# PARTIALLY ORDERED ABELIAN SEMIGROUPS II. ON THE STRONGNESS OF THE LINEAR ORDER DEFINED ON ABELIAN SEMIGROUPS

By

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In my previous paper[(1)], I note that a strong linearly ordered abelian semigroup (strong l.o. semigroup) is always normal, its elements are of infinite order except the unit element (if there exists) and the product cancellation law is held in it. In this Part, I shall seek for the other conditions which are exchangeable for the cancellation law. But the some theorems in this note will be expressed in the form of a partially ordered abelian semigroup.

*Definition 1.* Let $a$ be any element of an abelian semigroup $S$. We denote the set of all positive powers of $a$, in other words, the sub-semigroup of $S$ generated by $a$, by $S(a)$, which is called the *sector* by $a$. And let $T(a)$ be the set of all elements of $S$ whose some positive powers belong to $S(a)$, that is, the set of all elements $x$ of $S$ such that $x^m = a^n$ for some positive integers $m$ and $n$. $T(a)$ is called the *complete sector* by $a$.

Let $a$ and $b$ be any two elements of $S$. Then the two complete sectors $T(a)$ and $T(b)$ are either disjoint or identical. Hence, $S$ is the union of mutually disjoint complete sectors, and clearly $T(a)$ is a sub-semigroup of $S$.

*Definition 2.*[(2)] Let $a$ be any element of an abelian semigroup $S$, and we shall consider the sector $S(a)$ by $a$. There are two possible cases.

(1) Partially ordered abelian semigroups. I. On the extension of the strong partial order defined on abelian semigroups. Journal of the Faculty of Science, Hokkaido University, Series I, vol. XI, No. 4 (1951) pp. 181–189; this is referred to hereafter as "O.I."

A set $S$ is said to be a *partially ordered abelian semigroup* (p.o. semigroup), when $S$ is (I) an abelian semigroup (not necessarily contains the unit element), (II) a partially ordered set, and satisfies (III) the *homogeneity*: $a \geqq b$ implies $ac \geqq bc$ for any $c$ of $S$. When a partial order $P$ of an abelian semigroup $S$ satisfies the condition (III), we call $P$ a *partial order defined on* an abelian semigroup $S$. (Definition 1, O.I.)

(2) Cf. D. REES: On semi-groups, Proc. Cambridge Phil. Soc., vol. 36 (1940), pp. 387–400.

First, $a$ may satisfy no equation of the form $a^m = a^n\ (m \neq n)$. In this case, the element $a$ is said to be of *infinite order*.

Secondly, $a$ may satisfy a relation of the form $a^m = a^n\ (n > m)$. If $a$ satisfies such a relation, choose one for which $n$ is a minimum. Then clearly $a, a^2, \cdots a^{n-1}$ are distinct elements of $S(a)$. We write $n = m + r$, $r \geqq 1$. Then, if $q$ is any positive integer, by simple induction we have immediately $a^m = a^{m+qr}$. Now, if $N$ is any integer not less than $n$, we may write it in the form $N = m + qr + s\ (1 \leqq q,\ 0 \leqq s < r)$. Then, by the above, we have $a^N = a^{m+s}$. Accordingly, it follows that the sector $S(a)$ just consists of the $n-1$ elements $a, a^2, \cdots, a^{n-1}$. Further, let $t$ be the least integer of the form $qr+1$ greater than $m$ and put $d = a^t$. Then, since $t$ is prime to $r$, the elements $d, d^2, \cdots d^r$ are distinct from one another and coincide with the elements $a^m, a^{m+1}, \cdots, a^{n-1}$, and it is easily seen that

$$d^r d^s = d^s d^r = d^{r+s} = d^s \qquad (s \leqq r),$$
$$d^{r-s} d^s = d^s d^{r-s} = d^r \qquad (s < r).$$

Then the above shows that $a^m, a^{m+1}, \cdots, a^{n-1}$ form a cyclic group, generated by $d$, with the identity $d^r$. Such an element $a$ is said to be of *finite order*. We shall term $m$ the *length* and $r$ the *period* of the element $a$ of finite order.

An element $a$ of finite order is called *cyclic, quasi-idempotent* or *idempotent*, when $m = 1$, $r = 1$ or $m = 1$ and $r = 1$ respectively. The immediate consequence of the above consideration is that any semigroup containing elements of finite order contains idempotent elements.

Moreover, from the definition of the complete sector we get readily the following: The complete sector by the element of finite order does not contain the element of infinite order and contains one and only one idempotent element, and the complete sector by the element of infinite order does not contain the element of finite order.

*Definition 3.* We consider the following conditions for an abelian semigroup $S$:

($\alpha$): $a^n = b^n$ for some positive integer $n$ implies $a = b$.

($\beta$): All elements of $S$ are of infinite order except the unit element (if there exists).

($\gamma$): If $a$ is not the unit element, then $ab \neq b$ for any $b$ of $S$.

