ON THE WIRTINGER'S CONNECTIONS IN HIGHER ORDER SPACES

By

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Contents

§ 0. Introduction

The concept of connection introduced by W. WIRTINGER [1][1] was given as a generalization of WEYL's connection [2] under the consideration of the possibility of its application to physics and astronomy. He thought at that time that his theory was ample enough to stand for any mathematical requirement in some branches of them.

Nowadays, the progress of physics and astronomy is so remarkable that his idea can not be fully accepted, but from the geometrical point of view his concept of connection itself is very interesting, various generalizations have been performed by many students and moreover the study of his concept is being carried on even now.

Geometrically the WIRTINGER's connection contains two important concepts, the one is that of double vectors, the other is that of non-linear connections.

In the former case H. EYRAUD [3] generalized the parallel displacement of WEYL but his research has no direct relation with WIRTINGER. Later from a different point of view A. KAWAGUCHI [4] pointed out that as a special case of his general theory, the WIRTINGER's connection was derived. In addition to these papers the present author [5] has de-

(1) Numbers in brackets refer to the references at the end of the paper.

veloped the theory in higher order spaces.

V. V. VAGNER [6] in the Soviet Union has established the theory of Strip as a method of discussing the differential geometry. His method seems to us to be a special case of WIRTINGER's point of view.

In the matter of non-linear connections, after WIRTINGER, G. Y. RAINICH [7] studied it and H. FRIESECKE [8], E. BORTOLOTTI [9], M. MIKAMI [10] and others have followed.

Recently KAWAGUCHI [11] has established the foundation of non-linear connections.

The purpose of the present paper is to develop a general theory of WIRTINGER's connections introducing some kinds of WIRTINGER's connections in higher order spaces and discussing the relations between KAWAGUCHI's connections and WIRTINGER's in view of the previous papers of the present author.

§1 is devoted to the abridgment of the classical theory of WIRTINGER's connections. In §2 WIRTINGER's connections in higher order spaces are introduced and in addition, the modified forms will be found. §3 involves the discussion of WIRTINGER's connections in KAWAGUCHI spaces. The definition of covariant derivatives in our spaces is given in §4. The covariant derivatives in a special KAWAGUCHI spaces will be stated in detail in §5 and in §6 various tensors and identities are derived in the space.

§ 1. Theory of the WIRTINGER's connection

With each point of an n-dimensional space $X_n(x^\alpha)$ we associate a fibre $(\eta^\alpha, \upsilon_\alpha)$ consisting of a pair of a contravariant vector η^α and a covariant vector υ_α provided that our fibre is characterized by the incidence condition

$$\eta^\alpha \cdot \upsilon_\alpha = 0 .$$

In other words, our manifold consists of a bundle where $(\eta^\alpha, \upsilon_\alpha)$ with incidence condition is an element of the fibre in the base manifold X_n. Hereafter, such manifold will be called WIRTINGER manifold. The $(\eta^\alpha, \upsilon_\alpha)$ with the condition $\eta^\alpha \cdot \upsilon_\alpha = 0$ is called double vector by SCHOUTEN, and WIRTINGER named it E_{n-1}-element. E_{n-1}-elements form a $(2n-3)$-dimensional manifold.

Let us consider the infinitesimal displacement of any point in the base manifold X_n then $\eta^\alpha, \upsilon_\alpha$ which are attached to that point are affected by infinitesimal variations. We indicate such variations by

the symbols $\delta_\xi \eta^\alpha$, $\delta_\xi \upsilon_\alpha$ where δ_ξ means the variation of ξ^α-direction.

On the other hand, differentials of η^α, υ_α which belong to double vectors of any fixed point are indicated by the symbols $d\eta^\alpha$, $d\upsilon_\alpha$. It is remarkable that δ-differential and d-differential are commutative:

$$\delta_\xi d\eta^\alpha = d\delta_\xi \eta^\alpha \ , \qquad \delta_\xi d\upsilon_\alpha = d\delta_\xi \upsilon_\alpha \ . \tag{1.1}$$

Double vectors $(\eta^\alpha, \upsilon_\alpha)$ associated with a point 0 remain double vectors after all under a contact transformation in accordance with the infinitesimal displacement of the point 0. Therefore, in order to introduce our connection we take the contact condition

$$\delta_\xi (\eta^\alpha \cdot d\upsilon_\alpha) = \delta_\xi \eta^\alpha \cdot d\upsilon_\alpha + \eta^\alpha \cdot \delta_\xi d\upsilon_\alpha = \rho \eta^\alpha \cdot d\upsilon_\alpha + \sigma \upsilon_\alpha d\eta^\alpha \ . \tag{1.2}$$

From the incidence condition we have

$$\delta_\xi \eta^\alpha \cdot \upsilon_\alpha + \eta^\alpha \cdot \delta_\xi \upsilon_\alpha = 0 \ , \quad d\eta^\alpha \cdot \upsilon_\alpha + \eta^\alpha \cdot d\upsilon_\alpha = 0 \ , \quad \delta_\xi (d\eta^\alpha \cdot \upsilon_\alpha) = \delta_\xi (-\eta^\alpha \cdot d\upsilon_\alpha),$$

and by use of (1.2)

$$\delta_\xi (\upsilon_\alpha d\eta^\alpha) = \delta_\xi \upsilon_\alpha \cdot d\eta^\alpha + \upsilon_\alpha \cdot \delta_\xi d\eta^\alpha = -\rho \eta^\alpha \cdot d\upsilon_\alpha - \sigma d\eta^\alpha \cdot \upsilon_\alpha$$

are given.

Where $\delta_\xi \eta^\alpha$, $\delta_\xi \upsilon_\alpha$, ρ, σ are linear homogeneous functions of ξ^α, $\delta_\xi \eta^\alpha$ are homogeneous of degree one in η^α and of degree zero in υ_α. Vice versa, $\delta_\xi \upsilon_\alpha$ are homogeneous of degree one in υ_α, homogeneous of degree zero in η^α. On the other hand ρ, σ are both homogeneous function of degree zero in η^α, υ_α. Such assumptions are all valid in general.

We put

$$\eta^\alpha \delta_\xi \upsilon_\alpha \equiv W(x, \eta, \upsilon, \xi) \ . \tag{1.3}$$

then under the above assumptions, $W(x, \eta, \upsilon, \xi)$ is linear homogeneous of ξ^α and homogeneous of degree one in η^α, υ_α. Differentiate (1.3) partially by η^β, υ_β,

$$\begin{aligned} \frac{\partial W(x, \eta, \upsilon, \xi)}{\partial \eta^\beta} &= \delta_\xi \upsilon_\beta + \eta^\alpha \frac{\partial \delta_\xi \upsilon_\alpha}{\partial \eta^\beta} \ , \\ \frac{\partial W(x, \eta, \upsilon, \xi)}{\partial \upsilon_\beta} &= \eta^\alpha \frac{\partial \delta_\xi \upsilon_\alpha}{\partial \upsilon_\beta} \ , \end{aligned} \tag{1.4}$$

are obtained.

From (1.1) we get

$$\begin{aligned} d_\xi \delta \eta^\alpha &= \frac{\partial \delta_\xi \eta^\alpha}{\partial \eta^\beta} d\eta^\beta + \frac{\partial \delta_\xi \eta^\alpha}{\partial \upsilon_\beta} d\upsilon_\beta \ , \\ \delta_\xi d\upsilon_\alpha &= \frac{\partial \delta_\xi \upsilon_\alpha}{\partial \eta^\beta} d\eta^\beta + \frac{\partial \delta_\xi \upsilon_\alpha}{\partial \upsilon_\beta} d\upsilon_\beta \ , \end{aligned}$$

therefore (1.2) is rewritten as

$$\delta_\xi \eta^\alpha \cdot d\upsilon_\alpha + \eta^\alpha \frac{\partial \delta_\xi \upsilon_\alpha}{\partial \eta^\beta} d\eta^\beta + \eta^\alpha \frac{\partial \delta_\xi \upsilon_\alpha}{\partial \upsilon_\beta} d\upsilon_\beta = \rho \eta^\alpha \cdot d\upsilon_\alpha + \sigma d\eta^\alpha \cdot \upsilon_\alpha .$$

In the above expression, comparing the coefficients of $d\eta^\alpha$, $d\upsilon_\alpha$ in both sides and by use of (1.4) one obtains

$$\delta_\xi \eta^\alpha = -\frac{\partial W(x, \eta, \upsilon, \xi)}{\partial \upsilon_\alpha} + \rho \eta^\alpha ,$$

$$\delta_\xi \upsilon_\alpha = \frac{\partial W(x, \eta, \upsilon, \xi)}{\partial \eta^\alpha} - \sigma \upsilon_\alpha ,$$

The final WIRTINGER's formulae are given putting $\rho = \sigma$ in the above formulae as follows:

$$(1.5) \quad \delta_\xi \eta^\alpha = -\frac{\partial W(x, \eta, \upsilon, \xi)}{\partial \upsilon_\alpha} + \rho \eta^\alpha, \quad \delta_\xi \upsilon_\alpha = \frac{\partial W(x, \eta, \upsilon, \xi)}{\partial \eta^\alpha} - \rho \upsilon_\alpha .$$

Formulae (1.5) satisfy the most general assumption for a δ-transformation affecting the relation in double vectors in the infinitesimally near points (0) and (0′) subjected to the contact condition.

By the point transformation, W and ρ vary as

$$\bar{W}(\bar{x}, \bar{\eta}, \bar{\upsilon}, \bar{\xi}) = W(x, \eta, \upsilon, \xi) - \frac{\partial^2 \bar{x}^\alpha}{\partial x^\beta \partial x^\gamma} \bar{\upsilon}_\alpha \eta^\beta \xi^\gamma ,$$

$$\bar{\rho}(\bar{x}, \bar{\eta}, \bar{\upsilon}, \bar{\xi}) = \rho(x, \eta, \upsilon, \xi) .$$

In use of (1.5) we can define the WIRTINGER's covariant differential as follows:

$$(1.6) \quad \begin{aligned} D\eta^\alpha &= \delta_\xi \eta^\alpha + \frac{\partial W(x, \eta, \upsilon, \xi)}{\partial \upsilon_\alpha} - \rho \eta^\alpha , \\ D\upsilon_\alpha &= \delta_\xi \upsilon_\alpha - \frac{\partial W(x, \eta, \upsilon, \xi)}{\partial \eta^\alpha} + \rho \upsilon_\alpha . \end{aligned}$$

Clearly formulae (1.6) are a contact transformation in the double vector $(\eta^\alpha, \upsilon_\alpha)$ and as connection they are one kind of non-linear connections.

If we put the assumption that $W(x, \eta, \upsilon, \xi)$ is a linear homogeneous function in η^α, υ_α then (1.6) gives a linear connection and is reduced to an incidence invariant displacement, therefore in our case the WIRTINGER's connections seem to be a generalization of an incidence invariant displacement.

§ 2. Generalized WIRTINGER's connections

Elements of the base manifold in a WIRTINGER manifold are points (x^α), but here we assume the elements of our base manifold are line-elements of order m, that is, $(x^\alpha, x^{(1)\alpha}, \cdots, x^{(m)\alpha})$ and we again call them points.

With each point of the base manifold we associate double vectors consisting of a contravariant vector η^α and a covariant υ_α, therefore the incidence condition

$$\eta^\alpha \cdot \upsilon_\alpha = 0$$

holds good always.
We call such manifold the WIRTINGER manifold in higher order and indicate it by the symbol $W_n^{(m)}$. The symbol $W_n^{(0)}$ means the classical WIRTINGER manifold in our case.

We indicate the differentials of η^α, υ_α in double vectors associated to a point $(x^\alpha, x^{(1)\alpha}, \cdots, x^{(m)\alpha})$ by the symbols $d\eta^\alpha$, $d\upsilon_\alpha$ and the differentials of η^α, υ_α caused by x^α of that point by the symbols $\delta_{\xi^{(0)}}\eta^\alpha$, $\delta_{\xi^{(0)}}\upsilon_\alpha$, where $\xi^{(0)\alpha}$ means the variation of x^α, that is dx^α.

On the other hand, as the base manifold of $W_n^{(m)}$ is a manifold of line-elements of m-th order, we have to consider the variations of $x^{(1)\alpha}, x^{(2)\alpha}, \cdots, x^{(m)\alpha}$, that is $dx^{(1)\alpha}, dx^{(2)\alpha}, \cdots, dx^{(m)\alpha}$. Now we take the symbols $\xi^{(1)\alpha}, \xi^{(2)\alpha}, \cdots, \xi^{(m)\alpha}$ instead of the above symbols, then

$$\delta_{\xi^{(1)}}\eta^\alpha,\ \delta_{\xi^{(2)}}\eta^\alpha, \cdots, \delta_{\xi^{(m)}}\eta^\alpha\ ;\quad \delta_{\xi^{(1)}}\upsilon_\alpha,\ \delta_{\xi^{(2)}}\upsilon_\alpha, \cdots, \delta_{\xi^{(m)}}\upsilon_\alpha$$

are variations of η^α, υ_α respectively caused by $\xi^{(1)\alpha}, \xi^{(2)\alpha}, \cdots, \xi^{(m)\alpha}$. It is needless to say that d-difierential and δ-differential are commutative undoubtedly as in §1, that is to say

$$d\delta_{\xi^{(i)}}\eta^\alpha = \delta_{\xi^{(i)}}d\eta^\alpha, \quad d\delta_{\xi^{(i)}}\upsilon_\alpha = \delta_{\xi^{(i)}}d\upsilon_\alpha \qquad (i=0, 1, 2, \cdots, m).$$

We denote the variations of η^α, υ_α by $\delta_\xi\eta^\alpha$, $\delta_\xi\upsilon_\alpha$ respectively corresponding to an infinitesimal displacement from an element $(x, x^{(1)}, \cdots, x^{(m)})$ to a neighboring element $(x+dx, x^{(1)}+dx^{(1)}, \cdots, x^{(m)}+dx^{(m)})$, then the relation

$$\delta_\xi = \delta_{\xi^{(0)}} + \delta_{\xi^{(1)}} + \cdots + \delta_{\xi^{(m)}}$$

holds good always.

