A CHARACTERIZATION OF THE MODULARS OF L_p TYPE

By

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A modular on a universally continuous semi-ordered linear space R is, as defined in [1], a functional $m(x)\,(x \in R)$ satisfying the following conditions:

(i) $0 \leqq m(x) \leqq +\infty$, $m(0)=0$;

(ii) $m(\xi x)$ is a convex function of ξ which is finite in a neighbourhood of 0 and not identically zero, if $x \neq 0$;

(iii) $|x| \leqq |y|$ implies $m(x) \leqq m(y)$;

(iv) $x \perp y$ implies $m(x+y)=m(x)+m(y)$;

(v) $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $m(x)=\sup_{\lambda \in \Lambda} m(x_\lambda)$.

Since the set of elements $\{x\colon m(x) \leqq 1\}$ is convex, we can define a norm $|||x|||$ such that $|||x||| \leqq 1$ is equivalent to $m(x) \leqq 1$. This norm is said to be the *modular norm.* On the other hands, putting

$$\| x \| = \inf_{\xi > 0} \frac{1+m(\xi x)}{\xi} ,$$

we obtain another norm which is conjugate to the modular norm of the conjugate modular in case that the space R is semi-regular. We have a relation between these two norms, that is,

$$|||x||| \leqq \|x\| \leqq 2\, |||x||| .$$

In the space $L_p\,(p \geqq 1)$, putting

$$m(x) = \int_0^1 | x(t) |^p dt ,$$

we obtain a modular and we have in this case

(1) $$m(x) = |||x|||^p$$

and

(2) $$\|x\| = \alpha\, |||x||| ,$$

where α is the number such that

(3) $$\alpha = p^{\frac{1}{p}} q^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q}=1 \text{ (if } p=1, \text{ we put } \alpha=1).$$

The constancy of the ratio of the norms is a consequence of the property (1) of the modular, also in abstract case. The converse to this was conjectured by H. NAKANO, and for the case $\alpha=1$, S. YAMAMURO answered the problem in [2], proving that if the norms coincide each other, then the modular is either *linear*[1] or *singular*,[2] or in other words, the space is L_1-type or M-type.

Such a precision as this can not be attained in general case, and in this paper we shall prove the following

Theorem. *If the norms of an infinite-dimensional modulared semi-ordered linear space R satisfy the equality* (2) *for some $\alpha>1$, then there exists a normal manifold*[3] *N such that $N^{\perp}$*[4] *is at most two dimensional and the norm*[4] *is either L_p-type*[5] *or L_q-type in N, where p and q are as given in* (3), *and we have* (*in case of L_p-type*) *the equality* (1) *for every $x \in N$ such that $|||x||| \leqq 1$.*

The proof will be accomplished through the paper with an additional result about the relation between the norm and modular. In the sequel, if not mentioned the contrary, let R be an infinite-dimensional modulared semi-ordered linear space where the norms satisfy the equality (2) for some $\alpha>1$ and p, q be as in (3).

1. (2) is equivalent to

$$\inf_{\xi>0} \frac{1+m(\xi x)}{\xi} = \alpha ,$$

if $|||x|||=1$, or in other words, the curve $\eta=m(\xi x)$ is, in (ξ,η)-plane, in the upper side of the line $\eta=\alpha\xi-1$ which is either a tangent to the curve or parallel at infinity[6] to it.

Now we shall prove that

$$|||x|||=1 \quad \textit{implies} \quad m(x)=1 .$$

1) A modular is said to be linear if we have $m(\xi x)=\xi m(x)$.

2) A modular is said to be singular if it takes no value other than 0 or $+\infty$.

3) A manifold of R is said to be normal if R is decomposed into a direct sum: $R=N+M$, where $N \ni x$, $M \ni y$ imply $x \perp y$, and then M is denoted by $N^{\perp}$.

4) This means one of the two norms; since they are different only by some constant multiplier. This expression will be used often in the sequel.

5) A norm $\|x\|$ is said to be L_p-type if for every mutually orthogonal elements x, y we have $\|x+y\|^p = \|x\|^p + \|y\|^p$.

6) This means—"parallel to the asymptote of".