($\delta$): If two complete sectors $T(a)$ and $T(b)$ are disjoint, then $c\cdot T(a)$(3)

(3) $c\,T(a)$ is the set of all elements $cx$, $x \in T(a)$.

and $c\cdot T(b)$ are disjoint for any $c$ of $S$.

Moreover, we consider the following conditions for a p.o. semigroup $S$:

$(A)$: If $a$ is positive or negative[4], then $ab \geqq b$ or $b \geqq ab$ for any $b$ of $S$ respectively.

$(B)$: If $a$ ($\neq$ the unit element) is positive or negative, then $ab > b$ or $b > ab$ for any $b$ of $S$ respectively.

*Theorem 1.* If an abelian semigroup $S$ satisfies the condition $(\alpha)$, then the element of $S$ is idempotent or of infinite order.

*Proof.* Let $a$ be an element of finite order, $m$ and $r$ be the length and the period of $a$ respectively. And let $Z = \{a^m, \cdots, a^{m+r-1}\}$, then $Z$ is a cyclic group. Hence there exists a positive integer $q$ such that $a^q = (a^2)^q = y$, where $y$ is the identity of the cyclic group $Z$. Therefore, by the condition $(\alpha)$ we have $a = a^2$, that is, $a$ is idempotent.

*Corollary.* An element of a normal[5] p.o. semigroup $S$ is idempotent or of infinite order.

*Proof.* For, $S$ satisfies the condition $(\alpha)$ by Theorem 12, O.I.

*Example 1.* Let $S_1'$ be a free abelian semigroup generated by one element $a$ and $S_1'' = \{b, e = \text{the identity}\}$ be a group of order 2, which does not satisfy the conditions $(\alpha)$ and $(\beta)$. And let $S_1$ be a direct product of $S_1'$ and $S_1''$ in the usual sense. Then $S_1$ becomes the abelian semigroup which satisfies the conditions $(\beta)$, $(\gamma)$ and $(\delta)$, but, since $(xe)^2 = (xb)^2$ in spite of $xe \neq xb$ $(x \in S_1')$, does not satisfy the condition $(\alpha)$.

Clearly an abelian semigroup, all elements of which are idempotent, satisfies the condition $(\alpha)$. For instance, see Example 8, which does not satisfy the conditions $(\beta)$, $(\gamma)$ and $(\delta)$.

The conditions $(\alpha)$ and $(\beta)$ are equivalent to each other in abelian groups but not in semigroups.

*Theorem 2.* If an abelian semigroup $S$ satisfies the condition $(\alpha)$, then any complete sector $T(a)$ is isomorphic with a sub-semigroup of the additive semigroup of the non-negative rational numbers.

*Proof.* By Theorem 1, an element of $S$ is idempotent or of infinite order.

(4) An element $a$ of a p.o. semigroup $S$ is called *positive* or *negative*, when $a^2 \geqq a$ or $a \geqq a^2$ respectively. (Definition 4, O.I.)

(5) A partial order defined on an abelian semigroup $S$ is called *normal*, when the following condition is satisfied:

$a^n \geqq b^n$ for some positive integer $n$ implies $a \geqq b$. (Definition 6, O.I.)

i) $a$ is idempotent. Let $x$ be any element of $T(a)$, then $x$ is of finite order, and hence $x$ is idempotent. Therefore, $x=a$. Then the complete sector $T(a)$ consists of only $a$. Consider the correspondence $a \longleftrightarrow 0$.

ii) $a$ is of infinite order. Let $x$ be any element of the complete sector $T(a)$. Then $x$ is of infinite order and there exist positive integers $n$ and $m$ such that $x^m=a^n$. We consider the correspondence $x \longrightarrow n/m$.

First, let $x^m=a^n$ and $x^{m'}=a^{n'}$, then $x^{mn'}=a^{nn'}=x^{nm'}$. Hence $mn'=nm'$ because $x$ is of infinite order. Therefore we have $n/m=n'/m'$, i.e., the correspondence is unique.

Now, let $y \in T(a)$, $y \longrightarrow j/i$, that is, $y^i=a^j$. If $n/m=j/i$, then $x^{mi}=a^{ni}=a^{mj}=y^{mi}$, and hence $x=y$ by the condition $(\alpha)$, i.e., this correspondence is one-to-one.

Finally, since $x^{mi}=a^{ni}$ and $y^{mi}=a^{mj}$, we have $(xy)^{mi}=a^{ni+mj}$, therefore, $xy \longrightarrow (ni+mj)/mi=n/m+j/i$.

Consequently, the correspondence $x \longleftrightarrow n/m$ is the isomorphism of $T(a)$ into the additive semigroup of the positive rational numbers.

*Corollary 1.* If an abelian semigroup $S$ satisfies the condition $(\alpha)$, then the product cancellation law is held in a (complete) sector by any element of $S$.

*Corollary 2.* If an abelian semigroup $S$ satisfies the condition $(\alpha)$, then a strong(6) linear order may be defined on a (complete) sector by any element of $S$(7).

