C.), but this contradicts the induction hypothesis. Q. E. D.

Lemma 5. *If $x = \overset{n}{\underset{i=1}{\oplus}} x_i = \overset{m}{\underset{j=1}{\oplus}} y_j \oplus y_0$ with $x_i \in S$, $y_j \in S$ $(i=1, 2, \cdots, n;\ j=1, 2, \cdots, m)$ and furthermore x is not a complete*[10] *element, then $n \geqq m$ holds.*

Proof. Suppose contrarily $n<m$. Since R is non-atomic, we can find a set of mutually orthogonal elements $\{z_i\}_{i=1}^{\rho} \subset S$ such that $z_i \perp x$ $(1 \leqq i \leqq \rho)$ and

10) $x \in R$ is called a complete element if $\{x\}^{\perp} = \{0\}$ holds.

$n+\rho=2^{\mu}$ for some $\mu\geqq 1$. Then $T^{\mu}(\bigoplus_{i=1}^{\rho} z_i \oplus x)=T^{\mu}(z_1\oplus\cdots\oplus z_\rho\oplus x_1\oplus\cdots\oplus x_n)\in S$. On the other hand, we get $T^{\mu}(\bigoplus_{i=1}^{\rho} z_i\oplus x)=T^{\mu}(z_1\oplus\cdots\oplus z_\rho\oplus y_1\oplus\cdots\oplus y_m\oplus y_0)=T^{\mu}(y_1\oplus\cdots\oplus y_n\oplus z_1\oplus\cdots\oplus z_\rho)+T^{\mu}(y_{n+1}\oplus\cdots\oplus y_m)+T^{\mu}(y_0)$, which implies $1=\|T^{\mu}(x\oplus z)\|\geqq\|w+T^{\mu}y_m\|$, where $w=T^{\mu}(y_1\oplus\cdots\oplus y_n\oplus z_1\oplus\cdots\oplus z_\rho)$ belongs to S. However, this is inconsistent with the preceding lemma. Q. E. D.

Lemma 6. *If x is not a complete element and $x=x_1\oplus\cdots\oplus x_k=y_1\oplus\cdots\oplus y_l\oplus y_0$, where $x_\nu\in S_{m_\nu}$, $y_\mu\in S_{n_\mu}$ and $y_0\in V$ $(1\leqq\nu\leqq k,\ 1\leqq\mu\leqq l,\ 0\leqq m_\nu,\ n_\mu)$, then $\sum_{\nu=1}^{k}\frac{1}{2^{m_\nu}}\geqq\sum_{\mu=1}^{l}\frac{1}{2^{n_\mu}}$ holds.*

Proof. We put $N=\operatorname{Max}_{\substack{1\leqq\nu\leqq k\\1\leqq\mu\leqq l}}\{m_\nu, n_\mu\}$. Then, for each ν $(1\leqq\nu\leqq k)$ x is decomposed into $x_\nu=x_{\nu,1}\oplus x_{\nu,2}\oplus\cdots\oplus x_{\nu,2^{N-m_\nu}}$ with $x_{\nu,i}\in S_N$ $(1\leqq i\leqq 2^{N-m_\nu})$. Similarly $y_\mu=y_{\mu,1}\oplus\cdots\oplus y_{\mu,2^{N-n_\mu}}$ with $y_{\mu,j}\in S_N$ holds for each j $(1\leqq j\leqq 2^{N-n_\mu})$. Hence both $x=\bigoplus_{\nu=1}^{k}\bigoplus_{i=1}^{2^{N-m_\nu}}x_{\nu,i}$ and $x=\bigoplus_{\mu=1}^{l}\bigoplus_{j=1}^{2^{N-n_\mu}}y_{\mu,j}\oplus y_0$ holds, which implies $T^{-N}x=\bigoplus_{\nu}\bigoplus_{i}T^{-N}x_{\nu,i}=\bigoplus_{\mu}\bigoplus_{j}T^{-N}y_{\mu,j}\oplus T^{-N}y_0$ with $T^{-N}x_{\nu,i}\in S$ and $T^{-N}y_{\mu,j}\in S$ for each ν,μ,i, and j. In view of the preceding lemma we find

$$\sum_{\nu=1}^{k}2^{N-m_\nu}\geqq\sum_{\mu=1}^{l}2^{N-n_\mu}.$$

Thus we obtain $\sum_{\nu=1}^{k}\frac{1}{2^{m_\nu}}\geqq\sum_{\mu=1}^{l}\frac{1}{2^{n_\mu}}$. Q. E. D.