If there exists x such that $\||x\||=1$ and $m(x)<1$, then x is an *critical* element, that is, by definition, $m(\xi x)=+\infty$ for every $\xi>1$. Then for every orthogonal decomposition of x: $x=y+z$, at least one of them, say y, is also critical and hence we have also $\||y\||=1$ and $m(y)>1$. Thus we can suppose that for x above, there exists an orthogonal sequence of elements $y_\nu \neq 0$ $(\nu=1,2,\cdots)$ where every y_ν is orthogonal to x. Moreover we can take such y_ν as to satisfy the inequalities:

$$m(x+y_\nu)\leqq 1 \quad\text{and}\quad m(y_\nu)\geqq \varepsilon \qquad \text{for every } \nu=1,2,\cdots,$$

where ε is some positive number, because if y_ν is critical and $m(y_\nu)\leqq 1$, then we have $\||y_\nu\||=1$ and $m(y_\nu)+1\geqq\|y_\nu\|=\alpha$, or $m(y_\nu)\geqq\alpha-1$.

Since $m(\xi x)$ goes to $+\infty$ for $\xi>1$, the curve $\eta=m(\xi x)$ must have as a tangent the line $\eta=\alpha\xi-1$. Let ξ_0 be the least number satisfying the equality:

$$m(\xi_0 x)=\alpha\xi_0-1\,, \tag{4}$$

then we have $m(\xi_0 y_\nu)=0$ for every $\nu=1,2,\cdots$, because the same line $\eta=\alpha\xi-1$ must be also a tangent to the curve $\eta=m(\xi(x+y_\nu))$. Put $y=y_1+y_2+\cdots+y_n$ where n is an integer such that $n\varepsilon>1$, then we have

$$1=m(\xi_0 y)+1\geqq\xi_0\|y\|=\alpha\xi_0\||y\||\,,$$

and since $\alpha\xi_0\geqq 1$ by (4),

$$1\geqq\||y\||\,,$$

contradicting the fact that $m(y)\geqq n\varepsilon>1$.

2. We put

$$D=\left\{x:\ m(x)=\frac{1}{2}\right\}\,,$$

and for x in D,

$$\varphi_x(\xi)=m(\xi x)-\frac{1}{2}(\alpha\xi-1)\,.$$

Here we shall prove that

There exists an infinite-dimensional normal manifold N of R such that for every x in $N\frown D$, φ_x is not upper bounded as a function of ξ.

If x is in D and $m(\gamma x)=1$, then we have

$$\alpha\xi\left(1-\frac{\gamma}{2}\right)-\frac{1}{2}=\varphi_x(\gamma\xi)-\left\{m(\gamma\xi x)-(\alpha\xi-1)\right\}\leqq\varphi_x(\gamma\xi)\,.$$

Therefore if φ_x is upper bounded, then $\frac{\gamma}{2} \geqq 1$ and since $\frac{\gamma}{2} = \gamma m(x) \leqq m(\gamma x) = 1$, we have $\gamma = 2$, that is, $m(2x) = 1$.

If we can not find any normal manifold N mentioned above, then there exists an orthogonal sequence of elements x_ν in $D\,(\nu = 1, 2, \cdots)$ such that φ_{x_ν} are all upper bounded. Now putting

$$x = \frac{2}{n}(x_1 + x_2 + \cdots + x_n)\,,$$

we have $m(x) = 1$ and $m(nx) = n$,
and hence

$$\alpha \leqq \frac{1 + m(nx)}{n} = 1 + \frac{1}{n}\,.$$

Since n is arbitrary, we have $\alpha = 1$, a contradiction.

3. We put for every x in D,

$$S(x) = \{\xi :\ \varphi_x(\xi) \leqq 0\}$$

and

$$I(x) = \{\xi :\ \varphi_x(\xi) < 0\}\,,$$

then the first is a closed interval and the second an open one.

In the sequel we suppose that φ_x is not upper bounded for every x in D, and here we shall show that

If R is decomposed into mutually orthogonal normal manifolds N_1 and N_2 (denoting $R = N_1 \oplus N_2$), then for at least one of them, say N_1, we have $S(x) \neq \emptyset$ for every x in $N_1 \frown D$.