*Definition 4.* Let $S$ be an abelian semigroup and $P$ be a partial order defined on $S$. And let $T$ be a subset of $S$. If any two elements of $T$ are comparable in $P$, then $T$ is called a *comparable subset* (in $P$) of $S$, on the contrary if any two distinct elements of $T$ are non-comparable in $P$, then $T$ is called a *non-copmarable subset* (in $P$) of $S$.

Next, let $U$ and $V$ be two disjoint subsets of $S$. If any elements $u \in U$ and $v \in V$ are comparable in $P$, then $U$ and $V$ are called *comparable subsets to each other* (in $P$) of $S$, and if any elements $u$ and $v$ are non-comparable in $P$, then they are called *non-comparable subsets to each other* (in $P$) of $S$ (or we say that $U$ is comparable (or non-comparable) to $V$ in $P$).

If $U$ and $V$ are comparable subsets to each other in $P$ and $u>v$ in $P$ for any $u \in U$ and $v \in V$, then we denote $U>V$ in $P$.

(6) A p.o. semigroup $S$ is called *strong*, when the following condition is satisfied in $S$: $ac \geqq bc$ implies $a \geqq c$. (Definition 2, O.I.)

(7) Cf. Theorem 15, O.I.

If a complete sector by an element of *infinite order* contains a positive element, then it does not contain a negative element. Accordingly, we shall call such a complete sector *positive*. Similarly, we may define a *negative* complete sector.

*Theorem 3.* Let $S$ be a normal p.o. semigroup. Then the positive or negative complete sector is the comparable subset of $S$.

*Proof.* Let $T(a)$ be a positive complete sector, where we assume that $a$ is positive without loss of generality. Then clearly the sector $S(a)$ is the comparable subset. Now let $u$ and $v$ be any two elements of the complete sector $T(a)$, then there exist positive integers $i, j$ and $k$ such that $u^k = a^i$ and $v^k = a^j$. Since $a^i$ and $a^j$ are comparable, say that $a^i \geqq a^j$, we have $u^k = a^i \geqq a^j = v^k$, therefore by the normality $u \geqq v$. Similarly, a negative complete sector is a comparable subset of $S$.

*Example 2.* Let $S_2$ be a free abelian semigroup generated by one element $a$ with the order-relation

$$a^{2n+1} > a^{2m+1}\ (n > m \geqq 0), \quad a^{2n} > a^{2m}\ (n > m > 0).$$

Then $S_2$ is a p.o. semigroup which satisfies the condition $(\alpha)$ and $T(a) = S_2$. But $S_2$ is not normal because $a^2$ and $a^3$ are non-comparable in spite of $(a^3)^2 = a^6 > a^4 = (a^2)^2$. Since $(a^2)^2 = a^4 > a^2$, $a^2$ is positive, that is, $T(a)$ is the positive complete sector but is not the comparable subset.

Or, by putting the other order-relation

$$a^{n+1} > a^n \text{ for any positive integer } n > N \geqq 1,$$

$S_2$ becomes a p.o. semigroup which is not normal. And $T(a) = S_2$ is the positive complete sector but is not the comparable subset.

Or, we define the order-relation in $S_1$ (Example 1) as follows:

$$a^{n+1}e > a^n e, \quad a^{n+1}b > a^n b \text{ for any positive integer } n.$$

Then $S_1$ becomes the p.o. semigroup which is not normal because $S_1$ does not satisfy the condition $(\alpha)$, and $T(ae) = S_1$ is the positive complete sector but is not the comparable subset.

*Example 3.* Let $S_3'$ be a free abelian semigroup generated by one element $a$ and $S_3''$ be an abelian semigroup generated by two elements $b$ and $0$ with the relations

$$b^2 = 0, \qquad 0^2 = 0, \qquad b0 = 0b = 0.$$

And let $S_3$ be a direct product of $S_3'$ and $S_3''$. Then $S_3$ satisfies the conditions $(\beta)$, $(\gamma)$ and $(\delta)$.

By putting the order-relation

$$a0 < ab < a^2 0 < \cdots\cdots < a^n 0 < a^n b < a^{n+1} 0 < \cdots,$$

$S_3$ becomes a l.o. semigroup. Clearly $S_3$ does not satisfy the condition $(\alpha)$, i.e., $S_3$ is not normal, but $T(a0) = S_3$ is a comparable positive complete sector.

We note that $S_3''$ satisfies the condition $(\delta)$ but does not the conditions $(\alpha)$, $(\beta)$ and $(\gamma)$.