As the condition to introducing connections in our manifold involving the case of $W_n^{(0)}$ we take the following, that is

$$
(2.1)\qquad
\begin{aligned}
\delta_{\xi^{(0)}}(\eta^\alpha\cdot d\upsilon_\alpha) &= \delta_{\xi^{(0)}}\eta^\alpha\cdot d\upsilon_\alpha+\eta^\alpha\cdot\delta_{\xi^{(0)}}d\upsilon_\alpha=\overset{(0)}{\rho}\eta^\alpha\cdot d\upsilon_\alpha+\overset{(0)}{\sigma}\upsilon_\alpha\cdot d\eta^\alpha\ ,\\
\delta_{\xi^{(1)}}(\eta^\alpha\cdot d\upsilon_\alpha) &= \delta_{\xi^{(1)}}\eta^\alpha\cdot d\upsilon_\alpha+\eta^\alpha\cdot\delta_{\xi^{(1)}}d\upsilon_\alpha=\overset{(1)}{\rho}\eta^\alpha\cdot d\upsilon_\alpha+\overset{(1)}{\sigma}\upsilon_\alpha\cdot d\eta^\alpha\ ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\delta_{\xi^{(m)}}(\eta^\alpha\cdot d\upsilon_\alpha) &= \delta_{\xi^{(m)}}\eta^\alpha\cdot d\upsilon_\alpha+\eta^\alpha\cdot\delta_{\xi^{(m)}}d\upsilon_\alpha=\overset{(m)}{\rho}\eta^\alpha\cdot d\upsilon_\alpha+\overset{(m)}{\sigma}\upsilon_\alpha\cdot d\eta^\alpha\ .
\end{aligned}
$$

Conditions in (2.1) indicate that the incidence condition holds good always not only in whole variations together of line-elements but also in one by one variations. Moreover, it is assumed that $\delta_{\xi^{(i)}}\eta^\alpha$ $(i=0,1,2,\cdots,m)$ are homogeneous functions of degree one in η^α, of degree zero in υ_α, linear homogeneous functions in $\xi^{(i)\alpha}$,
$\delta_{\xi^{(i)}}\upsilon_\alpha$ $(i=0,1,2,\cdots,m)$ are homogeneous functions of degree one in υ_α, of degree zero in η^α, linear homogeneous functions in $\xi^{(i)\alpha}$,
$\overset{(i)}{\rho}$, $\overset{(i)}{\sigma}$ $(i=0,1,\cdots,m)$ are homogeneous functions of degree zero in η^α, υ_α, linear homogeneous functions in $\xi^{(i)\alpha}$.

By a transformation of coordinates $\delta_{\xi^{(i)}}\eta^\alpha$, $\delta_{\xi^{(i)}}\upsilon_\alpha$, $\overset{(i)}{\rho}$ or $\overset{(i)}{\sigma}$ transform in similar manner to the transformation laws of $\dfrac{\partial T^\alpha}{\partial x^{(i)\beta}}dx^{(i)\beta}$, $\dfrac{\partial T_\alpha}{\partial x^{(i)\beta}}dx^{(i)\beta}$, $\dfrac{\partial T}{\partial x^{(i)\beta}}dx^{(i)\beta}$ provided that T^α, T_α, T are respectively contravariant, covariant, scalar in higher order spaces.

We put

$$
(2.2)\qquad \eta^\alpha\cdot\delta_{\xi^{(i)}}\upsilon_\alpha\equiv\overset{(i)}{W}(x,x^{(1)},x^{(2)},\cdots,x^{(m)},\eta,\upsilon,\xi^{(i)})\qquad (i=0,1,\cdots,\mathrm{m})
$$

then $\overset{(i)}{W}(x,x^{(1)},\cdots,x^{(m)},\eta,\upsilon,\xi^{(i)})$ are homogeneous functions in η^α, υ_α, linear homogeneous functions in $\xi^{(i)\alpha}$.
Differentiate (2.2) partially in η^β, υ_β respectively. We obtain

$$
(2.3)\qquad
\begin{aligned}
\eta^\alpha\frac{\partial\delta_{\xi^{(i)}}\upsilon_\alpha}{\partial\eta^\beta} &= \frac{\partial\overset{(i)}{W}}{\partial\eta^\beta}-\delta_{\xi^{(i)}}\upsilon_\alpha\ ,\\
\eta^\alpha\frac{\partial\delta_{\xi^{(i)}}\upsilon_\alpha}{\partial\upsilon_\beta} &= \frac{\partial\overset{(i)}{W}}{\partial\upsilon^\beta}
\end{aligned}
$$

By use of commutative law between the d-differential and δ-differential

$$
(2.4)\qquad
\begin{aligned}
\delta_{\xi^{(i)}}d\eta^\alpha &= d\delta_{\xi^{(i)}}\eta^\alpha=\frac{\partial\delta_{\xi^{(i)}}\eta^\alpha}{\partial\eta^\beta}d\eta^\beta+\frac{\partial\delta_{\xi^{(i)}}\eta^\alpha}{\partial\upsilon_\beta}d\upsilon_\beta\ ,\\
\delta_{\xi^{(i)}}d\upsilon_\alpha &= d\delta_{\xi^{(i)}}\upsilon_\alpha=\frac{\partial\delta_{\xi^{(i)}}\upsilon_\alpha}{\partial\eta^\beta}d\eta^\beta+\frac{\partial\delta_{\xi^{(i)}}\upsilon_\alpha}{\partial\upsilon_\beta}d\upsilon_\beta
\end{aligned}
$$

are given.

Substituting (2.4) in (2.1) and applying (2.3), we obtain from

$$\left(\frac{\partial \overset{(i)}{W}}{\partial \eta^\beta} - \delta_{\xi^{(i)}} \upsilon_\beta - \overset{(i)}{\sigma} \upsilon_\beta\right) d\eta^\beta + \left(\delta_{\xi^{(i)}} \eta^\alpha + \frac{\partial \overset{(i)}{W}}{\partial \upsilon_\alpha} - \overset{(i)}{\rho} \eta^\alpha\right) d\upsilon_\alpha = 0 ,$$

the formulae

$$\delta_{\xi^{(i)}} \eta^\alpha = -\frac{\partial \overset{(i)}{W}(x, x^{(1)}, \cdots, x^{(m)}, \eta, \upsilon, \xi^{(i)})}{\partial \upsilon_\alpha} + \overset{(i)}{\rho}(x, x^{(1)}, \cdots, x^{(m)}, \eta, \upsilon, \xi^{(i)}) \eta^\alpha ,$$

$$\delta_{\xi^{(i)}} \upsilon_\alpha = \frac{\partial \overset{(i)}{W}(x, x^{(1)}, \cdots, x^{(m)}, \eta, \upsilon, \xi^{(i)})}{\partial \eta^\alpha} - \overset{(i)}{\sigma}(x, x^{(1)}, \cdots, x^{(m)}, \eta, \upsilon, \xi^{(i)}) \upsilon_\alpha$$

$$(i = 0, 1, 2, \cdots, m),$$

Therefore the covariant differentials η^α, υ_α in $W_n^{(m)}$ are

$$(2.5) \quad \begin{aligned} D\eta^\alpha &= \delta_\xi \eta^\alpha + \frac{\partial \overset{(0)}{W}_\beta}{\partial \upsilon_\alpha} dx^\beta + \frac{\partial \overset{(1)}{W}_\beta}{\partial \upsilon_\alpha} dx^{(1)\beta} + \cdots + \frac{\partial \overset{(m)}{W}_\beta}{\partial \upsilon_\alpha} dx^{(m)\beta} - \rho \eta^\alpha , \\ D\upsilon_\alpha &= \delta_\xi \upsilon_\alpha - \frac{\partial \overset{(0)}{W}_\beta}{\partial \eta^\alpha} dx^\beta - \frac{\partial \overset{(1)}{W}_\beta}{\partial \eta^\alpha} dx^{(1)\beta} - \cdots - \frac{\partial \overset{(m)}{W}_\beta}{\partial \eta^\alpha} dx^{(m)\beta} + \sigma \upsilon_\alpha , \end{aligned}$$

where $\overset{(i)}{W} = \overset{(i)}{W}_\beta dx^{(i)\beta}$, $\rho = \sum_{i=0}^{m} \overset{(i)}{\rho}$, $\sigma = \sum_{i=0}^{m} \overset{(i)}{\sigma}$.

The covariant differentials in $W_n^{(m)}$ corresponding to (1.6) are

$$(2.6) \quad \begin{aligned} D\eta^\alpha &= \delta_\xi \eta^\alpha + \sum_{i=0}^{m} \frac{\partial \overset{(i)}{W}_\beta}{\partial \upsilon_\alpha} dx^{(i)\beta} - \rho \eta^\alpha , \\ D\upsilon_\alpha &= \delta_\xi \upsilon_\alpha - \sum_{i=0}^{m} \frac{\partial \overset{(i)}{W}_\beta}{\partial \eta^\alpha} dx^{(i)\beta} + \rho \upsilon_\alpha . \end{aligned}$$

The formalae (2.5) or (2.6) seem to be the most natural generalization of the classical WIRTINGER's connection, but as these formulae are not applicable directly to the KAWAGUCHI space when we intend later to introduce WIRTINGER's connections in the space, we have to consider the following special case for that.

In $W_n^{(m)}$ we assume especially that the double vector $(\eta^\alpha, \upsilon_\alpha)$ has such relation as

$$\upsilon_\alpha \equiv \upsilon_\alpha(x, x^{(1)}, \cdots, x^{(m)}, \eta) ,$$

where υ_α are homogeneous functions of degree k in η^α. The d-differential and δ-differential of υ_α are respectively

$$
(2.7)\qquad
\begin{aligned}
d\upsilon_\alpha &= \frac{\partial \upsilon_\alpha}{\partial \eta^\beta} d\eta^\beta ,\\
\delta_{\xi^{(i)}} \upsilon_\alpha &= \frac{\partial \upsilon_\alpha}{\partial x^{(i)\beta}} \xi^{(i)\beta} + \frac{\partial \upsilon_\alpha}{\partial \eta^\beta} \delta_{\xi^{(i)}} \eta^\beta \qquad (i=0, 1, \cdots, m) ,
\end{aligned}
$$

accordingly putting

$$
\upsilon_\alpha \delta_{\xi^{(i)}} \eta^\alpha \equiv -\overset{(i)}{W}(x, x^{(1)} \cdots, x^{(m)}, \eta, \upsilon, \xi^{(i)}) ,
$$

and differentiating them partially by η^α, υ_α we can easily obtain

$$
(2.8)\qquad
\begin{aligned}
\frac{\partial \upsilon_\alpha}{\partial \eta^\beta} \delta_{\xi^{(i)}} \eta^\alpha + \upsilon_\alpha \frac{\partial \delta_{\xi^{(i)}} \eta^\alpha}{\partial \eta^\beta} &= -\frac{\partial \overset{(i)}{W}}{\partial \eta^\beta} - \frac{\partial \overset{(i)}{W}}{\partial \upsilon_\gamma} \frac{\partial \upsilon_\gamma}{\partial \eta^\beta} ,\\
\delta_{\xi^{(i)}} \eta^\beta + \upsilon_\alpha \frac{\partial \delta_{\xi^{(i)}} \eta^\alpha}{\partial \upsilon_\beta} &= -\frac{\partial \overset{(i)}{W}}{\partial \upsilon_\beta} .
\end{aligned}
$$

On the other hand, under the contact condition

$$
\delta_{\xi^{(i)}} (\eta^\alpha \cdot d\upsilon_\alpha) = \delta_{\xi^{(i)}} (-\upsilon_\alpha d\eta^\alpha) = -\delta_{\xi^{(i)}} \upsilon_\alpha d\eta^\alpha - \upsilon_\alpha \delta_{\xi^{(i)}} d\eta^\alpha = \overset{(i)}{\rho} \eta^\alpha \cdot d\upsilon_\alpha + \overset{(i)}{\sigma} \upsilon_\alpha d\eta^\alpha ,
$$

we get

$$
(2.9)\qquad
\begin{aligned}
&\delta_{\xi^{(i)}} \upsilon_\alpha \cdot d\eta^\alpha + \upsilon_\alpha \left(\frac{\partial \delta_{\xi^{(i)}} \eta^\alpha}{\partial \eta^\beta} d\eta^\beta + \frac{\partial \delta_{\xi^{(i)}} \eta^\alpha}{\partial \upsilon_\beta} d\upsilon_\beta \right)\\
&= -\overset{(i)}{\rho} \eta^\alpha \cdot d\upsilon_\alpha - \overset{(i)}{\sigma} \upsilon_\alpha d\eta^\alpha .
\end{aligned}
$$

applying (2.8). Substitution of (2.7) in (2.9) gives rise to

$$
(2.10)\qquad
\begin{aligned}
&\left(\frac{\partial \upsilon_\alpha}{\partial x^{(i)\beta}} \xi^{(i)\beta} + \frac{\partial \upsilon_\alpha}{\partial \eta^\beta} \delta_{\xi^{(i)}} \eta^\beta \right) d\eta^\alpha - \left(\frac{\partial \overset{(i)}{W}}{\partial \eta^\beta} + \frac{\partial \overset{(i)}{W}}{\partial \upsilon_\gamma} \frac{\partial \upsilon_\gamma}{\partial \eta^\beta} \right.\\
&\qquad \left. + \frac{\partial \upsilon_\alpha}{\partial \eta^\beta} \delta_{\xi^{(i)}} \eta^\alpha \right) d\eta^\beta - \left(\frac{\partial \overset{(i)}{W}}{\partial \upsilon_\beta} + \delta_{\xi^{(i)}} \eta^\beta \right) d\upsilon_\beta\\
&= -\overset{(i)}{\rho} \eta^\beta \frac{\partial \upsilon_\beta}{\partial \eta^\alpha} d\eta^\alpha - \overset{(i)}{\sigma} \upsilon_\alpha d\eta^\alpha .
\end{aligned}
$$

By the use of the incidence condition, we get the following relation

$$
(2.11)\qquad \eta^\beta \frac{\partial \upsilon_\beta}{\partial \eta^\alpha} = -\upsilon_\alpha ,
$$

and substitute (2.11) in (2.10) then (2.10) yields

$$
(2.12)\qquad
\begin{aligned}
\left(\frac{\partial \upsilon_\alpha}{\partial \eta^\beta} - 2 \frac{\partial \upsilon_\beta}{\partial \eta^\alpha} \right) \delta_{\xi^{(i)}} \eta^\beta &= -\frac{\partial \upsilon_\alpha}{\partial \eta^\beta} \xi^{(i)\beta} + \frac{\partial \overset{(i)}{W}}{\partial \eta^\alpha} + 2 \frac{\partial \overset{(i)}{W}}{\partial \upsilon_\beta} \frac{\partial \upsilon_\beta}{\partial \eta^\alpha}\\
&\quad - (\overset{(i)}{\rho} - \overset{(i)}{\sigma}) \frac{\partial \upsilon_\beta}{\partial \eta^\alpha} \eta^\beta .
\end{aligned}
$$

From (1.5) and (2.12) we are led at once to

$$(2.13)\qquad \delta_{\xi^{(i)}}\eta^{\alpha} = g^{\gamma\alpha}\left(-\frac{\partial v_{\gamma}}{\partial x^{(i)\beta}}\xi^{(i)\beta} + \frac{\partial \overset{(i)}{W}}{\partial \eta^{\gamma}} + 2\frac{\partial \overset{(i)}{W}}{\partial v_{\beta}}\frac{\partial v_{\beta}}{\partial \eta^{\gamma}}\right),$$

where $\quad g_{\alpha\beta} \equiv \dfrac{\partial v_{\alpha}}{\partial \eta^{\beta}} - 2\dfrac{\partial v_{\beta}}{\partial \eta^{\alpha}}, \quad g_{\alpha\beta}g^{\gamma\beta} = \delta_{\alpha}^{\gamma},$

provided that the determinant $|g_{\alpha\beta}|$ does not vanish identically. Substitute (2.7) in (2.13)

$$(2.14)\qquad \delta_{\xi^{(i)}}v_{\alpha} = \frac{\partial v_{\alpha}}{\partial x^{(i)\beta}}\xi^{(i)\beta} + \frac{\partial v_{\alpha}}{\partial \eta^{\gamma}}g^{\delta\gamma}\left(-\frac{\partial v_{\delta}}{\partial x^{(i)\beta}}\xi^{(i)\beta} + \frac{\partial \overset{(i)}{W}}{\partial \eta^{\delta}} + 2\frac{\partial \overset{(i)}{W}}{\partial v_{\beta}}\frac{\partial v_{\beta}}{\partial \eta^{\delta}}\right)$$

is obtained easily.