Here we turn to define an orthogonal additive functional (i.e. $\rho(x+y)=\rho(x)+\rho(y)$ for $x\perp y$) on R from $\|\cdot\|$. Let R_0 be the set of all non-complete elements of R and $\mathfrak{A}$ be the totality of elements of R_0 which can be represented as $x_1\oplus\cdots\oplus x_n$ with $x_i\in S_{m_i}$ $(i=1,2,\cdots,n;\ n=1,2,\cdots)$. On $\mathfrak{A}$ we define a functional ρ' as follows:

$$\rho'(x)=\sum_{i=1}^{n}\frac{1}{2^{m_i}},\tag{3.5}$$

where $x=x_1\oplus\cdots\oplus x_n$ with $x_i\in S_{m_i}$ $(1\leqq i\leqq n)$. According to Lemma 6 we see that this definition has a sense. It is evident from the definition that ρ' is orthogonally additive on $\mathfrak{A}$. Next, we put for each $x\in R$

$$\rho(x)=\begin{cases}\sup\limits_{|y|\leqq|x|,\,y\in\mathfrak{A}}\rho'(y),\\[1ex] 0, & \text{if there exists no element } y\in\mathfrak{A} \text{ with } |y|\leqq|x|.\end{cases}\tag{3.6}$$

In the succeeding section we shall show that ρ thus defined is in fact a modular on R and that $\|\cdot\|$ is nothing but the modular norm by ρ.

4. Properties of ρ and the proofs of Theorems. In view of construction of ρ and Lemma 6 we see easily that ρ satisfies the modular conditions M. 1) and M. 2). Since R contains no atomic element, we have also

$$\rho(x)=\rho'(x) \qquad \text{for each} \quad x\in\mathfrak{A}. \tag{4.1}$$

In order to prove the remaining conditions M. 3),~M. 6), we need some lemmas.

Lemma 7. *We have*

$$\rho(x)>\frac{1}{2^m} \text{ and } \rho(x)<\frac{1}{2^m} \text{ imply } \|T^{-m}x\|>1 \text{ and } \|T^{-m}x\| \tag{4.2}$$

$\leqq 1$ *respectively* $(m=0,1,2,\cdots)$;

$$\rho(x)<+\infty, \qquad \text{for each} \quad x\in R; \tag{4.3}$$

$$\rho(x)=\sup_{[p]x\in\mathfrak{A}}\rho'([p]x), \qquad \text{if} \quad \rho(x)>0. \tag{4.4}$$

Proof. (4.2) follows immediately from the definition of ρ. Since $\|\cdot\|$ is continuous, each element $x\in R$ can be represented as $x=\bigoplus_{i=1}^{n}x_i$ with $\|x_i\|\leqq 1$ $(1\leqq i\leqq n)$ for some $n\geqq 1$. From this we have $\rho(x)\leqq n$ in view of (3.5), (4.2) and M.2). Thus (4.3) is valid. Next, we shall show that if $\rho(x)>\frac{k}{2^m}$ x is written as $x=\bigoplus_{i=1}^{k}x_i\oplus x_0$ with $\|T^{-m}x_i\|>1$ for each i $(1\leqq i\leqq k)$. By (3.6) there exists $0\leqq x'\in\mathfrak{A}$ such that $|x|\geqq x'=\bigoplus_{i=1}^{k}x'_i\oplus x'_0$ with $x'_i\in S_m$ $(1\leqq i\leqq k)$ and $x_0\in\mathfrak{A}$. Now we decompose x'_0 into $x'_0=\bigoplus_{i=1}^{k}x''_i$ with $x''_i\in\mathfrak{A}$ for each i. On the ground of Lemma 4 $\|T^{-m}(x'_i\oplus x''_i)\|>1$ $(1\leqq i\leqq k)$ must hold. Putting $x_i=[x'_i\oplus x''_i]x$ and $x_0=x-\bigoplus_{i=1}^{k}x_i$, we obtain $x=\bigoplus_{i=1}^{k}x_i\oplus x_0$ with $\|T^{-m}x_i\|>1$ for each i $(1\leqq i\leqq k)$.