In fact, if for mutually orthogonal elements x, y in D, both $S(x)$ and $S(y)$ are void, then we have

$$m(\xi(x+y)) - (\alpha\xi - 1) = \varphi_x(\xi) + \varphi_y(\xi) > 0$$

for every ξ, and hence the line $\eta = \alpha\xi - 1$ must be parallel at infinity to the curve $\eta = m(\xi(x+y))$, contradicting the fact that φ_x is not upper bounded.

4. We suppose, moreover, in the sequel, that $S(x) \neq \emptyset$ for every x in D.

For every mutually orthogonal elements x, y of D, we have

$$I(x) \frown S(y) = \emptyset\,,$$

because

$$\varphi_x(\xi)+\varphi_y(\xi) = m(\xi(x+y))-(\alpha\xi-1) \geqq 0 .$$

Let $x_i (i=1,2,3)$ be mutually orthogonal elements of D, and suppose that $I(x_1)$, $I(x_2)$ and $I(x_3)$ are all non-void and in this order on the real line, then for every λ, μ such that

$$\lambda x_1+\mu x_3 \in D ,$$

we have

$$S(\lambda x_1+\mu x_3)\frown I(x_2)=\emptyset .$$

We can vary the pair (λ, μ) continuously from (1,0) to (0,1), while $S(\lambda x_1 +\mu x_3)$ varies also continuously from $S(x_1)$ to $S(x_2)$ being always disjoint with $I(x_2)$. But this is a contradiction and we conclude that at least one of $I(x_i)$ above must be void.

Thus we have proved that

If R is decomposed as: $R=N_1\oplus N_2\oplus N_3$, then for at least one N_i, we have $I(x)=\emptyset$ for every x in $N_i\frown D$.

5. We suppose that for every x in D, $S(x)\neq\emptyset$, $I(x)=\emptyset$ and φ_x is not upper bounded.

Then for every mutually orthogonal elements x, y of D, we have

$$S(x)\frown S(y)\neq\emptyset ,$$

because $S(x)\frown S(y) = \emptyset$ implies

$$m(\xi(x+y))-(\alpha\xi-1)=\varphi_x(\xi)+\varphi_y(\xi)>0$$

for every ξ, a contradiction as shown in Article 3.

Now we shall prove that

If R is decomposed as: $R=N_1\oplus N_2$, then for at least one $N_i (i=1$ or $2)$ we have

$$\bigcap_{x\in N_i\frown D} S(x) \neq \emptyset . \tag{5}$$

If the intersection above is void for N_1, then there exists a pair of elements x, y in $N_1\frown D$ for which we have

$$S(x)\frown S(y) = \emptyset.^{7)}$$

Then for every z in $N_2\frown D$, since we have

$$S(z)\frown S(x) \neq \emptyset ,$$

7) Because $S(x)$ is an interval; we can say more generally that if a collection of closed intervals in a complete lattice has void intersection, then there exists a disjoint pair among them.

and

$$S(z)\frown S(y) \neq \emptyset ,$$

$S(z)$ contains every ξ which is between the two disjoint intervals $S(x)$ and $S(y)$, and hence (5) is true for $i=2$.

6. We put

$$\pi_+(x)= \inf_{\varepsilon>0} \frac{1}{\varepsilon} \{m((1+\varepsilon)x)-m(x)\} ,$$

and

$$\pi_-(x)= \sup_{\varepsilon>0} \frac{1}{\varepsilon} \{m(x)-m((1-\varepsilon)x)\}$$

for every $x\in R$ such that $m(x)< +\infty$. We have always $\pi_-(x)\leqq\pi_+(x)$ and both $\pi_+(\xi x)$ and $\pi_-(\xi x)$ are non-decreasing functions of ξ, and $\pi_\pm(x)$ is *orthogonally additive*, that is, $\pi_\pm(x+y)=\pi_\pm(x)+\pi_\pm(x)$ if $x \perp y$.

If $\pi_+(x)=\pi_-(x)$, then we write it $\pi(x)$, and such an element x is said to be a *regular* element. Since $m(\xi x)$ is a convex function, ξx is regular (providing $m(\xi x)< +\infty$) except for a countable number of ξ.

Now we suppose that the line $\eta=\frac{1}{2}(\alpha\xi-1)$ is a tangent at a definite point (ξ_0, η_0) to the curve $\eta=m(\xi x)$ for every x in D.