The covariant differentials in such special WIRTINGER manifold $W_n^{(m)}$ are defined by use of (2.13) and (2.14) as

$$(2.15)\qquad \begin{aligned} D\eta^{\alpha} &= \delta_{\xi}\eta^{\alpha} + g^{\gamma\alpha}\sum_{i=0}^{m}\left(\frac{\partial v_{\gamma}}{\partial x^{(i)\beta}} - \frac{\partial \overset{(i)}{W}_{\beta}}{\partial \eta^{\gamma}} - 2\frac{\partial \overset{(i)}{W}_{\beta}}{\partial v_{\delta}}\frac{\partial v_{\delta}}{\partial \eta^{\gamma}}\right)dx^{(i)\beta}, \\ Dv_{\alpha} &= \delta_{\xi}v_{\alpha} - \sum_{i=0}^{m}\left\{\frac{\partial v_{\alpha}}{\partial x^{(i)\beta}} - \frac{\partial v_{\alpha}}{\partial \eta^{\delta}}g^{\gamma\delta}\left(\frac{\partial v_{\gamma}}{\partial x^{(i)\beta}} - \frac{\partial \overset{(i)}{W}_{\beta}}{\partial \eta^{\gamma}}\right.\right. \\ &\quad \left.\left. -2\frac{\partial \overset{(i)}{W}}{\partial v_{\varepsilon}}\frac{\partial v_{\varepsilon}}{\partial \eta^{\gamma}}\right)\right\}dx^{(i)\beta}. \end{aligned}$$

§ 3. WIRTINGER's connections introduced in KAWAGUCHI spaces

An n-dimensional space with the arc length

$$s = \int F(x, x^{(1)}, x^{(2)}, \cdots, x^{(m)})\,dt$$

is named a KAWAGUCHI space by J. L. SYNGE and H. V. CRAIG, hereafter we denote it by the symbol $K_n^{(m)}$.

In order that the arc length should be related intrinsically to the curve, we have the so-called ZERMELO's conditions, namely

$$(3.1)\qquad \sum_{\lambda=1}^{m}\lambda x^{(\lambda)i}F_{(\lambda)i} = F, \qquad \sum_{\lambda=s}^{m}\binom{\lambda}{s}x^{(\lambda-s+1)i}F_{(\lambda)i} = 0 \qquad (m \geqq s \geqq 1)$$

where we put $\quad F_{(r)i} \equiv \dfrac{\partial F}{\partial x^{(r)i}}\ (r = 1, 2, \cdots, m)$

Putting $m = s$ in the second expression in (3.1) one obtains

$$x^{(1)i} F_{(m)i} = 0 \,. \tag{3.2}$$

The relation (3.2) is nothing but an incidence condition between the contravariant vector $x^{(1)i}$ and the covariant vector $F_{(m)i}$. This fact means in general that, in the KAWAGUCHI space, with each point (x^i) the double vector $(x^{(1)i}, F_{(m)i})$ is associated always.

In some special KAWAGUCHI spaces, we find that $F_{(2)i}$ involves x^i and $x^{(1)i}$ only and the incidence condition

$$x^{(1)i} F_{(2)i} = 0$$

exists. It is clear that our such case is almost the same as the classical WIRTINGER's case.

In such a point of view, the KAWAGUCHI spaces give various examples of $W_n^{(m)}$, therefore various WIRTINGER's connections are introduced according to various points of view. We shall take up some interesting cases later on.

With the object of introducing a connection in $K_n^{(m)}$, KAWAGUCHI [12] considered a manifold with line-elements of order $2n-1$ and adopted

$$g_{ij} \equiv mF^{2m-1} F_{(m)i\,(m)j} + \overset{m}{\mathfrak{E}}_i \overset{m}{\mathfrak{E}}_j + \overset{1}{\mathfrak{E}}_i \overset{1}{\mathfrak{E}}_j \,, \tag{3.3}$$

as the fundamental tensor in the space.

$\overset{1}{\mathfrak{E}}_i$, $\overset{m}{\mathfrak{E}}_i$ in (3.3) are intrinsic SYNGE's vectors modified by

$$\overset{r}{E}_i \equiv \sum_{\lambda=r}^{m} (-1)^\lambda \binom{\lambda}{r} (F_{(\lambda)i})^{(\lambda-r)} \qquad (r=0, 1, 2, \cdots, m)$$

Here we assume that the rank of a matrix

$$((mF_{(m)i\,(m)j} + \overset{m}{\mathfrak{E}}_i \overset{m}{\mathfrak{E}}_j))$$

is $n-1$, then the determinant $|g_{ij}|$ does not vanish identically, so we can derive the contravariant tensor g^{ij} from which a geometrical qantity $\Gamma^i{}_j$ of class 1, and of order $2m-1$ is defined as

$$\Gamma^i{}_j \equiv g^{ik} (F^{2m-1} F_{(m)k(m)j} + \frac{1}{m} F^{2m-2} F_{(m)k} F_{(2m-1)j})$$
$$+ \frac{1}{m} \frac{F^{(1)}}{F} (\delta^i{}_j + \frac{1}{F} x^{(1)i} \overset{1}{\mathfrak{E}}_j) + \frac{x^{(1)i}}{F} \Xi_j \,.$$

This geometrical quantity $\Gamma^i{}_j$ is transformed in the same way as an affine parameter by the extended point transformation, that is

$$\Gamma^{\alpha}{}_{\beta} = \Gamma^{i}{}_{j} \frac{\partial x^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial x^{\beta}} - \frac{\partial^2 x^{\alpha}}{\partial x^{i} \partial x^{k}} \frac{\partial x^{i}}{\partial x^{\beta}} x^{(1)k} .$$

By means of the $\Gamma^{i}{}_{j}$ we can introduce connection parameters as follows. If we put

$$\overset{p}{D}\Gamma^{i}{}_{j} \equiv \sum_{\lambda=p}^{2m-1} \binom{\lambda}{p} \Gamma^{i}{}_{j(\lambda)k} dx^{(\lambda-p)k} \qquad (p=1, 2, \cdots, 2m-1),$$

then except $\overset{1}{D}\Gamma^{i}_{j}$, all of $\overset{p}{D}\Gamma^{i}{}_{j}$ are quantities with tensor character, therefore we can define the connection parameters as

$$F^{-1}\left(\sum_{p=1}^{m} F^{(p-1)p} D\Gamma^{i}{}_{j} + \sum_{p=1}^{m-1} \overset{pm+p}{\Psi} \overset{p}{D} \Gamma^{i}_{j}\right) \equiv \sum_{\alpha=0}^{2m-2} \overset{\alpha}{\Gamma}{}^{i}{}_{jk} dx^{(\alpha)k} ,$$

where $\overset{p}{\Psi}$ $(p=1, 2, \cdots, 2m-1)$ are scalar quantities of order $2m-1$.

Making use of thess quantities, the intrinsic covariant differential δX^{i} of an intrinsic vector X^{i} of order $2m-1$ can be defined as

$$\delta X^{i} = dX^{i} + \sum_{\alpha=0}^{2m-2} \overset{\alpha}{\Gamma}{}^{i}{}_{jk} X^{j} dx^{(\alpha)k} . \tag{3.4}$$

The above stated is the connection theory in the line-elements space of order $2m-1$ given by KAWAGUCHI.

In the first place, we shall embody the special WIRTINGER's connection in a KAWAGUCHI space making use of the general KAWAGUCHI's connection.

Let a pair $(\eta^{\alpha}, \upsilon_{\alpha})$ be the double vector in the KAWAGUCHI space $K_n^{(m)}$ in $W_n^{(m)}$, then the KAWAGUCHI's connections concerning the vectors η^{α}, υ_{α} are

$$\begin{aligned} \delta\eta^{\alpha} &= d\eta^{\alpha} + \sum_{i=0}^{2m-2} \overset{i}{\Gamma}{}^{\alpha}_{\beta\gamma} (x, x^{(1)}, \cdots, x^{(2m-1)}) \eta^{\beta} dx^{(i)\gamma} , \\ \delta\upsilon_{\alpha} &= d\upsilon_{\alpha} - \sum_{i=0}^{2m-2} \overset{i}{\Gamma}{}^{\beta}_{\alpha\gamma} (x, x, \cdots, x^{(2m-1)}) \upsilon_{\beta} dx^{(i)\gamma} , \end{aligned} \tag{3.5}$$

with the aid of (3.4) and we find at once

$$\begin{aligned} \upsilon_{\gamma} \delta_{\xi^{(i)}} \eta^{\gamma} &= -\overset{i}{\Gamma}{}^{\gamma}_{\alpha\beta} \eta^{\alpha} \upsilon_{\gamma} dx^{(i)\beta} = -\overset{(i)}{W}(x, x^{(1)}, \cdots, x^{(2m-1)}, \eta, \upsilon, \xi^{(i)}) \\ &= -\overset{(i)}{W}_{\beta} dx^{(i)\beta} \qquad (i=0, 1, 2, \cdots, 2m-2) , \\ \upsilon_{\gamma} \delta_{\xi^{(2m-1)}} \eta^{\gamma} &= -\overset{(2m-1)}{W}_{\beta} dx^{(2m-1)\beta} = 0 , \end{aligned} \tag{3.6}$$

from (3.5).

The relations in (3.6) give rise to

$$(3.7)\qquad \frac{\partial \overset{(i)}{W}_\beta}{\partial \eta^\alpha} = \overset{i}{\Gamma}{}^\gamma_{\alpha\beta}\, \upsilon_\gamma\ , \quad \frac{\partial \overset{(i)}{W}_\beta}{\partial \upsilon_\alpha} = \overset{i}{\Gamma}{}^\alpha_{\gamma\beta}\, \eta^\gamma \qquad (i=0, 1, 2, \cdots, 2m-2)\ .$$

Accordingly the covariant differentials based on the special WIRTINGER's connection are

$$(3.8)\qquad \begin{aligned} D\eta^\alpha &= d\eta^\alpha + g^{\gamma\alpha} \sum_{i=0}^{2m-2} \left(\frac{\partial \upsilon_\gamma}{\partial x^{(i)\beta}} - \overset{i}{\Gamma}{}^\delta_{\gamma\beta}\upsilon_\delta - 2\overset{i}{\Gamma}{}^\delta_{\varepsilon\beta}\eta^\varepsilon \frac{\partial \upsilon_\delta}{\partial \eta^\gamma} \right) dx^{(i)\beta}\ , \\ D\upsilon_\alpha &= d\upsilon_\alpha - \sum_{i=0}^{2m-2} \left\{ \frac{\partial \upsilon_\alpha}{\partial x^{(i)\beta}} - \frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta} \left(\frac{\partial \upsilon_\gamma}{\partial x^{(i)\beta}} - \overset{i}{\Gamma}{}^\varepsilon_{\gamma\beta}\upsilon_\varepsilon \right.\right. \\ &\qquad \left.\left. - 2\overset{i}{\Gamma}{}^\sigma_{\varepsilon\beta}\eta^\varepsilon \frac{\partial \upsilon_\sigma}{\partial \eta^\gamma} \right) \right\} dx^{(i)\beta}\ , \end{aligned}$$

by substituion of (3.7) in (2.15).

In use of (3.8) we can find the relations between the WIRTINGER's connection and the KAWAGUCHI's connection, namely

$$\begin{aligned} D\eta^\alpha &= d\eta^\alpha + g^{\gamma\alpha} \sum_{i=0}^{2m-2} \frac{\partial \upsilon_\alpha}{\partial x^{(i)\beta}} dx^{(i)\beta} - g^{\gamma\alpha} \sum_{i=0}^{2m-2} \overset{i}{\Gamma}{}^\delta_{\alpha\beta}\upsilon_\delta\, dx^{(i)\beta} - 2g^{\gamma\alpha} \frac{\partial \upsilon_\delta}{\partial \eta^\gamma} \sum_{i=0}^{2m-2} \overset{i}{\Gamma}{}^\delta_{\varepsilon\beta}\eta^\varepsilon dx^{(i)\beta} \\ &= d\eta^\alpha + g^{\gamma\alpha}\left(d\upsilon_\gamma - \frac{\partial \upsilon_\gamma}{\partial \eta^\beta} d\eta^\beta \right) - g^{\gamma\alpha}(d\upsilon_\gamma - \delta\upsilon_\gamma) - 2g^{\gamma\alpha} \frac{\partial \upsilon_\beta}{\partial \eta^\gamma}(\delta\eta^\beta - d\eta^\beta) \\ &= g^{\gamma\alpha}\left(\delta\upsilon_\gamma - 2\frac{\partial \upsilon_\beta}{\partial \eta^\gamma}\, \delta\eta^\beta \right)\ , \end{aligned}$$

and

$$\begin{aligned} D\upsilon_\alpha &= d\upsilon_\alpha - \sum_{i=0}^{2m-2} \frac{\partial \upsilon_\alpha}{\partial x^{(i)\beta}} dx^{(i)\beta} + \frac{\partial \upsilon^\alpha}{\partial \eta^\delta} g^{\gamma\delta} \sum_{i=0}^{2m-2} \frac{\partial \upsilon_\gamma}{\partial x^{(i)\beta}} dx^{(i)\beta} \\ &\quad - \frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta} \sum_{i=0}^{2m-2} \overset{i}{\Gamma}{}^\varepsilon_{\gamma\beta}\upsilon_\varepsilon dx^{(i)\beta} - 2\frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta} \sum_{i=0}^{2m-2} \overset{i}{\Gamma}{}^\gamma_{\varepsilon\beta}\eta^\varepsilon \frac{\partial \upsilon_\sigma}{\partial \eta^\gamma} dx^{(i)\beta} \\ &= d\upsilon_\alpha - \left(d\upsilon_\alpha - \frac{\partial \upsilon_\alpha}{\partial \eta^\beta} d\eta^\beta \right) + \frac{\partial \upsilon_\alpha}{\partial \eta^\delta}\left(d\upsilon_\gamma - \frac{\partial \upsilon_\gamma}{\partial \eta^\beta} d\eta^\beta \right) - \frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta}(d\upsilon_\gamma - \delta\upsilon_\gamma) \\ &\quad - 2\frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta} \frac{\partial \upsilon_\sigma}{\partial \eta^\gamma}(\delta\eta^\sigma - d\eta^\sigma) \\ &= \frac{\partial \upsilon_\alpha}{\partial \eta^\beta} d\eta^\beta - \frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta}\left(\frac{\partial \upsilon_\gamma}{\partial \eta^\beta} - 2\frac{\partial \upsilon_\beta}{\partial \eta^\gamma} \right) d\eta^\beta + \frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta}\left(\delta\upsilon_\gamma - 2\frac{\partial \upsilon_\beta}{\partial \eta^\gamma}\delta\eta^\beta \right) \\ &= \frac{\partial \upsilon_\alpha}{\partial \eta^\beta} g^{\gamma\beta}\delta\upsilon_\gamma + \frac{\partial \upsilon_\alpha}{\partial \eta^\beta}\delta\eta^\beta - \frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta} \frac{\partial \upsilon_\gamma}{\partial \eta^\beta}\delta\eta^\beta \\ &= \delta\upsilon_\alpha - \left(\delta^\gamma_\alpha - \frac{\partial \upsilon_\alpha}{\partial \eta^\beta} g^{\gamma\beta} \right)\left(\delta\upsilon_\gamma - \frac{\partial \upsilon_\gamma}{\partial \eta^\beta}\, \delta\eta^\beta \right)\ . \end{aligned}$$