From this one derives easily that if $\rho(x)>\frac{k}{2^m}$ there exist projectors $\{[p_\nu]\}_{\nu=1}^{k}$ such that $[p_i]\leqq[x_i]$ and $\|T^{-m}[p_i]x_i\|=1$ hold $(1\leqq i\leqq k)$, where $\{x_i\}_{i=1}^{k}$ satisfies the above condition. Since $[p_i]x_i\in S_m$ and $\bigoplus_{i=1}^{k}[p_i]x_i=(\sum_{i=1}^{k}[p_i])x$, $\rho'([p]x)\geqq\frac{k}{2^m}$ follows and (4.4) is proved, where $[p]=\sum_{i=1}^{k}[p_i]$. Q. E. D.

Lemma 8. *ρ is orthogonally additive, i.e., it satisfies* M. 5).

Proof. From the definition of ρ it follows that

$$\rho(x\oplus y)\geqq\rho(x)+\rho(y)$$

holds. Now suppose $\rho(x \oplus y) > \rho(x) + \rho(y)$ for some $x, y \in R$ with $x + y \in R_0$. Then we can choose a natural number m such that $\rho(x \oplus y) > \rho(x) + \rho(y) + \frac{1}{2^m}$. By (4.4) there exist projectors $[p]$, $[q]$ for which $\rho(x) - \rho'([p]x) < \frac{1}{2^{m+2}}$, $\rho(y) - \rho'([q]y) < \frac{1}{2^{m+2}}$, $[p]x \in \mathfrak{A}$ and $[q]y \in \mathfrak{A}$ hold.[11] Since $\rho((1-[p])x) \leqq \rho(x) - \rho([p]x) < \frac{1}{2^{m+2}}$ and $\rho((1-[q])y) \leqq \rho(y) - \rho([q]y) < \frac{1}{2^{m+2}}$ hold, we can find α, $\beta \geqq 1$ such that both $\alpha(1-[p])x$ and $\beta(1-[q])y$ belong to S_{m+2} according to (4.2) and the fact that T is similar. Putting $x' = [p]x + \alpha(1-[p])x$ and $y' = [q]y + \beta(1-[q])y$, we obtain $x', y' \in \mathfrak{A}$ and $\rho'(x' \oplus y') = \rho'(x') + \rho'(y') = \rho'([p]x) + \rho'([q]y) + \frac{1}{2^{m+1}}$, since ρ' is orthogonally additive on $\mathfrak{A}$. Hence we get

$$\rho'(x' \oplus y') \geqq \rho(x \oplus y) > \rho(x) + \rho(y) + \frac{1}{2^m}$$

$$\geqq \rho'([p]x) + \rho'([q]y) + \frac{1}{2^m} = \rho'(x' \oplus y') + \frac{1}{2^{m+1}},$$

which is, however, a contradiction. Thus we see easily that ρ is orthogonally additive by virtue of Lemma 7. Q. E. D.