Then we have for every x in D,

$$\pi_-(\xi_0 z) \leqq \frac{1}{2} \alpha\xi_0 \leqq \pi_+(\xi_0 x) ,$$

and since $x\in D$ is equivalent to $m(\xi_0 x)= \eta_0$,

$$\pi_-(x) \leqq \gamma m(x) \leqq \pi_+(x) \tag{6}$$

for every x such that $m(x)=\eta_0$, where $\gamma = \frac{\alpha\xi_0}{2\eta_0}$.

Then we shall prove that *for every x and ξ such that $m(\xi x)\leqq\eta_0$, we have*

$$m(\xi x) = \xi^\gamma m(x) . \tag{7}$$

We put

$$\rho(x) = \pi(x)-\gamma m(x)$$

for every reguar element x, and decompose R as: $R=N\oplus M$, where N is at least two dimensional.

For two regular elements x, y in N such that

$$m(x) = m(y) \leqq \eta_0 ,$$

if there exists a regular element z in M such that

$$m(z) = \eta_0 - m(x) ,$$

then we have by (6)

$$\rho(x+z) = \rho(y+z) = 0 ,$$

and hence by the orthogonal additivity of ρ,

$$\rho(x) = \rho(y) .$$

We shall here make use of the term "almost all" to mean "except a countable number of". Then there exists a function $\omega(\xi)$ which is defined for *almost all* ξ in the interval $[0, \eta_0]$ and for which we have

$$\rho(x) = \omega\{m(x)\}$$

for every regular element x in N such that $m(x) \leqq \eta_0$ and $\omega\{m(x)\}$ is defined. Since N contains mutually orthogonal elements both different from 0, we have for almost all ξ_1, ξ_2,

$$\omega(\xi_1 + \xi_2) = \omega(\xi_1) + \omega(\xi_2) .$$

Since $\omega(\xi)$ is obviously bounded at a neighbourhood of 0, and $\omega(\eta_0) = 0$ by (6), we can see easily that for almost all ξ,

$$\omega(\xi) = 0 .$$

Then for every x in N such that $m(x) < \eta_0$, we can find ξ_1, ξ_2, arbitrarily near to 1, such that $\xi_1 < 1 < \xi_2$ and $\xi_i x$ is regular and $\rho(\xi_i x) = \omega(m(\xi_i x)) = 0$ for $i = 1, 2$. Then we have

$$\gamma m(\xi_1 x) = \pi(\xi_1 x) \leqq \pi_-(x) \leqq \pi_+(x) \leqq \pi(\xi_2 x) = \gamma m(\xi_2 x) ,$$

and hence we can conclude that x is regular and

$$(8) \qquad \pi(x) = \gamma m(x) .$$

Similarly we can prove that every element x of R such that $m(x) \leqq \eta_0$ is regular and satisfies (8).

For an element x of R, put

$$f(\xi) = m(\xi x)$$

for every ξ such that $m(\xi x) \leqq \eta_0$, then $f(\xi)$ is differentiable and we have by (8)

$$f'(\xi) = \frac{1}{\xi}\pi(\xi x) = \frac{\gamma}{\xi} f(\xi) ,$$

and hence

$$f(\xi) = f(1)\,\xi^{\gamma} .$$

Thus (7) was proved.

Here γ must be either p or q, because, if not, the line $\eta = \frac{1}{2}(\alpha\xi - 1)$ would not be a tangent to the curve $\eta = \frac{1}{2}\xi^{\gamma}$.

If we have (7) for $\gamma = p$, then we can see easily that

$$|||x|||^{p} = m(x)$$

for every x such that $m(x) \leqq \eta_0$, and hence that the norm is L_p-type.

7. The case: $\alpha = 2$, is especially simple (then $p = q = 2$), because

$$\inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} = 2\,|||x|||$$

implies that the line $\eta = 2\xi - 1$ is always a tangent at the point (1,1) to the curve $\eta = m(\xi x)$ for every x such that $m(x) = 1$ (it is here equivalent to $|||x||| = 1$).

Since all the arguments in the last article is valid if R is at least three dimensional, we can say, more precisely than in the theorem for general case, that

If R is at least three dimensional and we have

$$||x|| = 2|||x|||$$

for every x in R, then the norm is L_2-type and

$$m(x) = |||x|||^{2}$$

for every x such that $|||x||| \leqq 1$.