So that

$$(3.9)\qquad \begin{aligned} D\eta^{\alpha} &= \delta\eta^{\alpha} + g^{\gamma\alpha}\left(\delta\upsilon_{\gamma} - \frac{\partial\upsilon_{\gamma}}{\partial\eta^{\beta}}\delta\eta^{\beta}\right), \\ D\upsilon_{\alpha} &= \delta\upsilon_{\alpha} - \left(\delta^{\gamma}_{\alpha} - \frac{\partial\upsilon_{\alpha}}{\partial\eta^{\beta}}g^{\gamma\beta}\right)\left(\delta\upsilon_{\gamma} - \frac{\partial\upsilon^{\gamma}}{\partial\eta^{\beta}}\delta\eta^{\beta}\right) \end{aligned}$$

are relations between them.
The formulae (3.8) are the special (or modified) WIRTINGER's connection introduced in the general KAWAGUCHI space.

In a special KAWAGUCHI space with the arc length

$$s = \int \left\{A_i(x, x')\, x''^i + B(x, x')\right\}^{1/p} dt\,,$$

KAWAGUCHI [13] introduced two kinds of connections, say connection C and connection C'.

Connection C:
We put

$$F \equiv A_i(x, x')\, x''^i + \mathrm{B}(x, x')\,,$$

where F is a scalar and covariant vector A_i is a homogeneous function of degree $p-2$ in x'^i.
With the exception of $2p=3$, the determinant of the tensor

$$G_{ij} \equiv 2A_{i(j)} - A_{j(i)}$$

does not vanish identically.
Let us introduce a covariant vector defined by

$$(3.10)\qquad T_i \equiv -2\frac{d}{dt}\frac{\partial F}{\partial x''^i} + \frac{\partial F}{\partial x'^i} = (A_{k(i)} - 2A_{i(k)})\, x''^k - 2A_{ik}x'^k + B_{(i)}\,,$$

where $\quad A_{k(i)} \equiv \dfrac{\partial A_k}{\partial x'^i}\,, \quad A_{ik} \equiv \dfrac{\partial A_i}{\partial x^k}\,, \quad B_{(i)} \equiv \dfrac{\partial B}{\partial x'^i}\,,$

and contract T_i with G^{ij} then we have at once

$$(3.11)\qquad x^{[2]j} \equiv -T_i G^{ij} = x''^j + 2\Gamma^j\,,$$

where $\quad 2\Gamma^j \equiv (2A_{ik}x'^k - B_{(i)})\, G^{ij}\,, \quad G_{ij}G^{ik} = \delta^k_j\,.$

The vector T_i defined by (3.10) is nothing but the so-called CRAIG vector and among $x^{[2]j}$, A_i and F there exists the following relation

$$A_i x^{[2]i} = F\,.$$

It is remarkable that although this space is a KAWAGUCHI space of order 2, it can be treated in the same manner as $K_n^{(l)}$,

The covariant differentials of a contravarint vector v^i and a covariant vector v_i are defined as

$$\delta v^i = dv^i + \Gamma^i_{(j)(k)} v^j dx^k \ , \qquad (3.12)$$
$$\delta v_i = dv_i - \Gamma^j_{(i)(k)} v_j dx^k \ ,$$

with the base connection

$$\delta x'^i = dx'^i + 2\Gamma^i \ . \qquad (3.13)$$

The connection given by (3.12) is the connection C in our space.

Connection C':

Theorem. *Let Φ_i be a covariant vector subjected to line-elements $(x, x^{(1)}, \cdots, x^{(m)})$ then*

$$D^{[m-\rho]}_{ij}(\Phi) v^j \equiv \sum_{\alpha=\rho}^{m} \binom{\alpha}{\rho} \frac{\partial \Phi_i}{\partial x^{(\alpha)j}} \frac{d^{\alpha-\rho} v^j}{dt^{\alpha-\rho}}$$

are components of a covariant vector provided v^i be a contravariant vector.

From this theorem given by Kawaguchi we see

$$-D^{[1]}_{ij}(T) v^j = 2G_{ij} \frac{dv^j}{dt} + G_{ik(l)} x''^k v^l + 2\Gamma_{i(l)} v^l \ ,$$

where $\Gamma_i \equiv G_{ik}\Gamma^k$, and by use of the above formulae we can introduce, excepting the case $2p=3$, an absolute differential along a curve as follows. Putting

$$D^{[1]}_i v^i \equiv \frac{dv^i}{dt} + G^{hi}\left(\frac{1}{2} G_{hk(l)} x''^k + \Gamma_{h(l)}\right) v^l \ ,$$

and being aided by

$$G^{hi}\left(\frac{1}{2} G_{hk(l)} x''^k + \Gamma_{h(l)}\right) = \frac{1}{2} G^{hi} G_{hk(l)} (x''^k + 2\Gamma^k) + \Gamma^i_{(l)}$$
$$= \frac{1}{2} G^{hi} G_{hk(l)} \delta x'^k + \Gamma^i_{(l)} \ ,$$

we can define the absolute differential of a vector corresponding to a displacement from a line element (x, x') to a neighboring line element $(x+dx, x'+dx')$ as

$$D^{[1]} v^i = dv^i + \Gamma^i_{(k)(j)} v^k dx^j + \frac{1}{2} G^{hi} G_{hj(k)} v^k \delta x'^j \ . \qquad (3.14)$$

This reduces to (3.12), when $\delta x'^i = 0$.

The absolute differential defined by (3.14) is not intrinsic, so that we have to introduce an intrinsic absolute differential. For the purpose, we adopt the following notations

$$
\begin{aligned}
I_{lji} &\equiv A_{l(j)(i)} + A_{j(l)(i)} + (p-3)\, A_{i(j)(l)}\,, \\
J_{li} &\equiv A_{l(i)} + (p-2)\, A_{i(l)}\,, \\
\widetilde{C}^{k}_{\cdot ij} &\equiv (p-3)\, G^{lk} I_{lji} + (p-4)\, G^{lk}_{\cdot\cdot(j)} J_{li}\,, \\
C^{k}_{\cdot ij} &\equiv 1/(p-3)^2\, \widetilde{C}^{k}_{\cdot ij} \quad p \neq 3\,, \qquad \equiv \widetilde{C}^{k}_{\cdot ij} \quad p=3\,,
\end{aligned}
$$

where I_{lji} is homogeneous of degree $p-4$ in x'^i and J_{li} is homogeneous of degree $p-3$ in x'^i.

The tensor $C^{k}_{\cdot ii}$ is constructed from only A_i. By the use of this tensor an absolute differential is defined as

$$
\begin{aligned}
\delta^* \eta^\alpha &= d\eta^\alpha + (\Gamma^{\alpha}_{(\beta)(\gamma)} + C^{\alpha}_{\cdot\beta\delta}\, \Gamma^{\delta}_{(\gamma)})\, \eta^\beta dx^\gamma + C^{\alpha}_{\cdot\beta\gamma}\, \eta^\beta dx'^\gamma\,, \\
\delta^* \upsilon_\alpha &= d\upsilon_\alpha - (\Gamma^{\beta}_{(\alpha)(\gamma)} + C^{\beta}_{\cdot\alpha\delta}\, \Gamma^{\delta}_{(\gamma)})\, \upsilon_\beta\, dx^\gamma - C^{\beta}_{\cdot\alpha\gamma}\, \upsilon_\beta\, dx'^\gamma\,,
\end{aligned}
\tag{3.15}
$$

provided η^α, υ_α are contravariant and covariant vector in our space. The connection given by (3.15) is the connection C' in our space.

The concrete forms of WIRTINGER's connections corresponding to (3.8) are introduced in two ways, the one is obtained by use of the connection C, the other making use of the connection C'.

Now we are going to explain these cases. The ZERMELO's conditions applied to the special KAWAGUCHI space are

$$
\begin{aligned}
A_i(x, x')\, x'^i &= 0\,, \\
2A_i x''^i + (A_{k(i)} x''^k + B_{(i)})\, x'^i &= p\,(A_i x''^i + B)\,,
\end{aligned}
$$

in which the former is nothing but an incidence condition between x'^i and A_i associated to each point in our space, therefore all conditions are satisfied for introduction of the WIRTINGER's connection provided that we have the (x'^i, A_i) as double vectors.

For the sake of introducing a concrete WIRTINGER's connection based on the connection C, we put $m=1$ in (2.15) and especially put $\upsilon_\alpha = \upsilon_\alpha\,(x, \eta)$ then the connection is defined as

$$
\begin{aligned}
D\eta^\alpha = \delta_\xi \eta^\alpha &+ g^{\gamma\alpha}\Big(\frac{\partial \upsilon_\gamma}{\partial x^\beta} - \frac{\partial \overset{(0)}{W}_\beta}{\partial \eta^\gamma} - 2\,\frac{\partial \overset{(0)}{W}_\beta}{\partial \upsilon_\delta}\,\frac{\partial \upsilon_\delta}{\partial \eta^\gamma}\Big)\, dx^\beta \\
&+ g^{\gamma\alpha}\Big(-\frac{\partial \overset{(1)}{W}_\beta}{\partial \eta^\gamma} - 2\,\frac{\partial \overset{(1)}{W}_\beta}{\partial \upsilon_\delta}\,\frac{\partial \upsilon_\delta}{\partial \eta^\gamma}\Big)\, dx^{(1)\beta}\,,
\end{aligned}
\tag{3.16}
$$

$$D\upsilon_\alpha = \delta_\xi \upsilon_\alpha - \left\{ \frac{\partial \upsilon_\alpha}{\partial x^\beta} - \frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta} \left(\frac{\partial \upsilon_\gamma}{\partial x^\beta} - \frac{\partial \overset{(0)}{W}_\beta}{\partial \eta^\gamma} - 2 \frac{\partial \overset{(0)}{W}_\beta}{\partial \upsilon_\varepsilon} \frac{\partial \upsilon_\varepsilon}{\partial \eta_\gamma} \right) \right\} dx^\beta$$

$$+ \frac{\partial \upsilon_\alpha}{\partial \eta^\delta} g^{\gamma\delta} \left(- \frac{\partial \overset{(1)}{W}_\beta}{\partial \eta^\gamma} - 2 \frac{\partial \overset{(1)}{W}_\beta}{\partial \upsilon_\varepsilon} \frac{\partial \upsilon_\varepsilon}{\partial \eta^\gamma} \right) dx^{(1)\beta} .$$

On the other hand, since KAWAGUCHI's connections in the case are

$$\delta\eta^\alpha = d\eta^\alpha + \Gamma^\alpha_{(\beta)(\gamma)} \eta^\beta dx^\gamma , \quad \delta\upsilon_\alpha = d\upsilon_\alpha - \Gamma^\beta_{(\alpha)(\gamma)} \upsilon_\beta dx^\gamma ,$$

the relations

$$\upsilon_\alpha \delta_{\xi^{(0)}} \eta^\alpha = - \Gamma^\alpha_{(\beta)(\gamma)} \eta^\beta \upsilon_\alpha dx^\gamma = - \overset{(0)}{W}(x, x', \eta, \upsilon, \xi^{(0)}) ,$$

$$\upsilon_\alpha \delta_{\xi^{(1)}} \eta^\alpha = - \overset{(1)}{W}(x, x', \eta, \upsilon, \xi^{(1)}) = 0$$

which lead to

$$\frac{\partial \overset{(0)}{W}}{\partial \eta^\gamma} = \Gamma^\alpha_{(\gamma)(\beta)} \upsilon_\alpha dx^\beta , \qquad \frac{\partial \overset{(0)}{W}}{\partial \upsilon_\delta} = \Gamma^\delta_{(\alpha)(\beta)} \eta^\alpha dx^\beta ,$$

accordingly

$$(3.17) \quad \begin{aligned} D\eta^\alpha &= d\eta^\alpha + g^{\gamma\alpha} \left(\frac{\partial \upsilon_\gamma}{\partial x^\beta} - \upsilon_\delta \Gamma^\delta_{(\beta)(\gamma)} - 2\upsilon_{\delta(\gamma)} \Gamma^\delta_{(\varepsilon)(\beta)} \eta^\varepsilon \right) dx^\beta , \\ D\upsilon_\alpha &= d\upsilon_\alpha - \left\{ \upsilon_{\alpha\beta} - \upsilon_{\alpha(\delta)} g^{\gamma\delta} (\upsilon_{\gamma\beta} - \upsilon_\varepsilon \Gamma^\varepsilon_{(\beta)(\gamma)} - 2\Gamma^\varepsilon_{(\rho)(\beta)} \eta^\rho \upsilon_{\varepsilon(\gamma)}) \right\} dx^\beta \end{aligned}$$

are obtained at once.
Substituting $\eta^\alpha = x'^\alpha$, $\upsilon_\alpha = A_\alpha$ in (3.17) we have

$$(3.18) \quad \begin{aligned} Dx'^\alpha &= dx'^\alpha + g^{\gamma\alpha} A_{\gamma\beta} - A_\delta \Gamma^\delta_{(\beta)(\gamma)} - 2A_{\delta(\gamma)} \Gamma^\delta_{(\beta)}) dx^\beta , \\ DA_\alpha &= dA_\alpha - \left\{ A_{\alpha\beta} - A_{\alpha(\delta)} g^{\gamma\delta} (A_{\gamma\beta} - A_\varepsilon \Gamma^\varepsilon_{(\beta)(\gamma)} - 2\Gamma^\varepsilon_{(\beta)} A_{\varepsilon(\gamma)}) \right\} dx^\beta . \end{aligned}$$