Lemma 9. *We have*

$$\rho(x) \leqq 1 \quad \textit{if and only if} \quad \|x\| \leqq 1. \tag{4.5}$$

Proof. The fact that $\|x\| \leqq 1$ implies $\rho(x) \leqq 1$ is ovbious by virtue of Lemma 4. On the other hand, for any x with $\rho(x) \leqq 1$ we can find a sequence of projectors $\{[p_\nu]\}_{\nu=1}^{\infty}$ such that $[p_\nu] \uparrow_{\nu=1}^{\infty} [x]$, $[p_\nu]x \in \mathfrak{A}$ and $\rho([p_\nu]x) \uparrow_{\nu=1}^{\infty} \rho(x) \leqq 1$ on account of (4.4) and the orthogonal additivity of ρ. By (4.1) and the definition of ρ', we now get $\|[p_\nu]x\| \leqq 1$ for each $\nu \geqq 1$, hence $\|x\| \leqq 1$ because of the semi-continuity of $\|\cdot\|$. Q. E. D.

Lemma 10. *ρ is semi-continuous, i.e., it satisfies* M. 6).

Proof. Let $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ and $\rho(x) > \frac{k}{2^m}$. As is shown in the proof of (4.4), there exists $p \in R$ such that $[p]x \in \mathfrak{A}$, $[p]x = \bigoplus_{i=1}^{k} w_i$ and $\|T^{-m} w_i\| > 1$ $(1 \leqq i \leqq k)$. Then, since $[w_i]x_\lambda \uparrow_{\lambda \in \Lambda} [w_i]x = w_i$ holds for each i and $\|\cdot\|$ is semi-continuous, we have for a sufficiently large λ_0 that $\|T^{-m}[w_i]x_{\lambda_0}\| > 1$ stands for every i $(1 \leqq i \leqq k)$. Therefore we have

11) In case of $\rho(x) = 0$ (or $\rho(y) = 0$), we choose $p = 0$ (resp. $q = 0$).

$$\rho(x_{\lambda_0}) \geqq \rho([p]x_{\lambda_0}) \geqq \frac{k}{2^m},$$

which shows the semi-continuity of ρ. Q. E. D.

Lemma 11. *ρ satisfies* M.3). *i.e.*, $\lim_{\xi\to 0}\rho(\xi x)=0$.

Proof. If $\rho(\xi x) > \frac{1}{2^m}$ holds for each $\xi>0$, we have $\|T^{-m}\xi x\|>1$. Since $\bigcap_{\xi>0}\xi|x|=0$ stands, $\bigcap_{\xi>0}T^{-m}\xi|x|=0$ holds. Hence it follows that $\|T^{-m}\xi x\|\to 0$ as $\xi\to 0$, because of the continuity of $\|\cdot\|$. This is a contradiction. Q. E. D.

Summing up the above results, we see that ρ satisfies all the conditions of modular except M.4). Next lemma shall show that ρ fulfils M.4) too.

Lemma 12. *$\rho(\xi x)$ is a convex function of ξ ($\xi \geqq 0$) for each $x\in R$.*

Proof. We shall first show that the set $B_\xi=\{x\colon \rho(x)\leqq\xi\}$ is convex for every ξ with $0\leqq\xi\leqq 1$. Let $x, y\in B_\xi$ and $\alpha,\beta>0$ with $\alpha+\beta=1$. By virtue of semi-continuity of ρ, we may assume without loss of generality that there exists $0\neq z\in R$ belonging to $\{x, y\}^{\perp}$. Furthermore we may choose z as $\rho(z)=1-\xi$, since ρ satisfies (4.3) and R has no atom. It follows that both $x+z$ and $y+z$ belong to V, hence $\alpha(x+z)+\beta(y+z)$ does also. Consequently, we obtain $\rho(\alpha x+\beta y)+\rho(z)\leqq 1$ by Lemma 9, hence $\alpha x+\beta y\in B_\xi$. Therefore B_ξ is convex.