*Example 4.* Let $S_4'$ and $S_4''$ be free abelian semigroups generated by elements $a$ and $b$ respectively. And by defining the order-relations

$$a^{m+1} > a^m \ (m \geqq 1), \qquad b^{n+1} > b^n \ (n > 1),$$

$S_4'$ and $S_4''$ becomes a l.o. semigroup and a p.o. semigroup respectively. Let $S_4$ be a direct product of $S_4'$ and $S_4''$. Then $S_4$ is an abelian semigroup which satisfies the conditions $(\alpha)$, $(\beta)$, $(\gamma)$ and $(\delta)$, and $S_4$ is the union of mutually disjoint complete sectors:

$$S_4 = \bigcup_{i,j} T(a^i b^j),$$

where $i$ and $j$ run over all relatively prime positive integers. For, let $x$ be any element of $S_4$, then $x = a^m b^n = (a^p b^q)^d$, where $(m, n) = d$ and $(p,q) = 1$[8], and hence $x \in T(a^p b^q)$. Next, suppose theat $T(a^i b^j) = T(a^m b^n)$ and $(i,j) = (m,n) = 1$. Then we get easily $i = m$ and $j = n$.

Now, we define the order-relation in $S_4$ as follows:

$$a^i b^j > a^m b^n$$

$$\text{if and only if } a^i > a^m \text{ or } a^i = a^m \text{ and } b^j > b^n.$$

Then $S_4$ becomes a p.o. semigroup. Since $ab^2$ and $ab$ are non-comparable in spite of $(ab^2)^2 = a^2b^4 > a^2b^2 = (ab)^2$, $S_4$ is not normal. On the other hand, any complete sector $T(a^i b^j)$, where without loss of generality $(i,j) = 1$, is a comparable subset and is positive. For, if $x$ and $y$ are any two elements of $T(a^i b^j)$, then they are expressed in the form $x = (a^i b^j)^p$ and $y = (a^i b^j)^q$. Hence, as can easily be seen $x$ and $y$ are comparable, and, since $(a^i b^j)^2 > a^i b^j$, $T(a^i b^j)$ is a positive complete sector.

*Theorem 4.* Let $S$ be a normal p.o. semigroup. Then a complete sector $T(a)$ is order-isomorphic or anti-order-isomorphic with a subsemigroup of the additive semigroup of the non-negative rational numbers, when $a$ is positive or negative respectively.

(8) $(i,j)$ is G.C.M. of $i$ and $j$.

*Proof.* An element $a$ of $S$ is of finite order, i.e., $a$ is idempotent by Theorem 1, if and only if $a$ is positive and negative. In such a case, $T(a)=a$, consequently the theorem is trivial.

Suppose now that $a$ is of infinite order and positive. Then $a^2>a$, and hence $a^n>a^m$ if and only if $n>m$. Further, by Theorem 3 the complete sector $T(a)$ is a comparable subset of $S$. Let $x$ be any element of $T(a)$, then there exist positive integers $m$ and $n$ such that $x^m=a^n$. By Theorem 2, the correspondence $x\longleftrightarrow n/m$ is the algebraic isomorphism of $T(a)$ into the additive semigroup of positive rational numbers.

Let $y\longleftrightarrow j/i$, $y\in T(a)$, that is, $y^i=a^j$.

i) Let $n/m>j/i$, i.e., $ni>mj$. Since $x^{mi}=a^{ni}$ and $y^{mi}=a^{mj}$, we have $x^{mi}=a^{ni}>a^{mj}=y^{mi}$. By the normality we have $x>y$.

ii) Conversely, let $x>y$. Then we have $x^{mi}>y^{mi}$ by the normality. And hence $a^{ni}>a^{mj}$. Therefore, $ni>mj$, i.e., $n/m>j/i$.

Similarly, a complete sector $T(a)$, where $a$ is of infinite order and negative, is anti-order-isomorphic with a sub-semigroup of the additive semigroup of the positive rational numbers.

*Corollary.* If a l.o. semigroup $S$ satisfies the condition ($\alpha$), then *any* complete sector $T(a)$ is order-isomorphic or anti-order-isomorphic with a sub-semigroup of the additive semigroup of the non-negative rational numbers when $a$ is positive or negative respectively.

*Proof.* By the linearity of $S$, any elements $a$ of $S$ is positive or negative. And, since the following property is held always in a l.o. semigroup: $a^n>b^n$ for some positive integer $n$ implies $a>b$, then the condition ($\alpha$) and the normality are equivalent to each other in a l.o. semigroup.

*Theorem 5.* Let $S$ be a l.o. semigroup. And let $T(a)$ be a positive, $T(b)$ be a negative complete sector and $g$ be an element of finite order. Then

i) $T(a)>T(b)$ or $T(b)>T(a)$,

ii) if $T(u)$ is any complete sector different from $T(g)$, then $T(u)>T(g)$ or $T(g)>T(u)$,

iii) if $T(b)>T(a)$, then $b>ab>a$ and $ab=a^mb^n$ for any positive integers $m$ and $n$.