If we adopt the notation

$$\Phi_{\alpha\beta} \equiv A_{\alpha\beta} - A_\gamma \Gamma^\gamma_{(\alpha)(\beta)} - 2A_{\gamma(\alpha)} \Gamma^\gamma_{(\beta)} ,$$

the relation

$$\Phi_{\alpha\beta} = \nabla_\beta A_\alpha + g_{\alpha\gamma} \Gamma^\gamma_{(\beta)} ,$$

can be easily obtained with the aid of KAWAGUCHI's covariant derivatives

$$\nabla_\beta A_\alpha = A_{\alpha\beta} - A_{\alpha(\gamma)} \Gamma^\gamma_{(\beta)} - A_\gamma \Gamma^\gamma_{(\alpha)(\beta)} .$$

Therefore (3.18) gives rise to

$$
\begin{aligned}
Dx'^{\alpha} &= dx'^{\alpha} + g^{\gamma\alpha}\Phi_{\gamma\beta}dx^{\beta} = dx'^{\alpha} + g^{\gamma\alpha}(\nabla_{\beta}A_{\gamma} + g_{\gamma\delta}\Gamma^{\delta}_{(\beta)})\,dx^{\beta} \\
&= dx'^{\alpha} + \Gamma^{\alpha}_{(\beta)}dx^{\beta} + g^{\gamma\alpha}\nabla_{\beta}A_{\gamma}dx^{\beta}\ , \\
DA_{\alpha} &= dA_{\alpha} - (A_{\alpha\beta} - A_{\alpha(\gamma)}g^{\delta\gamma}\Phi_{\delta\beta})\,dx^{\beta} \\
&= dA_{\alpha} - \left\{A_{\alpha\beta} - A_{\alpha(\gamma)}g^{\delta\gamma}(\nabla_{\beta}A_{\delta} + g_{\delta\varepsilon}\Gamma^{\varepsilon}_{(\beta)})\right\}dx^{\beta} \\
&= dA_{\alpha} - \Gamma^{\gamma}_{(\alpha)(\beta)}A_{\gamma}dx^{\beta} + A_{\alpha(\delta)}g^{\gamma\delta}\nabla_{\beta}A_{\gamma}dx^{\beta} - \nabla_{\beta}A_{\alpha}dx^{\beta}\ .
\end{aligned}
$$

Finally the relations between both connections are obtained as follows

$$
(3.19)\qquad
\begin{aligned}
Dx'^{\alpha} &= \delta x'^{\alpha} + g^{\gamma\alpha}\nabla_{\beta}A_{\gamma}\,dx^{\beta}\ , \\
DA_{\alpha} &= \delta A_{\alpha} - (\delta^{\gamma}_{\alpha} - A_{\alpha(\delta)}g^{\gamma\delta})\,\nabla_{\beta}A_{\gamma}dx^{\beta}\ .
\end{aligned}
$$

The above obtained formulae (3.19) may be found from (3.9) directly, because if we put $\eta^{\alpha} = x'^{\alpha}$, $\upsilon_{\alpha} = A_{\alpha}$ in (3.9) it follows that

$$
\begin{aligned}
Dx'^{\alpha} &= \delta x'^{\alpha} + g^{\alpha\gamma}(\delta A_{\gamma} - A_{\gamma(\beta)}\delta x'^{\beta})\ , \\
DA_{\alpha} &= \delta A_{\alpha} - (\delta^{\gamma}_{\alpha} - A_{\alpha(\beta)}g^{\gamma\beta})(\delta A_{\gamma} - A_{\gamma(\beta)}\delta x'^{\beta})\,.
\end{aligned}
$$

Then paying attention to the following relation

$$
\begin{aligned}
\delta A_{\gamma} - A_{\gamma(\beta)}\delta x'^{\beta} &= dA_{\gamma} - \Gamma^{\alpha}_{(\gamma)(\beta)}A_{\alpha}dx^{\beta} - A_{\gamma(\beta)}(dx'^{\beta} + \Gamma^{\beta}_{(\delta)}dx^{\delta}) \\
&= (A_{\gamma(\beta)} - \Gamma^{\alpha}_{(\gamma)(\beta)}A_{\alpha} - A_{\gamma(\delta)}\Gamma^{\delta}_{(\beta)})\,dx^{\beta} \\
&= \nabla_{\beta}A_{\gamma}dx^{\beta}\ ,
\end{aligned}
$$

we can attain our object.

As the general form of the WIRTINGER's connection is indicated in (3.16) and that of the KAWAGUCHI's in (3.15) we obtain

$$
\begin{aligned}
\upsilon_{\alpha}\delta_{\xi}{}^{(0)}\eta^{\alpha} &= -(\Gamma^{\alpha}_{(\gamma)(\beta)} + C^{\alpha}_{\cdot\gamma\delta}\Gamma^{\delta}_{(\beta)})\eta^{\gamma}\upsilon_{\alpha} = -\overset{(0)}{W}_{\beta}(x, x', \eta, \upsilon)\ , \\
\upsilon_{\alpha}\delta_{\xi}{}^{(1)}\eta^{\alpha} &= -C^{\alpha}_{\cdot\gamma\beta}\eta^{\gamma}\upsilon_{\alpha} = -\overset{(1)}{W}_{\beta}(x, x', \eta, \upsilon)\ ,
\end{aligned}
$$

form which

$$
(3.20)\qquad
\begin{aligned}
&\frac{\partial\overset{(0)}{W}_{\beta}}{\partial\eta^{\gamma}} = (\Gamma^{\alpha}_{(\gamma)(\beta)} + C^{\alpha}_{\cdot\gamma\delta}\Gamma^{\delta}_{(\beta)})\,\upsilon_{\alpha}\ , && \frac{\partial\overset{(0)}{W}_{\beta}}{\partial\upsilon_{\delta}} = (\Gamma^{\delta}_{(\gamma)(\beta)} + C^{\delta}_{\cdot\gamma\varepsilon}\Gamma^{\varepsilon}_{(\beta)})\,\eta^{\gamma}\ , \\
&\frac{\partial\overset{(1)}{W}_{\beta}}{\partial\eta^{\gamma}} = C^{\alpha}_{\cdot\gamma\beta}\upsilon_{\alpha}\ , && \frac{\partial\overset{(1)}{W}_{\beta}}{\partial\upsilon_{\delta}} = C^{\delta}_{\cdot\gamma\beta}\eta^{\gamma}
\end{aligned}
$$

are derived.

By the substitution of (3.20) in (3.16)

$$
\begin{aligned}
D\eta^{\alpha} &= \delta_{\varepsilon}\eta^{\alpha} + g^{\gamma\alpha}\Big\{\frac{\partial v_{\gamma}}{\partial x^{\beta}} - (\Gamma^{\rho}_{(\gamma)(\beta)} + C^{\rho}_{\cdot\gamma\delta}\Gamma^{\delta}_{(\beta)})\, v_{\rho} - 2(\Gamma^{\delta}_{(\rho)(\beta)} \\
&\quad + C^{\delta}_{\cdot\rho\varepsilon}\Gamma^{\varepsilon}_{(\beta)})\eta^{\rho}\frac{\partial v_{\delta}}{\partial \eta^{\gamma}}\Big\} dx^{\beta} + g^{\gamma\alpha}\Big(-C^{\delta}_{\cdot\gamma\beta}v_{\delta} - 2C^{\delta}_{\cdot\varepsilon\beta}\eta^{\varepsilon}\frac{\partial v_{\delta}}{\partial \eta^{\gamma}}\Big) dx^{(1)\beta}, \\
D v_{\alpha} &= \delta_{\varepsilon} v_{\alpha} - \Big[\frac{\partial v_{\alpha}}{\partial x^{\beta}} - \frac{\partial v_{\alpha}}{\partial \eta_{\delta}} g^{\gamma\delta}\Big\{\frac{\partial v_{\gamma}}{\partial x^{\beta}} - (\Gamma^{\rho}_{(\gamma)(\beta)} + C^{\rho}_{\cdot\gamma\delta}\Gamma^{\delta}_{(\beta)})\, v_{\rho} \\
&\quad - 2(\Gamma^{\lambda}_{(\rho)(\beta)} + C^{\lambda}_{\cdot\rho\varepsilon}\Gamma^{\varepsilon}_{(\beta)})\eta^{\rho}\frac{\partial v_{\lambda}}{\partial \eta^{\gamma}}\Big\}\Big] dx^{\beta} \\
&\quad + \frac{\partial v_{\alpha}}{\partial \eta^{\delta}} g^{\gamma\delta}\Big(-C^{\lambda}_{\cdot\gamma\beta}v_{\lambda} - 2C^{\lambda}_{\cdot\varepsilon\beta}\eta^{\varepsilon}\frac{\partial v_{\lambda}}{\partial \eta^{\gamma}}\Big) dx^{(1)\beta}
\end{aligned}
\tag{3.20$'$}
$$

are given. Puting $\eta^{\alpha} = x'^{\alpha}$, $v_{\alpha} = A_{\alpha}$ then the final formulae of the connection are

$$
\begin{aligned}
Dx'^{\alpha} &= dx'^{\alpha} + g^{\beta\alpha}\Big\{A_{\beta\varepsilon} - (\Gamma^{\gamma}_{(\beta)(\varepsilon)} + C^{\gamma}_{\cdot\beta\delta}\Gamma^{\delta}_{(\varepsilon)})\, A_{\gamma} \\
&\quad - 2\Gamma^{\gamma}_{(\rho)(\varepsilon)} x'^{\rho} A_{\gamma(\beta)}\Big\} dx^{\varepsilon} - g^{\beta\alpha} C^{\delta}_{\cdot\beta\varepsilon} A_{\delta} dx^{(1)\varepsilon}, \\
DA_{\alpha} &= dA_{\alpha} - \Big[A_{\alpha\varepsilon} + A_{\alpha(\beta)} g^{\delta\beta}\Big\{-A_{\delta\varepsilon} + (\Gamma^{\rho}_{(\delta)(\varepsilon)} + C^{\rho}_{\cdot\delta\sigma}\Gamma^{\sigma}_{(\varepsilon)})\, A_{\rho} \\
&\quad + 2\Gamma^{\gamma}_{(\rho)(\varepsilon)} x'^{\rho} A_{\gamma(\delta)}\Big\}\Big] dx^{\varepsilon} - A_{\alpha(\beta)} g^{\delta\beta} C^{\sigma}_{\delta\varepsilon} A_{\sigma} dx^{(1)\varepsilon}.
\end{aligned}
\tag{3.21}
$$

They clearly demonstrate that the formulae (3. 21) are a concrete form of the WIRTINGER's connection in the special KAWAGUCHI space.
In order to obtain the relations between the WIRTINGER's connection and the KAWAGUCHI's we adopt again the notation δ^* as the KAWAGUCHI's covariant differential based on the connection C', then

$$
\begin{aligned}
\delta^* x'^{\alpha} &= dx'^{\alpha} + (\Gamma^{\alpha}_{(\beta)(\gamma)} + C^{\alpha}_{\cdot\beta\delta}\Gamma^{\delta}_{(\gamma)})\, x'^{\beta} dx^{\gamma} + C^{\alpha}_{\cdot\beta\gamma} x'^{\beta} dx'^{\gamma} \\
&= dx'^{\alpha} + \Gamma^{\alpha}_{(\gamma)} dx^{\gamma} = \delta x'^{\alpha}, \\
\delta^* A_{\alpha} &= dA_{\alpha} - (\Gamma^{\beta}_{(\alpha)(\gamma)} + C^{\beta}_{\cdot\alpha\delta}\Gamma^{\delta}_{(\gamma)})\, A_{\beta} dx^{\gamma} - C^{\beta}_{\cdot\alpha\gamma} A_{\beta} dx'^{\gamma} \\
&= dA_{\alpha} - \Gamma^{\beta}_{(\alpha)(\gamma)} A_{\beta} dx^{\gamma} - C^{\beta}_{\cdot\alpha\gamma} A_{\beta}\delta x'^{\gamma} = \delta A_{\alpha} - C^{\beta}_{\cdot\alpha\gamma} A_{\beta}\delta x'^{\gamma}
\end{aligned}
\tag{3.22}
$$

are the covariant differentials defined by KAWAGUCHI. Formulae (3.21) are rewritten as

$$
\begin{aligned}
Dx'^{\alpha} &= dx'^{\alpha} + g^{\beta\alpha}\Phi_{\beta\varepsilon} dx^{\varepsilon} + g^{\beta\alpha}(\tilde{\Phi}_{\beta\varepsilon} - A_{\beta(\varepsilon)})\, dx'^{\varepsilon} \\
&\quad + g^{\beta\alpha}\Gamma^{\delta}_{(\varepsilon)}(\tilde{\Phi}_{\beta\delta} - A_{\beta(\delta)})\, dx^{\varepsilon}, \\
DA_{\alpha} &= dA_{\alpha} + [-A_{\alpha\varepsilon} + A_{\alpha(\beta)} g^{\delta\beta}\Phi_{\delta\varepsilon}]\, dx^{\varepsilon} \\
&\quad - A_{\alpha(\beta)} g^{\delta\beta} C^{\rho}_{\cdot\delta\sigma} A_{\rho}\,(dx'^{\sigma} + \Gamma^{\sigma}_{(\varepsilon)} dx^{\varepsilon}),
\end{aligned}
$$

by the use of

$$\Phi_{\beta\varepsilon} \equiv A_{\beta\varepsilon} - 2A_{\gamma(\beta)}\Gamma^{\gamma}_{(\varepsilon)} - A_{\gamma}\Gamma^{\gamma}_{(\beta)(\varepsilon)} ,$$
$$\tilde{\Phi}_{\beta\varepsilon} \equiv A_{\beta(\varepsilon)} - A_{\gamma}C^{\gamma}_{\cdot\beta\varepsilon} .$$

On the other hand, (3.21) leads us to

$$\begin{aligned} Dx'^{\alpha} &= dx'^{\alpha} + \Gamma^{\alpha}_{(\beta)}dx^{\beta} + g^{\beta\alpha}\nabla_{\gamma}A_{\beta}dx^{\gamma} - g^{\beta\alpha}A_{\gamma}C^{\gamma}_{\cdot\beta\varepsilon}\delta x'^{\varepsilon} , \\ (3.23) \qquad DA_{\alpha} &= dA_{\alpha} - \Gamma^{\beta}_{(\alpha)(\gamma)}A_{\beta}dx^{\gamma} - \nabla_{\beta}A_{\alpha}dx^{\beta} \\ &\quad - A_{\alpha(\beta)}g^{\delta\beta}(\nabla_{\gamma}A_{\delta}dx^{\gamma} - C^{\gamma}_{\cdot\delta\varepsilon}A_{\gamma}\delta x'^{\varepsilon}) , \end{aligned}$$

by use of $\nabla_{\varepsilon}A_{\alpha}$.