Next, suppose that $\rho(\xi x)\leqq 1$ and $\rho(\eta x)\leqq 1$ for some $x\in R$ and $\xi>\eta\geqq 0$. Since ρ is finite, orthogonally additive and semi-continuous, we can find $p\in R$ for which $\rho(\xi[p]x)=\rho(\xi(1-[p])x)$ holds. If $\rho(\eta[p]x)<\rho(\eta(1-[p])x)$ stands for such $[p]$, there can be constructed a system of projectors $\{[p_\alpha]\}_{(0\leqq\alpha\leqq 1)}$ and $\{[p'_\alpha]\}_{(0\leqq\alpha\leqq 1)}$ such that $[p_\alpha]\downarrow([p'_\alpha]\uparrow)$ as $\alpha\downarrow 0$, $[p_\alpha]\leqq[p]$, $[p'_\alpha]\leqq(1-[p])$ with $[p_1]=[p]$, $[p'_0]=(1-[p])$, and $\rho(\xi[p_\alpha]x)=\rho(\xi(1-[p])(1-[p'_\alpha])x)=\alpha\,\rho(\xi[p]x)$ holds for each $0\leqq\alpha\leqq 1$. Putting $[q_\alpha]=[p_\alpha]+(1-[p])[p'_\alpha]$, we obtain $[q_\alpha]\leqq[x]$ and $\rho(\xi[q_\alpha]x)=\rho(\xi(1-[q_\alpha])x)$ for every α. Furthermore we see easily that both $\rho(\eta[q_1]x)<\rho(\eta(1-[q_1])x)$ and $\rho(\eta[q_0]x)>\rho(\eta(1-[q_0])x)$ hold. From this it follows that $\rho(\eta[q_\alpha]x)=\rho(\eta(1-[q_\alpha])x)$ stands for some α. In consequence, we have shown that there exists $p\in R$ such that $\rho(\xi[p]x)=\frac{1}{2}\rho(\xi x)$ and $\rho(\eta[p]x)=\frac{1}{2}\rho(\eta x)$ hold simultaneously. Because $\rho(\xi[p]x+\eta(1-[p])x)=\rho(\eta[p]x+\xi(1-[p])x)=\frac{1}{2}\left\{\rho(\xi x)+\rho(\eta x)\right\}\leqq 1$, we have

$$\rho\left(\frac{1}{2}(\xi x+\eta y)\right)\leqq\frac{1}{2}\left\{\rho(\xi x)+\rho(\eta y)\right\} \tag{4.6}$$

by the fact shown just above.

Finally, since each x can be decomposed orthogonally into $x=\bigoplus_{i=1}^{n} x_i$ with $\rho(x_i)\leqq 1$ $(1\leqq i\leqq n)$, we see that (4.6) holds for any $x\in R$, i.e. $\rho(\xi x)$ is a convex function of ξ $(\xi\geqq 0)$ for each $x\in R$. Q. E. D.

Here we are in position to prove the theorems stated in **2.**

Proof of Theorem 1. *Sufficiency.* The functional ρ constructed in **3** is a modular satisfying (4.5), as is shown above. Hence we have $\|x\|=\inf\left\{\frac{1}{|\xi|};\ \rho(\xi x)\leqq 1\right\}$ i.e., $\|\cdot\|$ is the modular norm by the modular ρ.

Necessity. Let $\|\cdot\|$ be the modular norm by a modular m on R. m is necessarily finite since $\|\cdot\|$ is continuous, In the same manner as in the proof of Theorem 2 in [8], we can construct a similar transformation T_0 on R satisfying

$$m(T_0 x)=\frac{1}{2}m(x) \qquad \text{for every} \quad x\in R.$$

It is now clear that T_0 satisfies the condition (T.C.). Q. E. D.