*Proof.* i): Suppose that $a>b$. Let $x$ be any element of $T(a)$ and $y$ be any element of $T(b)$, then there exist positive integers $i, j$ and $k$ such that $x^k=a^i$ and $y^k=b^j$. But, since $a^i\geqq a>b\geqq b^j$, we have $x^k=a^i>b^j=y^k$ and hence by the linearity $x>y$, that is, $T(a)>T(b)$. Next, let $b > a$. If there exist the elements $x \in T(a)$ and $y \in T(b)$ such that $x > y$, $x^k = a^i$ and $y^k = b^j$, then we have $x^k > y^k$ because $T(a)$ and $T(b)$ are disjoint, and hence $a^i > b^j$. If $i \geqq j$, then $a^i > b^j \geqq b^i$, hence we have $a > b$, which is absurd. Similarly, $j \geqq i$ implies $a^j \geqq a^i > b^j$, i.e., we have $a > b$, contrary to $b > a$. Therefore, $T(b) > T(a)$.

ii): Let $f$ be the (only one) idempotent element belonging to $T(g)$. Clearly $T(g) = T(f)$. Suppose now that $u > f$. Since $T(u)$ and $T(f)$ are disjoint, we have $u^n > f$ for every positive integer $n$. If $x \in T(u)$ and $y \in T(f)$, then we get easily $x > y$, therefore $T(u) > T(g)$. Similarly, if $f > u$, then $T(g) > T(u)$.

iii): $b > a$ implies that $b > b^2 \geqq ab \geqq a^2 > a$, i.e., $b > ab > a$. Furthermore, we have

$$b > b^2 \geqslant b^3 \geqq ab^2 \geqq a^2b \geqq ab ,$$
$$ab \geqq ab^2 \geqq a^2b \geqq a^3 > a^2 > a .$$

Hence $ab^2 = ab = a^2b$. Now we assume that $a^m b^n = ab$, $m+n = \mathrm{N}$. Then $a^{m+1}b^n = a^2b = ab = ab^2 = a^m b^{n+1}$. Since $ab$ is idempotent and $b > ab > a$, we have $T(b) > T(ab) > T(a)$ by ii).

Let $f$ be an idempotent element and $T(f) \supsetneqq f$. If $g$ $(\neq f)$ of $T(f)$ is positive and $h$ $(\neq f)$ of $T(f)$ is negative, then

$$h > h^2 > \cdots > h^t = f = g^s > \cdots > g^2 > g ,$$
$$h^m g^n = f \text{ for any positive integers } m \text{ and } n ,$$

where $s$ and $t$ are the lengths of $g$ and $h$ respectively.

*Corollary.* If $S$ has the unit element $e$, then the above theorem is somewhat brief:

1) $T(a) > e > T(b)$, $T(e) = e$,

2) if $T(g) > e$, then the element $h$ of $T(g)$ differing from the idempotent element $f$ belonging to $T(g)$, is positive, i.e., $f \geqq h^2 > h$ and $fh = f$; and its dual.

The proof is obvious.

*Theorem 6.* Let $S$ be a p. o. semigroup which satisfies the condition $(A)$. Then the following properties are held in $S$:

1) If $a$ is positive and $b \geqq a$, then $b$ is positive.

2) If $a$ is negative and $a \geqq b$, then $b$ is negatiqe.

3) If $a$ is positive and $b$ is negative, then $a \geqq b$.

*Proof.* 1): $b \geqq a$ implies $b^2 \geqq ab$ by the homogeneity, and $ab \geqq b$ by the condition $(A)$. Therefore, we have $b^2 \geqq b$, which says that $b$ is positive. 2): Similarly to 1). 3): By the condition $(A)$, we have $a \geqq ab \geqq b$.

*Example 5.* Let $S_5$ be an abelian semigroup generated by two elements $a$ and $b$ with the relation

$$ab = a^m b^n \quad \text{for any positive integers } m \text{ and } n.$$

Then $S_5$ satisfies the condition $(\alpha)$ but does not the conditions $(\beta)$, $(\gamma)$ and $(\delta)$.

And by putting the order-relation

$$a^{m+1} > a^m > ab > b^n > b^{n+1}$$
$$\text{for any positive integers } m \text{ and } n,$$

$S_5$ becomes a l. o. semigroup which satisfies the properties 1), 2) and 3) of Theorem 6 but does not the conditian $(A)$ because $a > ab = (ab)a$ in spite of $(ab)^2 \geqq ab$.

*Theorem 7.* Let $S$ be a p.o. semigroup which satisfies the condition $(A)$. Then $S$ may contain only one idempotent element, or what is the same, only one positive and negative element. Further, if there exists an idempotent element $e$, then it is necessarily the unit element and all positive or negative elements are comparable to $e$.

*Proof.* If $S$ has a positive and negative element $e$, then by the condition $(A)$ $eb \geqq b$ and $b \geqq eb$, i.e., $eb = b$ for any $b$ of $S$, and hence $e$ is the unit element. Accordingly, all elements of finite order of $S$ belong to the complete sector $T(e)$. Let $a$ be any positive (or negative) element. By the condition $(A)$, we have $a = ae \geqq e$ (or $e \geqq ae = a$).

*Corollary 1.* If a p.o. semigroup $S$ satisfies the conditions $(\alpha)$ and $(A)$, then $S$ satisfies the condition $(\beta)$.