In consequence of (3.22) and (3.23) the relations between both connections are expressed as

$$(3.24) \qquad \begin{aligned} Dx'^{\alpha} &= \delta^{*}x'^{\alpha} + g^{\beta\alpha}(\nabla_{\gamma}A_{\beta}dx^{\gamma} - A_{\gamma}C^{\gamma}_{\cdot\beta\varepsilon}\delta x'^{\varepsilon}) , \\ DA_{\alpha} &= \delta^{*}A_{\alpha} - (\delta^{\rho}_{\alpha} - A_{\alpha(\beta)}g^{\rho\beta})(\nabla_{\varepsilon}A_{\rho}dx^{\varepsilon} - A_{\sigma}C^{\sigma}_{\cdot\rho\gamma}\delta x'^{\gamma}) . \end{aligned}$$

Of course, the formulae (3.24) are easily derived from (3.9). Besides this there are many special KAWAGUCHI spaces in which such a connection as in (3.23) can be introduced, for example, in KAWAGUCHI speces with matrics

$$s = \int \left\{ a_i(x, x')\, a_j(x, x')\, x''^i x''^j + 2b(x, x')\, a_i(x, x')\, x''^i + c(x, x') \right\}^{1/p} dt ,$$
$$s = \int \left\{ A_i(x, x')\, x''^i + B(x, x', x'') \right\}^{1/p} dt ,$$

we can introduce concrete WIRTINGER's connections into the spaces.

§ 4. Covariant derivatives

To η^{α} and υ_{α} in double vectors $(\eta^{\alpha}, \upsilon_{\alpha})$, the covariant derivatives based on the special WIRTINGER's connection which is introduced in $K_n^{(m)}$ can be defined making use of KAWAGUCHI's covariant derivatives. The base connections in the KAWAGUCHI space are defined by

$$F^{2m-1}g^{ij}\delta\overset{1}{\mathfrak{S}}_i \equiv \delta x^{(2m-1)j} = \left(\delta_i^j + \overset{1}{\mathfrak{S}}_i \frac{x'^j}{F}\right) dx^{(2m-1)i} + \sum_{\alpha=0}^{2m-2} \overset{\alpha}{\underset{2m-1}{\Lambda}}{}_i^j\, dx^{(\alpha)i} ,$$

$$\binom{2m-1}{p} F^{2m-p-1}g^{ij}\sum_{\mu=p}^{2m-1} A^{p}_{\mu-p+1}\sum_{\lambda=\mu}^{2m-1}\binom{\lambda}{\mu}\overset{1}{\mathfrak{S}}_{i(\lambda)k}\, dx^{(\lambda-\mu)k} \equiv \delta x^{(2m-p-1)j}$$
$$= \left(\delta_i^j + \overset{1}{\mathfrak{S}}_i \frac{x'^j}{F}\right) dx^{(2m-p-1)i} + \sum_{\alpha=0}^{2m-p-2} \overset{\alpha}{\underset{2m-p-1}{\Lambda}}{}_i^j\, dx^{(\alpha)i} \quad (p=1, 2, \cdots, 2m-2) .$$

Making use of these connections, one may rewrite (3.4) as

$$(4.1) \qquad \delta X^i = \sum_{\alpha=0}^{2m-1} \nabla_j^{(\alpha)} X^i \cdot \delta x^{(\alpha)j} \qquad (\delta x^{(0)j} \equiv dx^j) ,$$

where

$$\nabla_j^{(2m-1)} X^i = X^i_{(2m-1)j} ,$$

$$(4.2) \qquad \nabla_j^{(p)} X^i = X^i_{(p)j} - \sum_{\lambda=p+1}^{2m-1} \nabla_k^{(\lambda)} X^i \overset{p}{\underset{\lambda}{A}}{}_j^k + \overset{p}{\Gamma}{}^i_{kj} X^k \quad (p=0, 1, 2, \cdots, 2m-1).$$

It should be noticed that

$$x'^j \nabla_j^{(p)} X^i = 0 \qquad (p=1, 2, \cdots, 2m-1) .$$

By use of KAWAGUCHI's covariant derivatives

$$\nabla_j^{(p)} X^i \qquad (p=0, 1, 2, \cdots, 2m-1) ,$$

the covariant derivatives in our case can be defined as follows.

From (3.9) the relations between the KAWAGUCHI's connection and the WIRTINGER's connection are

$$D\eta^\alpha = \delta\eta^\alpha + g^{\gamma\alpha}\left(\delta v_\gamma - \frac{\partial v_\gamma}{\partial \eta^\beta}\delta\eta^\beta\right) ,$$

$$Dv_\alpha = \delta v_\alpha - \left(\delta_\alpha^\gamma - \frac{\partial v_\alpha}{\partial \eta^\beta} g^{\gamma\beta}\right)\left(\delta v_\gamma - \frac{\partial v_\gamma}{\partial \eta^\beta}\delta\eta^\beta\right) .$$

Therefore, corresponding to (4.1) we have the following expressions

$$\delta\eta^\alpha = \sum_{\lambda=0}^{2m-1} \overset{\lambda}{\nabla}_\beta \eta^\alpha \cdot \delta x^{(\lambda)\beta} , \qquad \delta v_\alpha = \sum_{\lambda=0}^{2m-1} \overset{\lambda}{\nabla}_\beta v_\alpha \cdot \delta x^{(\lambda)\beta} .$$

Making use of the above expressions, one derives

$$\begin{aligned} D\eta^\alpha &= \sum_{\lambda=0}^{2m-1} \overset{\lambda}{\nabla}_\beta \eta^\alpha \delta x^{(\lambda)\beta} + g^{\gamma\alpha}\left(\sum_{\lambda=0}^{2m-1} \overset{\lambda}{\nabla}_\beta v_\gamma \cdot \delta x^{(\lambda)\beta} - \frac{\partial v_\gamma}{\partial \eta^\beta} \sum_{\lambda=0}^{2m-1} \overset{\lambda}{\nabla}_\delta \eta^\beta \cdot \delta x^{(\lambda)\beta}\right) \\ &= \sum_{\lambda=0}^{2m-1}\left(\overset{\lambda}{\nabla}_\delta \eta^\alpha + g^{\gamma\alpha}\overset{\lambda}{\nabla}_\delta v_\gamma - g^{\gamma\alpha}\frac{\partial v_\gamma}{\partial \eta^\beta}\overset{\lambda}{\nabla}_\delta \eta^\beta\right)\delta x^{(\lambda)\delta} \\ &= \sum_{\lambda=0}^{2m-1} {}^D\overset{\lambda}{\nabla}_\beta \eta^\alpha \cdot \delta x^{(\lambda)\beta} , \end{aligned}$$

where we put

$$(4.3) \qquad \begin{aligned} {}^D\overset{\lambda}{\nabla}_\beta \eta^\alpha &= \overset{\lambda}{\nabla}_\beta \eta^\alpha + g^{\gamma\alpha}\overset{\lambda}{\nabla}_\beta v_\gamma - g^{\gamma\alpha}\frac{\partial v_\gamma}{\partial \eta^\delta}\overset{\lambda}{\nabla}_\beta \eta^\delta \\ &= \overset{\lambda}{\nabla}_\beta \eta^\alpha + g^{\gamma\alpha}\left(\overset{\lambda}{\nabla}_\beta v_\gamma - \frac{\partial v_\gamma}{\partial \eta^\delta}\overset{\lambda}{\nabla}_\beta \eta^\delta\right) \\ &= \left(\delta_\delta^\alpha - g^{\gamma\alpha}\frac{\partial v_\gamma}{\partial \eta^\delta}\right)\overset{\lambda}{\nabla}_\beta \eta^\delta + g^{\gamma\alpha}\overset{\lambda}{\nabla}_\beta v_\gamma . \end{aligned}$$

Similarly

$$
\begin{aligned}
Dv_\alpha &= \sum_{\lambda=0}^{2m-1} \overset{\lambda}{\nabla}_\beta v_\alpha \, \delta x^{(\lambda)\beta} - \left(\delta_\alpha^\gamma - \frac{\partial v_\alpha}{\partial \eta^\beta} g^{\gamma\beta}\right) \sum_{\lambda=0}^{2m-1} \overset{\lambda}{\nabla}_\delta v_\gamma \cdot \delta x^{(\lambda)\delta} \\
&\qquad + \left(\delta_\alpha^\gamma - \frac{\partial v_\alpha}{\partial \eta^\beta} g^{\gamma\beta}\right) \frac{\partial v_\gamma}{\partial \eta^\delta} \sum_{\lambda=0}^{2m-1} \overset{\lambda}{\nabla}_\varepsilon \eta^\delta \cdot \delta x^{(\lambda)\varepsilon} \\
&= \sum_{\lambda=0}^{2m-1} \left\{ \overset{\lambda}{\nabla}_\varepsilon v_\alpha - \left(\delta_\alpha^\gamma - \frac{\partial v_\alpha}{\partial \eta^\beta} g^{\gamma\beta}\right) \overset{\lambda}{\nabla}_\varepsilon v_\gamma + \left(\delta_\alpha^\gamma - \frac{\partial v_\alpha}{\partial \eta^\beta} g^{\gamma\beta}\right) \frac{\partial v_\gamma}{\partial \eta^\delta} \overset{\lambda}{\nabla}_\varepsilon \eta^\delta \right\} \delta x^{(\lambda)\varepsilon} \\
&= \sum_{\lambda=0}^{2m-1} {}^D\overset{\lambda}{\nabla}_\beta v_\alpha \cdot \delta x^{(\lambda)\beta} ,
\end{aligned}
$$

where we put

$$
\begin{aligned}
{}^D\overset{\lambda}{\nabla}_\varepsilon v_\alpha &\equiv \overset{\lambda}{\nabla}_\varepsilon v_\alpha - \left(\delta_\alpha^\gamma - \frac{\partial v_\alpha}{\partial \eta^\beta} g^{\gamma\beta}\right) \overset{\lambda}{\nabla}_\varepsilon v_\gamma + \left(\delta_\alpha^\gamma - \frac{\partial v_\alpha}{\partial \eta^\beta} g^{\gamma\beta}\right) \frac{\partial v_\gamma}{\partial \eta^\delta} \overset{\lambda}{\nabla}_\varepsilon \eta^\delta \\
&= \overset{\lambda}{\nabla}_\varepsilon v_\alpha - \left(\delta_\alpha^\gamma - \frac{\partial v_\alpha}{\partial \eta^\beta} g^{\gamma\beta}\right) \left(\overset{\lambda}{\nabla}_\varepsilon v_\gamma - \frac{\partial v_\gamma}{\partial \eta^\delta} \overset{\lambda}{\nabla}_\varepsilon \eta^\delta \right) .
\end{aligned} \tag{4.4}
$$

It is valid to adopt these expressions defined in (4.3) and (4.4) as the covariant derivatives in our case; so, when they are applied to the special KAWAGUCHI space in the case of the connection C

$$
{}^D\overset{0}{\nabla}_\beta \eta^\alpha = \nabla_\beta \eta^\alpha + g^{\gamma\alpha} \left(\nabla_\beta v_\gamma - \frac{\partial v_\gamma}{\partial \eta^\delta} \nabla_\beta \eta^\delta \right) ,
$$

$$
{}^D\overset{1}{\nabla}_\beta \eta^\alpha = \nabla'_\beta \eta^\alpha + g^{\gamma\alpha} \left(\nabla'_\beta v_\gamma - \frac{\partial v_\gamma}{\partial \eta^\delta} \nabla'_\beta \eta^\delta \right)
$$

are obtained.

Putting

$$
\eta^\alpha \equiv x'^\alpha , \qquad v_\alpha \equiv A_\alpha ,
$$

and taking care of

$$
\nabla_\beta x'^\alpha = 0 , \qquad \nabla'_\beta x'^\alpha = \delta_\beta^\alpha , \qquad \nabla'_\beta A_\alpha = A_{\alpha(\beta)} ,
$$

one obtains

$$
\begin{aligned}
{}^D\overset{0}{\nabla}_\beta x'^\alpha &= \nabla_\beta x'^\alpha + g^{\gamma\alpha} (\nabla_\beta A_\gamma - A_{\gamma(\delta)} \nabla_\beta x'^\delta) = g^{\gamma\alpha} \nabla_\beta A_\gamma , \\
{}^D\overset{1}{\nabla}_\beta x'^\alpha &= \nabla'_\beta x'^\alpha + g^{\gamma\alpha} (\nabla'_\beta A_\gamma - A_{\gamma(\delta)} \nabla'_\beta x'^\delta) \\
&= \delta_\beta^\alpha + g^{\gamma\alpha} (A_{(\gamma)\beta} - A_{\gamma(\delta)} \delta_\beta^\delta) = \delta_\beta^\alpha ,
\end{aligned}
$$

therefore as the result we have

$$
Dx'^\alpha = g^{\gamma\alpha} \nabla_\beta A_\gamma \, dx^\beta + \delta_\beta^\alpha \delta x'^\beta . \tag{4.5}
$$

On the other hand, from

$$
{}^{D}\overset{0}{\nabla}_{\varepsilon} A_{\alpha} = \nabla_{\varepsilon} A_{\alpha} - (\delta_{\alpha}^{\gamma} - A_{\alpha(\beta)} g^{\gamma\beta})(\nabla_{\varepsilon} A_{\gamma} - A_{\gamma(\delta)} \nabla_{\varepsilon} x'^{\delta}) = A_{\alpha(\beta)} g^{\gamma\beta} \nabla_{\varepsilon} A_{\gamma} ,
$$

$$
{}^{D}\overset{1}{\nabla}_{\varepsilon} A_{\alpha} = \nabla'_{\varepsilon} A_{\alpha} - (\delta_{\alpha}^{\gamma} - A_{\alpha(\beta)} g^{\gamma\beta})(\nabla'_{\varepsilon} A_{\gamma} - A_{\gamma(\delta)} \nabla'_{\varepsilon} x'^{\delta}) = A_{\alpha(\varepsilon)} ,
$$

we can derive

$$
DA_{\alpha} = A_{\alpha(\beta)} g^{\gamma\beta} \nabla_{\varepsilon} A_{\gamma} dx^{\varepsilon} + \nabla'_{\varepsilon} A_{\alpha} \delta x'^{\varepsilon} . \tag{4.6}
$$