Proof of Theorem 2. *Sufficiency.* In view of Theorem 1 we find a finite modular ρ on a complete semi-normal manifold R_C of R, for which $\|\cdot\|$ is the modular norm on R_C. We extend now ρ on the whole space R as follows:

$$\rho_0(x)=\sup_{0\leqq y\leqq |x|,\ y\in R_C}\rho(y) \qquad (x\in R). \tag{4.7}$$

ρ_0 thus defined is an almost finite modular on R, as is easily seen, and $\rho_0(x)=\rho(x)$ if $x\in R_C$. Because of the semi-continuity of $\|\cdot\|$ and ρ, $\|x\|=\inf\left\{\frac{1}{|\xi|};\ \rho_0(\xi x)\leqq 1\right\}$ holds for each $x\in R$, that is, $\|\cdot\|$ is the modular norm by ρ_0. The nessecity is derived similarly as the proof of Theorem 1. Q. E. D.

5. Here let $(R, \|\cdot\|)$ be the same as in **3** and ρ be the modular defined, in the manner described above, from $\|\cdot\|$ and a similar transformation T on R satisfying the condition (T.C.). From the construction of ρ one derives easily

$$\rho(Tx)=\frac{1}{2}\rho(x) \qquad (x\in R).$$

Also this enables us obviously to restate properties of the modular ρ in terms of similar transformations T.[12] We describe below a few examples of such

12) Of course, we can state properties of ρ by means of $\|\cdot\|$, since there are found closed relations between modulars and their norms [1, 6, and 7].

restatements in terms of T. Being trivial, their proofs are omitted.

5.1. *ρ is simple (i.e. $\rho(x)=0$ implies $x=0$), if and only if $\bigcap_{m\geqq 1} T^m x=0$ for each $x\in R$.*

5.2. *ρ is uniformly simple (i.e. $\inf_{\|x\|\geqq\delta}\rho(x)>0$ for each $\delta>0$), if and only if for each $\varepsilon>0$ there exists $m\geqq 0$ with $\sup_{x\in S}\|T^m x\|<\varepsilon$.*

5.3. *ρ is uniformly finite (i.e. $\sup_{\|x\|\leqq\delta}\rho(x)<+\infty$ for each $\delta>0$), if and only if for each $\delta>0$ there exists $m\geqq 0$ with $\inf_{x\in S}\|T^{-m}x\|>\delta$.*

5.4. *ρ is upper bounded (i.e. $\rho(\alpha x)\leqq\gamma\rho(x)$ holds for every $x\in R$, where $1<\alpha$, γ are fixed constants), if and only if $T\leqq\left(\frac{1}{2}\right)^{\frac{1}{p}}I$[13] for some $p\geqq 1$.*

Finally let (E, Ω, μ) be a σ-finite non-atomic measure space with a countably additive non-negative measure μ on a σ-field Ω of E. A modulared space $(\boldsymbol{X}, m)$ consisting of measurable functions on E is a semi-normal manifold of *modulared function space* $\boldsymbol{L}_{M(\xi,t)}$ defined by a modular function $M(\xi, t)$[14] on $[0, \infty)\times E$, that is, $\boldsymbol{X}$ is contained in the totality of all measurable functions f such that $\int_E M(\alpha|f(t)|, t)\,d\mu(t)<+\infty$ for some $\alpha>0$, and

$$m(f)=\int_E M(|f(t)|, t)\,d\mu(t) \tag{5.1}$$

holds for each $f\in X$. Conversely, it is known [6] that each modulared semi-ordered linear space R can be considered as a modulared function space $\boldsymbol{L}_{M(\xi,t)}$ on a measure space (E, Ω, μ) suitably chosen, and m is represented by (5.1).

For any finite modulared function space[15] $(\boldsymbol{L}_{M(\xi,t)}(E), \|\cdot\|)$ we can obtain a similar transformation T with the condition (T.C.) directly as follows: We define for $(\xi, t)\in[0, \infty)\times E$

$$h(\xi, t)=\begin{cases} M_t^{-1}\left(\frac{1}{2}M(\xi, t)\right), & \text{if } M(\xi, t)>0\,; \\ \xi, & \text{if } M(\xi, t)=0, \end{cases} \tag{5.2}$$

where $M_t^{-1}(\xi)$ is the inverse of the function $M_t(\xi)=M(\xi, t)$ for each $t\in E$. Then $h(\xi, t)$ on $[0, \infty)\times E$ is a Carathéodory's function, and the transformation $\mathfrak{h}$ defined by

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13) I is the identity operator on R and **5.4** follows from Theorem **3.3** of [9].