8. We can see, as the consequence of the facts shown in the foregoing articles, that there exists an infinite-dimensional normal manifold N of R such that the norm is L_p-type in N. If the norm is L_p-type in every normal manifold $N_\lambda\,(\lambda \in \Lambda)$, then it is also so in the normal manifold which is the join of all N_λ, because the norm is semi-continuous, that is, $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $|||x||| = \sup_{\lambda \in \Lambda} |||x_\lambda|||$. Since the existence of two normal manifolds in which the norm is L_p-type and L_q-type respectively, contradicts (2), if $\alpha \neq 2$, there exists the maximum

normal manifold N having the property above. Moreover we can assume that there exists a normal manifold $N_0 \subset N$, also infinite-dimensional, and for which we have $S(x) \neq \emptyset$, $I(x) = \emptyset$ and φ_x is not upper bounded for every x in $D_\frown N_0$.

Now we shall show that

$N^\perp$ is at most two dimensional.

Decompose N as: $N = N_1 \oplus N_2 \oplus N_3$ where N_i has non-void intersection with N_0 for every $i = 1,2,3$. If we have a decomposition: $N^\perp = M_1 \oplus M_2 \oplus M_3$, then since $S(x) \neq \emptyset$ for every x in $D_\frown(N_i \oplus M_i)$ $(i = 1, 2, 3)$ as shown in 3, for at least one of $N_i \oplus M_i$, say $N_1 \oplus M_1$, we have $I(x) = \emptyset$ for every x in $D_\frown(N_1 \oplus M_1)$. Then we can see easily that the condition in Article 6 is satisfied for $N_1 \oplus M_1$, and hence the norm is L_p-type in it, a contradiction if $M_1 \neq 0$.

9. The most part of our theorem has been already proved and the rest to prove is that

$$x \in N \text{ and } |||x||| \leqq 1 \quad \text{imply} \quad m(x) = |||x|||^p .$$

This is an immediate consequence of the following more general fact:

(*) *If there exist two modulars m_1, m_2 on R which is at least three dimensional, and the modular norms of then coincide each other, and if moreover $|||x||| = 1$ implies $m_1(x) = m_2(x) = 1$, then we have $m_1(x) = m_2(x)$ for every x such that $|||x||| \leqq 1$.*

In fact, in our case, $|||x|||^p$ is a modular whose modular norm is $|||x|||$ itself, and for the two modulars $|||x|||^p$ and $m(x)$, we can apply (*).

To prove (*), we decompose R as: $R = N \oplus M$ where N is at least two dimensional. Then for every x, y in N, if $m_1(x) = m_1(y) \leqq 1$, then we have also $m_2(x) = m_2(y)$, because there exists an element z in M, such that $m_1(z) = 1 - m_1(x)$ and we have

$$m_1(x+z) = m_1(y+z) = 1 ,$$

and hence

$$m_2(x+z) = m_2(y+z) = 1 .$$

Therefore there exists an function $f(\xi)$ defined for $0 \leqq \xi \leqq 1$ and for which we have

$$m_2(x) = f(m_1(x))$$

for every x such that $m_1(x) \leqq 1$.

Since N contains mutually orthogonal elements, f is additive, and moreover it is continuous and $f(1)=1$. From this we can conclude that

$$f(\xi)=\xi\ .$$

Therefore (*) was proved for N, and since N is arbitrary and modulars have the orthogonal additivity, it is valid also for the whole space R.

10. Here we shall give some remarks and counter examples.

If R has no discrete element, then the exceptional manifold in the theorem, of course, can not exist. Moreover, in this case, if two modulars m_1, m_2 on R coincide each other for every sufficiently small values of their own, then, by virtue of the orthogonal additivity of modulars, we can prove easily that they coincide completely.

Therefore we have, for the last part of the theorem,

$$m(x)=|||x|||^p$$

without restrictions.