In like manner, applying (4.3), (4.4) to the special KAWAGUCHI space with the connection C',

$$
{}^{D}\overset{0}{\nabla}_{\beta} x'^{\alpha} = \nabla_{\beta} x'^{\alpha} + g^{\gamma\alpha} \left(\nabla_{\beta} A_{\gamma} - \frac{\partial A_{\gamma}}{\partial x'^{\delta}} \nabla_{\beta} x'^{\delta} \right) ,
$$

$$
{}^{D}\overset{1}{\nabla}_{\beta} x'^{\alpha} = \nabla'_{\beta} x'^{\alpha} + g^{\gamma\alpha} \left(\nabla'_{\beta} A_{\gamma} - \frac{\partial A_{\gamma}}{\partial x'^{\delta}} \nabla'_{\beta} x'^{\delta} \right) ,
$$

$$
{}^{D}\overset{0}{\nabla}_{\varepsilon} A_{\alpha} = \nabla_{\varepsilon} A_{\alpha} - (\delta_{\alpha}^{\gamma} - A_{\alpha(\beta)} g^{\gamma\beta})(\nabla_{\varepsilon} A_{\gamma} - A_{\gamma(\delta)} \nabla_{\varepsilon} x'^{\delta}) ,
$$

$$
{}^{D}\overset{1}{\nabla}_{\varepsilon} A_{\alpha} = \nabla'_{\varepsilon} A_{\alpha} - (\delta_{\alpha}^{\gamma} - A_{\alpha(\beta)} g^{\gamma\beta})(\nabla'_{\varepsilon} A_{\gamma} - A_{\gamma(\delta)} \nabla'_{\varepsilon} x'^{\delta})
$$

are derived, so that paying attention to

$$
\nabla_{j} x'^{i} = 0 , \quad \nabla'_{j} x'^{i} = \delta_{j}^{i}, \quad \nabla'_{j} A_{i} = A_{i(j)} - C_{\cdot ij}^{k} A_{k} ,
$$

we have

$$
{}^{D}\overset{0}{\nabla}_{\beta} x'^{\alpha} = g^{\gamma\alpha} \nabla_{\beta} A_{\gamma} ,
$$

$$
{}^{D}\overset{1}{\nabla}_{\beta} x'^{\alpha} = \delta_{\beta}^{\alpha} + g^{\gamma\alpha} (A_{\gamma(\beta)} - C_{\cdot\alpha\beta}^{\delta} A_{\delta} - A_{\gamma(\delta)} \delta_{\beta}^{\delta}) ,
$$

$$
{}^{D}\overset{0}{\nabla}_{\varepsilon} A_{\alpha} = \nabla_{\varepsilon} A_{\alpha} - (\delta_{\alpha}^{\gamma} - A_{\alpha(\beta)} g^{\gamma\beta})(\nabla_{\varepsilon} A_{\gamma} - A_{\gamma(\delta)} \times 0) = A_{\alpha(\beta)} g^{\gamma\beta} \nabla_{\varepsilon} A_{\gamma} ,
$$

$$
\begin{aligned}
{}^{D}\overset{1}{\nabla}_{\varepsilon} A_{\alpha} &= A_{\alpha(\varepsilon)} - C_{\cdot\alpha\varepsilon}^{\delta} A_{\delta} - (\delta_{\alpha}^{\gamma} - A_{\alpha(\beta)} g^{\gamma\beta})(A_{\gamma(\varepsilon)} - C_{\cdot\gamma\varepsilon}^{\delta} A_{\delta} - A_{\gamma(\delta)} \delta_{\varepsilon}^{\delta}) \\
&= A_{\alpha(\varepsilon)} - A_{\alpha(\beta)} g^{\gamma\beta} C_{\cdot\gamma\varepsilon}^{\delta} A_{\delta} .
\end{aligned}
$$

From these results we get at once

$$
\begin{aligned}
Dx'^{\alpha} &= g^{\gamma\alpha} \nabla_{\beta} A_{\gamma} dx^{\beta} + (\delta_{\beta}^{\alpha} - g^{\gamma\alpha} C_{\cdot\gamma\beta}^{\delta} A_{\delta}) \delta x'^{\beta} , \\
DA_{\alpha} &= A_{\alpha(\beta)} g^{\gamma\beta} \nabla_{\varepsilon} A_{\gamma} dx^{\varepsilon} + (A_{\alpha(\varepsilon)} - A_{\alpha(\beta)} g^{\gamma\beta} C_{\cdot\gamma\varepsilon}^{\delta} A_{\delta}) \delta x'^{\varepsilon} .
\end{aligned} \tag{4.7}
$$

Other covariant derivatives in our special KAWAGUHI space can be defined but the details will be stated in §5.

§ 5. Covariant derivatives in a special KAWAGUCHI space

On the connection of a special KAWAGUCHI space with the metrics

$$
s = \int \left\{ A_{i} x''^{i} + B \right\}^{1/p} dt ,
$$

we have developed the outline in §3 and on the other hand, on the covariant derivatives coming in contact with them we have touched very briefly in §4.

In §5 covariant derivatives subjected to the WIRTINGER's connection and relations with the KAWAGUCHI's derivatives will be stated somewhat in detail.

The KAWAGUCHI's differentials in the case of connection C are

$$\delta^{(C)}v^i = dv^i + \Gamma^i_{(j)(k)} v^j dx^k \,, \quad \delta^{(C)} v_i = dv_i - \Gamma^j_{(i)(k)} v_j dx^k \,,$$

$$\text{(base connection)} \qquad \delta x'^i = dx'^i + \Gamma^i_{(j)} dx^j \,,$$

so that we have

$$\delta^{(C)} v^i = \left(\frac{\partial v^i}{\partial x^k} - \frac{\partial v^i}{\partial x'^l} \Gamma^l_{(k)} + \Gamma^i_{(j)(k)} v^j \right) dx^k + \frac{\partial v^i}{\partial x'^k} \delta x'^k$$

$$= \overset{C}{\nabla}_j v^i dx^j + \overset{C}{\nabla}'_j v^i \delta x'^j \,,$$

where we put

$$\text{(5.1)} \qquad \overset{C}{\nabla}_j v^i \equiv \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^l} \Gamma^l_{(j)} + \Gamma^i_{(k)(j)} v^k \,, \qquad \overset{C}{\nabla}'_j v^i \equiv \frac{\partial v^i}{\partial x'^j} \,.$$

Similarly

$$\delta^{(C)} v_i = \left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_i}{\partial x'^l} \Gamma^l_{(k)} - \Gamma^j_{(i)(k)} v_j \right) dx^k + \frac{\partial v_i}{\partial x'^k} \delta x'^k$$

$$\equiv \overset{C}{\nabla}_j v_i dx^j + \overset{C}{\nabla}'_j v_i \delta x'^j$$

where we put

$$\text{(5.2)} \qquad \overset{C}{\nabla}_j v_i \equiv \frac{\partial v_i}{\partial x^j} - \frac{\partial v_i}{\partial x'^l} \Gamma^l_{(j)} - \Gamma^k_{(i)(j)} v_k \,, \qquad \overset{C}{\nabla}'_j v_i \equiv \frac{\partial v_i}{\partial x'^j} \,.$$

From (3.15) KAWAGUCHI's differentials in the case of connection C' expressed by

$$\delta^{(C')} v^i = dv^i + \Gamma^{*i}_{\cdot jk} v^j dx^k + C^i_{\cdot jk} v^j dx'^k \,,$$

$$\delta^{(C')} v_i = dv_i - \Gamma^{*j}_{\cdot ik} v_j dx^k - C^j_{\cdot ik} v_j dx'^k \,,$$

$$\text{(base connection)} \qquad \delta x'^i = dx'^i + \Gamma^i_{(j)} dx^j \,,$$

where we put

$$\Gamma^{*i}_{\cdot jk} \equiv \Gamma^i_{(j)(k)} + C^i_{\cdot jl} \Gamma^l_{(k)} \,.$$

Rewriting the above expressions we have

$$\delta^{(C')}v^i = dv^i + \Gamma^i_{(j)(k)} v^j dx^k + C^i_{\cdot jk} v^j \delta x'^k ,$$
$$\delta^{(C')}v_i = dv_i - \Gamma^j_{(i)(k)} v_j dx^k - C^j_{\cdot ik} v_j \delta x'^k ,$$

so that

$$\begin{aligned}
\delta^{(C')}v^i &= \left(\frac{\partial v^i}{\partial x^k} - \frac{\partial v^i}{\partial x'^l}\Gamma^l_{(k)} + \Gamma^i_{(j)(k)} v^j\right) dx^k + \left(\frac{\partial v^i}{\partial x'^k} + C^i_{\cdot jk} v^j\right) \delta x'^k \\
&= \overset{C'}{\nabla}_j v^i dx^j + \overset{C'}{\nabla}'_j v^i \delta x'^j , \\
\delta^{(C')}v_i &= \left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_i}{\partial x'^l}\Gamma^l_{(k)} + \Gamma^j_{(i)(k)} v_j\right) dx^k + \left(\frac{\partial v_i}{\partial x^k} - C^j_{\cdot ik} v_j\right) \delta x'^k \\
&= \overset{C'}{\nabla}_j v_i dx^j + \overset{C'}{\nabla}'_j v_i \delta x'^j
\end{aligned} \tag{5.2)'}$$

are given, where we put

$$\begin{aligned}
&\overset{C'}{\nabla}_j v^i \equiv \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^l}\Gamma^l_{(j)} + \Gamma^i_{(k)(j)} v^k , \quad \overset{C'}{\nabla}'_j v^i \equiv \frac{\partial v^i}{\partial x'^j} + C^i_{\cdot kj} v^k , \\
&\overset{C'}{\nabla}_j v_i \equiv \frac{\partial v_i}{\partial x^j} - \frac{\partial v_i}{\partial x'^l}\Gamma^l_{(j)} - \Gamma^k_{(i)(j)} v_k , \quad \overset{C'}{\nabla}'_j v_i \equiv \frac{\partial v_i}{\partial x'^j} - C^k_{\cdot ij} v_k .
\end{aligned} \tag{5.3}$$

Quantities given by (5.1), (5.2) and (5.3) are KAWAGUCHI's derivatives from which the next relations can be obtained easily

$$\begin{aligned}
&\overset{C}{\nabla}_j v^i = \overset{C'}{\nabla}_j v^i , \quad \overset{C'}{\nabla}'_j v^i = \overset{C'}{\nabla}'_j v^i + C^i_{\cdot kj} v^k , \\
&\overset{C}{\nabla}_j v_i = \overset{C'}{\nabla}_j v_i , \quad \overset{C'}{\nabla}'_j v_i = \overset{C'}{\nabla}_j v_i - C^k_{\cdot ij} v_k .
\end{aligned} \tag{5.4}$$

If we put

$$v_i \equiv A_i , \qquad v^i \equiv x'^i ,$$

the expressions

$$\begin{aligned}
&\overset{C}{\nabla}_j x'^i = \overset{C'}{\nabla}_j x'^i = 0 , \quad \overset{C}{\nabla}_j A_i = \overset{C'}{\nabla}_j A_i = A_{ij} - A_{i(l)}\Gamma^l_{(j)} - A_l \Gamma^l_{(i)(j)} , \\
&\overset{C'}{\nabla}'_j x'^i = \overset{C}{\nabla}'_j x'^i = \delta^i_j , \quad \overset{C'}{\nabla}'_j A_i = \overset{C}{\nabla}'_j A_i - C^k_{\cdot ij} A_k = A_{i(j)} - A_k C^k_{\cdot ij}
\end{aligned} \tag{5.5}$$

are given by (5.4).

Since

$$D^{(C)}\eta^i = d\eta^i + g^{ki}\left(\frac{\partial v_k}{\partial x^j} - v_l \Gamma^l_{(j)(k)} - 2\frac{\partial v_l}{\partial \eta^k}\Gamma^l_{(h)(j)}\eta^h\right) dx^j ,$$

$$D^{(C)}v_i = dv_i - \left\{\frac{\partial v_i}{\partial x^j} - \frac{\partial v_i}{\partial \eta^l} g^{kl}\left(\frac{\partial v_k}{\partial x^j} - v_m \Gamma^m_{(j)(k)} - 2\frac{\partial v_h}{\partial \eta^k}\Gamma^h_{(m)(j)}\eta^m\right)\right\} dx^j$$

are derived by use of (3.9) or (3.17) in our special KAWAGUHI space

applying the WIRTINGER's connection based on the connection C to a double vector $(\eta^i(x'), \upsilon_i(x, \eta))$, one obtains

$$D^{(C)}\eta^i = \frac{\partial \eta^i}{\partial x'^j} dx'^j + g^{ki}\left(\frac{\partial \upsilon_k}{\partial x^j} - \upsilon_l \Gamma^l_{(j)(k)} - 2\frac{\partial \upsilon_l}{\partial \eta^k}\Gamma^l_{(h)(j)}\eta^h\right) dx^j$$

$$= \left\{-\frac{\partial \eta^i}{\partial x'^l}\Gamma^l_{(j)} + g^{ki}\left(\frac{\partial \upsilon_k}{\partial x^j} - \upsilon_l \Gamma^l_{(j)(k)} - 2\frac{\partial \upsilon_l}{\partial \eta^k}\Gamma^l_{(h)(j)}\eta^h\right\} dx^j + \frac{\partial \eta^i}{\partial x'^j}\delta x'^j$$

$$= \overset{W}{\nabla}_j \eta^i dx^j + \overset{W}{\nabla}'_j \eta^i \delta x'^j ,$$

where we put

$$(5.6) \qquad \begin{aligned} \overset{W}{\nabla}_j \eta^i &\equiv -\frac{\partial \eta^i}{\partial x'^l}\Gamma^l_{(j)} + g^{ki}\left(\frac{\partial \upsilon_k}{\partial x^j} - \upsilon_l \Gamma^l_{(j)(k)} - 2\frac{\partial \upsilon_l}{\partial \eta^k}\Gamma^l_{(h)(j)}\eta^h\right) , \\ \overset{W}{\nabla}'_j \eta^i &\equiv \frac{\partial \eta^i}{\partial x'^j} . \end{aligned}$$

Similarly

$$D^{(C)}\upsilon_i = \frac{\partial \upsilon_i}{\partial x^j} dx^j - \frac{\partial \upsilon_i}{\partial \eta^l}\frac{\partial \eta^l}{\partial x'^j} dx'^j - \left\{\frac{\partial \upsilon_i}{\partial x^j} - \frac{\partial \upsilon_i}{\partial \eta^l} g^{kl}\left(\frac{\partial \upsilon_k}{\partial x^j} - \upsilon_m \Gamma^m_{(j)(k)} - 2\frac{\partial \upsilon_h}{\partial \eta^k}\Gamma^h_{(m)(j)}\eta^m\right)\right\} dx^j$$

$$= \left\{-\frac{\partial \upsilon_i}{\partial \eta^l}\frac{\partial \eta^l}{\partial x'^m}\Gamma^m_{(j)} + \frac{\partial \upsilon_i}{\partial \eta^l} g^{kl}\left(\frac{\partial \upsilon_k}{\partial x^j} - \upsilon_m \Gamma^m_{(j)(k)} - 2\frac{\partial \upsilon_h}{\partial \eta^k}\Gamma^h_{(m)(j)}\eta^m\right)\right\} dx^j + \frac{\partial \upsilon_i}{\partial \eta^l}\frac{\partial \eta^l}{\partial x'^j}\delta x'^j$$

$$\equiv \overset{W}{\nabla}_j \upsilon_i dx^j + \overset{W}{\nabla}'_j \upsilon_i \delta x'^j ,$$

where we put

$$(5.7) \qquad \begin{aligned} \overset{W}{\nabla}_j \upsilon_i &\equiv -\frac{\partial \upsilon_i}{\partial \eta^l}\frac{\partial \eta^l}{\partial x'^m}\Gamma^m_{(j)} + \frac{\partial \upsilon_i}{\partial \eta^l} g^{kl}\left(\frac{\partial \upsilon_k}{\partial x^j} - \upsilon_m \Gamma^m_{(j)(k)} - 2\frac{\partial \upsilon_h}{\partial \eta^k}\Gamma^h_{(m)(j)}\eta^m\right) , \\ \overset{W}{\nabla}'_j \upsilon_i &\equiv \frac{\partial \upsilon_i}{\partial \eta^l}\frac{\partial \eta^l}{\partial x'^j} . \end{aligned}$$