14) For the definition of modular functions see [3 or 6]. Roughly speaking, $M(\xi, t)$ is a N'-function of ξ for each $t\in E$. In $L_{M(\xi,t)}$ we consider $\int_E M(|f(t)|, t)\,d\mu(t)$ as a modular m always.

15) m on $L_{M(\xi,t)}$ is finite, if and only if $M(2\xi, t)\leqq\gamma M(\xi, t)+a(t)$ for all $(\xi, t)\in[0, \infty)\times E$, where $\gamma>0$ and $a(t)\in L_1(E)$[3]. m is almost finite if and only if $M(\xi, t)<+\infty$ a. e. in $[0. \infty)\times E$.

$$(5.3) \qquad \mathfrak{h}(f(t)) = h(f(t), t) \qquad (f \in \boldsymbol{L}_{M(\xi,t)})$$

forms a similar transformation satisfying the condition (T. C.) for the modular norm on $\boldsymbol{L}_{M(\xi,t)}$. Conversely, in view of Theorem 1 we have

Theorem 3. *If $(\boldsymbol{X}, \|\cdot\|)$ is a normed function space*[16] *with a continuous norm $\|\cdot\|$, and if a similar transformation $\mathfrak{h}$ from $\boldsymbol{X}$ onto $\boldsymbol{X}$, defined by a Carathéodory's function $h(\xi, t)$*[17] *on $[0, \infty) \times E$, satisfies the condition (T. C.), then there can be found a modular function $M(\xi, t)$ on $[0, \infty) \times E$ such that $\boldsymbol{X}$ is a semi-normal manifold of $\boldsymbol{L}_{M(\xi,t)}$, $M(\xi, t)$ satisfies (5.2),*[18] *and $\|\cdot\|$ coincides with the modular norm of the space $\boldsymbol{L}_{M(\xi,t)}$.*

Remark 1. In this theorem if moreover, $(\boldsymbol{X}, \|\cdot\|)$ is monotone complete (i. e. $0 \leqq f_\nu \uparrow$, $\sup_{\nu \geqq 1} \|f_\nu\| < +\infty$ implies $\bigcup_{\nu=1}^{\infty} f_\nu \in \boldsymbol{X}$), then $\boldsymbol{X} = \boldsymbol{L}_{M(\xi,t)}$ holds.

Remark 2. In Theorem 3, if $h(\xi, t) = h(\xi)$ for all $(\xi, t) \in [0, \infty) \times E$, then $L_{M(\xi,t)}$ can be replaced by an Orlicz space $\boldsymbol{L}_M$.

When $\|\cdot\|$ is almost continuous, we have a similar theorem as above on the basis of Theorem 2. In this case, $\mathfrak{h}$ acts from $\boldsymbol{L}^{(f)}_{M(\xi,t)}$, the finite manifold of $\boldsymbol{L}_{M(\xi,t)}$ (the totality of all $f \in \boldsymbol{L}_{M(\xi,t)}$ with $m(\xi f) < +\infty$ for every $\xi \geqq 0$), onto itself and satisfies (5.2), if $0 < M(\xi, t) < +\infty$.

On the basis of Theorems 1 and 2, a theorem characterizing the modular norms in terms of norms only can be obtained, and it shall be shown in a separate paper.

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16) We assume that X is semi-normal.

17) For $h(\xi, t)$ we assume $h(0, t) \equiv 0$ for all $t \in E$.

18) Strictly speaking, $h(\xi, t) = M_t^{-1}(\frac{1}{2}M(\xi, t))$ holds a. e. for (ξ, t) satisfying $M(\xi, t) > 0$. In general, $h(\xi, t) = \xi$ does not hold for (ξ, t) with $M(\xi, t) = 0$.

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