*Proof.* For, by Theorem 1 an element of $S$ is idempotent or of infinite order.

*Corollary 2.* If a l.o. semigroup $S$ satisfies the condition $(A)$, then $S$ satisfies the condition $(\beta)$.

*Proof.* If there exists an element $a$ of finite order, then $S$ contains an idempotent element $e$. By the above theorem, $e$ is the unit element of $S$ and $a \in T(e)$. Moreover, by the property 1) of Corollary, Theorem 5, $T(e) = e$. Therefore we have $a = e$, that is $S$ satisfies the condition $(\beta)$.

*Example 6.* Let $S_6$ be an abelian group of finite order $n \geqq 2$. And consider the vacuous order-relation of $S_6$. Then $S_6$ satisfies the condition $(A)$ but does not satisfy the conditions $(\alpha)$ and $(\beta)$.

*Example* 7. Let $S_7$ be an abelian semigroup generated by two elements $a$ and $e$ with the relations

$$e^2=e,\quad a^m e=ea^m=a^m \quad \text{for any positive integer } m,$$
$$a^n>a^m \quad \text{for} \quad n>m.$$

Then $S_7$ is a p.o. semigroup with the unit element $e$ and satisfies the conditions ($\alpha$) and ($\beta$). But, since $ae=a$ and $e$ are non-comparable and $a^2>a$, $S_7$ does not satisfy the condition ($A$). (Or, see Examples 1 and 2.)

*Theorem 8*. Let $S$ be a p.o. semigroup which satisfies the condition ($A$). If $S$ has not the unit element, then one can adjoin the new element $e$ to $S$, so that $S^*=S\cup e$ becomes a p.o. semigroup in which $e$ is the unit element, and $S$ is order-embedded(9) in $S^*$, moreover, all positive or negative elements are comparable to $e$ in $S^*$.

*Proof*. We define the multiplication in $S^*$ as follows:

$$e\cdot e=e, \qquad e\cdot a=a\cdot e=a \qquad \text{for any a of } S,$$

and for any two elements $a$ and $b$ of $S$ the product is the same as in $S$. Thus $S^*$ becomes the abelian semigroup with the unit element $e$ under the multiplication introduced above.

Let $P$ be the partial order defined on $S$. Let us now define the order-relatien $Q$ in $S^*$ as follows:

$a>b$ in $Q\,(a,b\in S)$ if and only if $a>b$ in $P$,
$a>e$ in $Q\,(a\in S)$ if and only if $a$ is positive in $P$,
$e>a$ in $Q\,(a\in S)$ if and only if $a$ is negative in $P$.

$S$ has not a positive and negative element, because if there exists such an element $a$, then $a$ is the unit element, by Theorem 7, which contradicts the hypothesis of the theorem. Therefore, $a>b$ and $b>a$ in $Q$ are contradictory.

The transitivity of $Q$ is verified by the properties 1), 2) and 3) of Theorem 6 and the homogeneity of $Q$ is easily verified by the condition ($A$) itself.

Then $S^*$ becomes a p.o. semigroup with the unit element $e$, and it is clear that $S$ is order-embedded in $S^*$ and also $S^*$ satisfies the condition ($A$).

*Example 8*. Let $S_8$ be a set of three elements $a, b$ and $c$ with the relations

(9) A p. o. semigroup $S$ will be said to be *order-embedded* in a p.o. semigroup $S'$, if there exists an order-isomorphism of $S$ into $S'$. (Definition 3, O.I.)

$$ab = ba = c, \qquad ac = ca = c, \qquad bc = cb = c;$$

$$a^2 = a, \qquad b^2 = b, \qquad c^2 = c, \qquad a > c > b.$$

Then $S_8$ is a l.o. semigroup without the unit element. It is easy to verify that $S_8$ does not satisfy the condition $(A)$ and that one cannot adjoin the new element $e$ to $S_8$, so that $S_8^* = S_8 \cup e$ becomes a l.o. semigroup in which $e$ is the unit element and $S_8$ is order-embedded in $S_8^*$.

*Example 9.* Let $S_9$ be a set of two elements 0 and $e$ with the relations

$$0^2 = 0, \qquad e^2 = e, \qquad 0e = e0 = 0, \qquad 0 > e.$$

Then $S_9$ is a l.o. semigroup with the unit element $e$. Since $0 \geqq 0^2$ and $0e = 0 > e$, $S_9$ does not satisfy the condition $(A)$. (Or, see Example 7.)

*Example 10.* Let $T$ be an abelian semigroup generated by two elements $a$ and $f$ with the relations

$$f^2 = f,$$
$$f > a > af > a^2 > \cdots > a^n > a^n f > a^{n+1} > \cdots.$$

Then $T$ is a l.o. semigroup without the unit element and does not satisfy the condition $(A)$ because $a > af$ in spite of $f^2 \geqq f$. But one can adjoin the new element $e$ to $T$, so that $T^* = T \cup e$ becomes a l.o. semigroup in which $e$ is the unit element, by adding the following relations:

$$e^2 = e, \quad ex = xe = x, \quad e > x \quad \text{for all } x \text{ in } T.$$

*Theorem 9.* Let $S$ be a p.o. semigroup which satisfies the condition $(\beta)$. If $S$ has the unit element $e$ and all positive or negative elements are comparable to $e$, then $S$ satisfies the conditoin $(A)$.