In general case, for the exceptional part of the theorem we do not know whether it is superfluous or not. But the last restriction: $|||x|||\leqq 1$ can not be wholly removed for the validity of (1), since, for instance, we can define a modular on the sequence space l_2 as:

$$m(x)=\begin{cases}\sum_{\nu=1}^{\infty}|\xi_\nu|^2 & \text{if } \sum_{\nu=1}^{\infty}|\xi_\nu|^2\leqq 1\\ +\infty & \text{otherwise}\end{cases}$$

for every $x\equiv\{\xi_\nu\}$, and the two norms of this modular both coincide with those of the usual modular of l_2 $\left(\text{that is, } \sum_{\nu=1}^{\infty}|\xi_\nu|^2 \text{ for } \{\xi_\nu\}\right)$ respectively. This is also an example for the necessity of the similar restriction in (*).

The theorem is also valid in case that R is finite-dimensional, if the number of its dimension is sufficiently large. Although the least number of dimension for this validity is not yet determined, it is sufficient if

$$\text{Dim.}(R)>\text{Max}\left\{12,\frac{1}{\alpha-1}\right\}\ .$$

The following example shows that this number can not remain bounded when α tends to 1.

Put

$$f(\xi)=\begin{cases} 0 & \text{if } 0\leqq\xi\leqq\dfrac{1}{\alpha} \\ \alpha\xi-1 & \text{if } \dfrac{1}{\alpha}\leqq\xi\leqq 1 \\ +\infty & \text{if } 1<\xi\,. \end{cases}$$

In the n-dimensional space where $n<\dfrac{1}{\alpha-1}$, if we define a modular as:

$$m(x)=\sum_{\nu=1}^{n} f(|\xi_\nu|),$$

where

$$x\equiv\{\xi_1,\xi_2,\cdots\xi_n\},$$

then we have

$$|||x|||=\underset{\nu}{\mathrm{Max}}\,|\xi_\nu|,$$

since there is no element x such that $1<m(x)<+\infty$. Then if $|||x|||=1$ and $x\equiv\{1,\xi,\cdots\xi_n\}$, then putting $x_1\equiv\{1,0,\cdots 0\}$ we have for every ξ such that $\dfrac{1}{\alpha}\leqq\xi\leqq 1$,

$$\frac{1+m(\xi x)}{\xi}\geqq\frac{1+m(\xi x_1)}{\xi}=\frac{1+f(\xi)}{\xi}=\alpha\,.$$

On the other hand we have also for every ξ such that $0\leqq\xi\leqq\dfrac{1}{\alpha}$,

$$\frac{1+m(\xi x)}{\xi}=\frac{1}{\xi}\geqq\alpha$$

and here the equality holds for $\xi=\dfrac{1}{\alpha}$, and hence $\|x\|=\alpha$.

Therefore we have (2) for this space while the norm is M-type.

As for (*), we can, to some extent, remove the assumption that $|||x|||=1$ implies $m_1(x)=m_2(x)=1$ and prove by the similar method as displayed there, that

If R is infinte-dimensional and m_1, m_2 are two modulars on it such that the modular norm of them coincide each other and, nevertheless, they do not coincide on the set $\{x: |||x|||\leqq 1\}$, then we can find a normal manifold N of

R such that $N^{\perp}$ is finite-dimensional and there is no element x in N such that

$$1 < m_1(x) < +\infty \ ,$$

and hence the modular norm is M-type in N.

On the other hand, even under the assumption mentioned above, (*) is not true for some two dimensional spaces, for instance, if we put

$$\varphi(\xi) = \begin{cases} \sin^2(\pi\xi^2) & \text{if } |\xi| \leqq 1 \\ 0 & \text{if } |\xi| > 1 \end{cases}$$

and

$$m(x) = \xi^2 + \eta^2 + \varepsilon\{\varphi(\xi) - \varphi(\eta)\} \ ,$$

where $x \equiv \{\xi, \eta\}$, and ε is a positive constant such that $\varepsilon \leqq \frac{1}{4\pi^2 + \pi}$, then we obtain a modular and for the modular norm of it we have

$$|||x||| = (\xi^2 + \eta^2)^{\frac{1}{2}}$$

because $\xi^2 + \eta^2 = 1$ implies $m(x) = 1$. Moreover if ε is sufficiently small, say $\varepsilon = \frac{1}{3000}$, then we have also

$$||x|| = 2(\xi^2 + \eta^2)^{\frac{1}{2}} \ .$$

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References.

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[2] S. YAMAMURO : On linear modulars. Proc. Japan Acad. vol. 27 (1951), pp. 623–624.