*Proof.* Let $a$ $(\neq e)$ be positive. If $a^2 \geqq a$ implies $a \not> e$, then we have $e > a$ by the hypothesis of the theorem, and hence $a \geqq a^2$, therefore $a = a^2$, which contradicts the condition $(\beta)$. Hence we have $a > e$, therefore $ab \geqq b$ for any $b$ of $S$. Similarly, if $a\,(\neq e)$ is negative then $b \geqq ab$ for any $b$ of $S$.

*Corollary 1.* Let $S$ be a p.o. semigroup with the unit element $e$ which satisfies the condition $(\alpha)$. Then $S$ satisfies the condition $(\beta)$ and all positive or negative elements are comparable to $e$ if and only if it satisfies the condition $(A)$. (Cf. Corollary 1 of Theorem 7.)

*Corollary 2.* Let $S$ be a l.o. semigroup with the unit element. Then $S$ satisfies the condition $(\beta)$ if and only if it satisfies the condition $(A)$. (Cf. Corollary2 of Theorem 7.)

*Example 11.* Let $T_1$ be an abelian semigroup generated by two elements $a$ and $e$ with the relations

$$e^2 = e, \quad a^m e = e a^m = a^m \quad \text{for any positive integer } m,$$
$$a^{2n+1} > a^{2m+1} \qquad (n > m \geqq 0),$$
$$a^{2n} > a^{2m} > e \qquad (n > m > 0.$$

Then $T_1$ is a p.o. semigroup with the unit element $e$ and satisfies the conditions ($\alpha$) and ($\beta$), but is not normal. All positive elements (there exist no negative elements except $e$) are comparable to $e$, and clearly $T_1$ satisfies the condition ($A$).

*Theorem 10.* If a l.o. semigroup $S$ satisfies the condition ($\beta$), then $S$ satisfies the properties 1), 2) and 3) of Theorem 6.

*Proof.* If $S$ has the unit element $e$, then by Corollary 2 of Theorem 9 $S$ satisfies the condition ($A$) and hence the properties 1), 2) and 3) by Theorem 6. Now we assume that $S$ has not the unit element, that is, all elements of $S$ are of infinite order. Let $a$ be positive, $b$ be negative, i.e., $a^2 > a$ and $b > b^2$. If $b > a$, then $T(b) > T(a)$ and hence $T(b) > ab > T(a)$ and $ab$ is idempotent by Theorem 5, iii). This contradicts the hypothesis. Therefore we have $a > b$. Similarly, let $c > a (b > c)$, then $c$ is positive (negative).

We note that a l.o. semigroup S, which does not satisfy the condition ($\beta$), may satisfy the properties 1), 2) and 3); see Example 5.

*Example 12.* Let $T_2$ be an abelian semigroup generated by two elements $a$ and $b$ with the relation

$$a^m b^n = a b^n.$$

Then $T_2$ satisfies the conditions ($\alpha$) and ($\beta$). And by putting the order-relation

$$a^{m+1} > a^m > b^n > ab^n > b^{n+1} > ab^{n+1}$$
$$\text{for any positive integers } m \text{ and } n,$$

$T_2$ becomes a l.o. semigroup which does not satisfy the condition ($A$) because $b > ab$ in spite of $a^2 > a$,

Or, by putting the other-relation

$$b^{n+1} > ab^{n+1} > b^n > ab^n > a^{m+1} > a^m$$
$$\text{for any positive integers } m \text{ and } n,$$

$T_2$ becomes a l.o. semigroup which does not satisfy the condition ($A$) because $b > ab$ in spite of $a^2 > a$.

*Theorem 11.* In a l.o. semigroup the conditions ($\gamma$) and ($B$) are equivalent to each other.

*Proof.* Let $S$ be a l.o. semigroup which satisfies the condition ($\gamma$) and let $a$ ($\neq$ the unit element) be a positive element of $S$. Then $a^2 > a$ and hence $a^2b = a(ab) > ab$ for any $b$ of $S$. By the linearity, we have $ab > b$. Similarly, if $a$ ($\neq$ the unit element) is negative, then $b > ab$ for any $b$ of $S$. The converse is obvious.

We note that if an abelian semigroup $S$ satisfies the condition ($\gamma$), then the element $a$ of finite order is cyclic. For, let $m$ be the length and $r$ be the period of $a$, then $a^m = a^m a^r$ and hence $a^r$ is the unit element, thereby $a = a \cdot a^r$, thus we have $m = 1$, i.e., $a$ is cyclic.

*Example 13.* Let $T_3$ be an abelian semigroup generated by two elements $a$ and $b$ with the relations

$$a^m b^n = b^n, \quad a^{m+1} > a^m > b^n > b^{n+1}$$

for any positive integers $m$ and $n$.

Then $T_3$ becomes a l.o. semigroup which satisfies the conditions ($\alpha$), ($\beta$), ($\delta$) and ($A$) but does not the conditions ($\gamma$) and ($B$).

*Theorem 12.* A l.o. semigroup $S$ is strong if and only if $S$ satisfies the conditions ($\alpha$), ($\delta$) and ($B$).

*Proof.* Necessity: For the condition ($\alpha$), see Theorem 10, O.I. Let $c \cdot T(a)$ and $c \cdot T(b)$ be intersect for some $c$ of $S$, then there exist two elements $x \in T(a)$ and $y \in T(b)$ such that $cx = cy$. By the strongness, we have $x = y$. Therefore $T(a) = T(b)$. Since in the strong l.o. semigroup the element of finite order is the unit element (Theorems 9 and 10, O.I.), if $a$ ($\neq$ the unit element $e$) is positive, then we have $a^2 > a$. Hence by the strongness we have $a^2b > ab$, thereby $ab > b$ for any $b$ of $S$. Similarly, if $a$ ($\neq e$) is negative, then we have $b > ab$ for any $b$ of $S$.

Sufficiency: Let $xz = yz$. i) $T(x) = T(y) = T(z)$. By Corollary 1 of Theorem 2, we have $x = y$. ii) $T(x) = T(y) \neq T(z)$. Since $S$ satisfies the condition ($B$), $S$ satisfies the condition ($\beta$). If $x$ is of finite order, that is, $x$ is the unit element, then $T(x) = T(y) = x$. Suppose that $x$ is of infinite order. Then there exist positive integers $n$ and $m$ such that $x^n = y^m$. And hence $y^n z^n = x^n z^n = y^m z^n$. If $n \neq m$, say that $n > m$, then we have $y^m z^n = y^{n-m}(y^m z^n)$, which contradicts the condition ($B$), since $y$ is not of finite order, that is, is not the unit element. Hence $n = m$, i.e., $x^n = y^n$, therefore we have $x = y$ by the condition ($\alpha$). iii) $T(x) \neq T(y)$. By the condition ($\delta$), $xz \neq yz$ for all $z$ of $S$. Therefore, the product cancellation law is held in $S$. By Theorem 4, O.I., $S$ is strong.

*Corollary 1.* A l.o. semigroup $S$ is strong if and only if $S$ satisfies the conditions ($\alpha$), ($\gamma$) and ($\delta$).

*Corollary 2.* A strong linear order may be defined on an abelian semigroup $S$ if and only if $S$ satisfies the conditions ($\alpha$), ($\gamma$) and ($\delta$).

*Proof.* Let $x$ be an element of finite order, then by the condition ($\alpha$) $x$ is idempotent, furthermore by the condition ($\gamma$) $x$ is the unit element. That is, $S$ satisfies the condition ($\beta$). Similarly to the proof of Theorem 12, we see that the product cancellation law is held in $S$. Therefore the proof is complete by Theorem 15, O.I.

*Example 14.* Let $T_4'$ be a free abelian semigroup generated by one element $a$ and $T_4''$ be an abelian semigroup generated by two elements 0 and $e$ with the relations

$$0^2=0, \qquad e^2=e, \qquad 0e=e0=0.$$

And let $T_4$ be a direct product of $T_4'$ and $T_4''$, then $T_4$ is an abelian semigroup which satisfies the conditions ($\alpha$), ($\beta$) and ($\gamma$) but does not the condition ($\delta$).

By putting the order-relation

$$a^n0<a^ne<a^{n+1}0<a^{n+1}e \quad \text{for any positive integer } n,$$

$T_4$ becomes a l.o. semigroup.

Examples 3, 13 and 14 show that the conditions ($\alpha$), ($\gamma$) and ($\delta$) are independent in l.o. semigroups.

*Theorem 13.* A l.o. semigroup $S$ with the unit element $e$ is strong if and only if $S$ satisfies the conditions ($\alpha$) and ($\delta$).

*Proof.* Necessity: It is clear.

Sufficiency: Suppose that $ab=b$ for some $b$ of $S$, that is, $ab=eb$. By the condition ($\delta$), $T(a)=T(e)$. Since $T(e)=e$ by the condition ($\alpha$), we have $a=e$, consequently $S$ satisfies the condition ($\gamma$).

*Corollary 1.* A strong linear order may be defined on an abelian semigroup $S$ with the unit element if and only if $S$ satisfies the conditions ($\alpha$) and ($\delta$).

*Corollary 2.* The product cancellation law is held in an abelian semigroup with the unit element, when it has the following properties:

1) $a^n=b^n$ for some positive integer $n$ implies $a=b$.

2) $ac=bc$ implies $a^m=b^n$ for some positive integers $m$ and $n$.

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