Put $\eta^i \equiv x'^i$, $\upsilon_i \equiv A_i$ and substitute them in (5.6) then we have

$$\overset{W}{\nabla}_j x'^i = -\Gamma^i_{(j)} + g^{ki}(A_{kj} - A_l \Gamma^l_{(j)(k)} - 2A_{l(k)}\Gamma^l_{(j)}) .$$

By the use of

$$\overset{C}{\nabla}_j A_k = A_{kj} - A_{k(l)}\Gamma^l_{(j)} - \Gamma^l_{(k)(j)} A_l ,$$

we have such other forms of the above expressions as

$$(5.8)\qquad \begin{aligned} \overset{W}{\nabla}_j x'^i &= -\Gamma^i_{(j)} + g^{ki}(\overset{C}{\nabla}_j A_k + \Gamma^l_{(j)} g_{kl}) = g^{ki}\overset{C}{\nabla}_j A_k\,, \\ \overset{W}{\nabla}'_j x'^i &= \delta^i_j = \overset{C}{\nabla}'_j x'^i\,. \end{aligned}$$

On the other hand, from (5.7) we are led to

$$(5.9)\qquad \begin{aligned} \overset{W}{\nabla}_j A_i &= -A_{i(l)}\Gamma^l_{(j)} + A_{i(l)} g^{kl}(A_{kj} - A_m \Gamma^m_{(j)(k)} - 2A_{h(k)}\Gamma^h_{(j)}) \\ &= -A_{i(l)}\Gamma^l_{(j)} + A_{i(l)} g^{kl}(\overset{C}{\nabla}_j A_k + \Gamma^h_{(j)} g_{kh}) \\ &= A_{i(l)} g^{kl}\overset{C}{\nabla}_j A_k\,, \\ \overset{W}{\nabla}'_j A_i &= A_{i(j)} = \overset{C}{\nabla}'_j A_i\,. \end{aligned}$$

The expressions (5.8) and (5.9) give rise to relations between KAWAGUCHI's derivatives and WIRTINGER's derivatives in the special KAWAGUCHI space, and it goes without saying that these relations coincide with those shown already in (4.5) and (4.6).

With the aid of

$$\delta^{(C)} x'^i = D^{(C)} x'^i - g^{ki}\overset{C}{\nabla}_j A_k$$

$D^{(C)}\eta^i$, $D^{(C)}\upsilon_i$ are expressed respectively by

$$(5.10)\qquad \begin{aligned} D^{(C)}\eta^i &= \overset{W}{\nabla}_j \eta^i dx^j + \overset{W}{\nabla}'_j \eta^i (D^{(C)} x'^j - g^{kj}\overset{C}{\nabla}_l A_k dx^l) \\ &= (\overset{W}{\nabla}_j \eta^i - \overset{W}{\nabla}'_l \eta^i g^{kl}\overset{C}{\nabla}_j A_k)\, dx^j + \overset{W}{\nabla}'_j \eta^i D^{(C)} x'^j\,, \end{aligned}$$

$$(5.11)\qquad \begin{aligned} D^{(C)}\upsilon_i &= \overset{W}{\nabla}_j \upsilon_i dx^j + \overset{W}{\nabla}'_j \upsilon_i (D^{(C)} x'^j - g^{kj}\overset{C}{\nabla}_l A_k dx^l) \\ &= (\overset{W}{\nabla}_j \upsilon_i - \overset{W}{\nabla}'_l \upsilon_i g^{kl}\overset{C}{\nabla}_j A_k)\, dx^j + \overset{W}{\nabla}'_j \upsilon_i D^{(C)} x'^j\,. \end{aligned}$$

The geometrical meaning of (5.10) and (5.11) is that if the WIRTINGER's differential $D^{(C)}x'^i$ is adopted as the base connection, the covariant derivatives are given by (5.10) and (5.11). Denoting covariant derivatives by symbols

$$\overset{W}{\tilde{\nabla}}_j \eta^i\,, \quad \overset{W}{\tilde{\nabla}}'_j \eta^i\,, \quad \overset{W}{\tilde{\nabla}}_j \upsilon_i\,, \quad \overset{W}{\tilde{\nabla}}'_j \upsilon_i\,,$$

we have

$$(5.12)\qquad \begin{aligned} \overset{W}{\tilde{\nabla}}_j \eta^i &= \overset{W}{\nabla}_j \eta^i - \overset{W}{\nabla}'_l \eta^i g^{kl}\overset{C}{\nabla}_j A_k\,, & \overset{W}{\tilde{\nabla}}'_j \eta^i &= \overset{W}{\nabla}'_j \eta^i \\ \overset{W}{\tilde{\nabla}}_j \upsilon_i &= \overset{W}{\nabla}_j \upsilon_i - \overset{W}{\nabla}'_l \upsilon_i g^{kl}\overset{C}{\nabla}_j A_k\,, & \overset{W}{\tilde{\nabla}}'_j \upsilon_i &= \overset{W}{\nabla}'_j \upsilon_i \end{aligned}$$

From (5.8), (5.9), (5.10) and (5.11) the relations among each kind of

covariant derivatives can be derived.
Now we put $\eta^i \equiv x'^i$, $\upsilon_i \equiv A_i$ in (5.12) and we have

$$(5.13)\qquad \begin{aligned} &\overset{W}{\tilde{\nabla}}_j x'^i = \overset{W}{\nabla}_j x'^i - \overset{W}{\nabla}'_i x'^i g^{kl} \overset{C}{\nabla}_j A_k = 0 , && \overset{W}{\tilde{\nabla}}'_j x'^i = \delta^i_j , \\ &\overset{W}{\tilde{\nabla}}_j A_i = \overset{W}{\nabla}_j A_i - \overset{W}{\nabla}'_i A_i g^{kl} \overset{C}{\nabla}_j A_k = 0 , && \overset{W}{\tilde{\nabla}}'_j A_i = A_{i(j)} . \end{aligned}$$

In the KAWAGUCHI's covariant differentials $\delta^{(C)}\eta^i$, $\delta^{(C)}\upsilon_i$ with the connection C we take the WIRTINGER's covariant differential $D^{(C)}x'^i$ in place of the base connection $\delta^{(C)}x'^i$. Then we have

$$\begin{aligned} \delta^{(C)}\eta^i &= \overset{C}{\nabla}_j \eta^i dx^j + \overset{C}{\nabla}'_j \eta^i \delta^{(C)} x'^j \\ &= \overset{C}{\nabla}_j \eta^i dx^j + \overset{C}{\nabla}'_j \eta^i (D^{(C)} x'^j - g^{kj} \overset{C}{\nabla}_l A_k dx^l) \\ &= \overset{C}{\nabla}_j \eta^i - \overset{C}{\nabla}'_l \eta^i g^{kl} \overset{C}{\nabla}_j A_k) dx^j + \overset{C}{\nabla}'_j \eta^i D^{(C)} x'^j , \\ \delta^{(C)}\upsilon_i &= \overset{C}{\nabla}_j \upsilon_i dx^j + \overset{C}{\nabla}'_j \upsilon_i \delta^{(C)} x'^j \\ &= \overset{C}{\nabla}_j \upsilon_i dx^j + \overset{C}{\nabla}'_j \upsilon_i (D^{(C)} x'^j - g^{kj} \overset{C}{\nabla}_l A_k dx^l) \\ &= \overset{C}{\nabla}_j \upsilon_i - \overset{C}{\nabla}'_l \upsilon_i g^{kl} \overset{C}{\nabla}_j A^k) dx^j + \overset{C}{\nabla}'_j \upsilon_i D^{(C)} x'^j . \end{aligned}$$

Putting

$$(5.14)\qquad \begin{aligned} \delta^{(C)}\eta^i &\equiv \overset{C}{\tilde{\nabla}}_j \eta^i dx^j + \overset{C}{\tilde{\nabla}}'_j \eta^i D^{(C)} x'^j , \\ \delta^{(C)}\upsilon_i &\equiv \overset{C}{\tilde{\nabla}}_j \upsilon_i dx^j + \overset{C}{\tilde{\nabla}}'_j \upsilon_i D^{(C)} x'^j , \end{aligned}$$

one obtains

$$(5.15)\qquad \begin{aligned} &\overset{C}{\tilde{\nabla}}_j \eta^i = \overset{C}{\nabla}_j \eta^i - \overset{C}{\nabla}'_l \eta^i g^{kl} \overset{C}{\nabla}_j A_k , && \overset{C}{\tilde{\nabla}}'_j \eta^i = \overset{C}{\nabla}'_j \eta^i , \\ &\overset{C}{\tilde{\nabla}}_j \upsilon_i = \overset{C}{\nabla}_j \upsilon_i - \overset{C}{\nabla}'_l \upsilon_i g^{kl} \overset{C}{\nabla}_j A_k , && \overset{C}{\tilde{\nabla}}'_j \upsilon_i = \overset{C}{\nabla}'_j \upsilon_i . \end{aligned}$$

Therefore, in the case of $\eta^i \equiv x'^i$ and $\upsilon_i \equiv A_i$ (5.15) becomes

$$(5.16)\qquad \begin{aligned} &\overset{C}{\tilde{\nabla}}_j x'^i = \overset{C}{\nabla}_j x'^i - \overset{C}{\nabla}'_l x'^i g^{kl} \overset{C}{\nabla}_j A_k = -g^{ki} \overset{C}{\nabla}_j A_k , \\ &\overset{C}{\tilde{\nabla}}'_j x'^i = \overset{C}{\nabla}'_j x'^i = \delta^i_j , \\ &\overset{C}{\tilde{\nabla}}_j A_i = \overset{C}{\nabla}_j A_i - A_{i(l)} g^{kl} \overset{C}{\nabla}_j A_k = (\delta^k_i - A_{i(l)} g^{kl}) \overset{C}{\nabla}_j A_k , \\ &\overset{C}{\tilde{\nabla}}'_j A_i = \overset{C}{\nabla}'_j A_i = A_{i(j)} . \end{aligned}$$

The above stated are relations between two kinds of covariant derivatives in the KAWAGUCHI space with the connection C.

When we take the connection C' in place of connection C, (3.9) or (3.20)′ gives rise to

$$
\begin{aligned}
D^{(C')}\eta^i &= d\eta^i + g^{ki}\Big\{\frac{\partial v_k}{\partial x^j} - (\Gamma^h_{(k)(j)} + C^h_{\cdot kl}\Gamma^l_{(j)})v_h - 2(\Gamma^l_{(h)(j)} + C^l_{\cdot hs}\Gamma^s_{(j)})\eta^h \\
&\qquad \times \frac{\partial v_l}{\partial \eta^k}\Big\} dx^j + g^{ki}\Big(-C^l_{\cdot kj}v_l - 2C^l_{\cdot sj}\eta^s\frac{\partial v_l}{\partial \eta^k}\Big) dx'^j \\
&= \overset{W'}{\nabla}_j \eta^i dx^j + \overset{W'}{\nabla}'_j \eta^i \delta x'^j ,
\end{aligned}
\tag{5.16)'}
$$

$$
\begin{aligned}
D^{(C')}v_i &= dv_i - \Big[\frac{\partial v_i}{\partial x^j} - \frac{\partial v_i}{\partial \eta^l} g^{kl}\Big\{\frac{\partial v_k}{\partial x^j} - (\Gamma^h_{(k)(j)} + C^h_{\cdot kl}\Gamma^l_{(j)})v_h \\
&\qquad - 2\Gamma^m_{(h)(j)} + C^m_{\cdot hs}\Gamma^s_{(j)})\eta^h\frac{\partial v_m}{\partial \eta^k}\Big\}\Big] dx^j \\
&\qquad + \frac{\partial v_i}{\partial \eta^l} g^{kl}\Big(-C^m_{\cdot kj}v_m - 2C^m_{\cdot sj}\eta^s\frac{\partial v_m}{\partial \eta^k}\Big) dx'^j \\
&= \overset{W'}{\nabla}_j v_i dx^j + \overset{W'}{\nabla}'_j v_i \delta x'^j ,
\end{aligned}
$$

where we put

$$
\begin{aligned}
\overset{W'}{\nabla}_j \eta^i &\equiv \frac{\partial \eta^i}{\partial x^j} + g^{ki}\Big\{\frac{\partial v_k}{\partial x^j} - (\Gamma^h_{(k)(j)} + C^h_{\cdot kl}\Gamma^l_{(j)})v_h - 2(C^l_{(h)(j)} + C^l_{\cdot hs}\Gamma^s_{(j)})\eta^h\frac{\partial v_l}{\partial \eta^k}\Big\} \\
&\qquad - \Big\{\frac{\partial \eta^i}{\partial x'^h} + g^{ki}\Big(-C^l_{\cdot kh}v_l - 2C^l_{\cdot sh}\eta^s\frac{\partial v_l}{\partial \eta^k}\Big)\Big\}\Gamma^h_{(j)} , \\
\overset{W'}{\nabla}'_j \eta^i &\equiv \frac{\partial \eta^i}{\partial x'^j} + g^{ki}\Big(-C^l_{\cdot kj}v_l - 2C^l_{\cdot sj}\eta^s\frac{\partial v_l}{\partial \eta^k}\Big) , \\
\overset{W'}{\nabla}_j v_i &\equiv \frac{\partial v_i}{\partial \eta^l} g^{kl}\Big\{\frac{\partial v_k}{\partial x^j} - (\Gamma^h_{(k)(j)} + C^h_{\cdot kl}\Gamma^l_{(j)})v_h - 2(\Gamma^m_{(h)(j)} + C^m_{\cdot hs}\Gamma^s_{(j)})\eta^h\frac{\partial v_m}{\partial \eta^k}\Big\} \\
&\qquad - \Big\{\frac{\partial v_i}{\partial x'^h} + \frac{\partial v_i}{\partial \eta^l} g^{kl}\Big(-C^m_{\cdot kh}v_m - 2C^m_{\cdot sh}\eta^s\frac{\partial v_m}{\partial \eta^k}\Big)\Big\}\Gamma^h_{(j)} , \\
\overset{W'}{\nabla}'_j v_i &\equiv \frac{\partial v_i}{\partial x'^j} + \frac{\partial v_i}{\partial \eta^l} g^{kl}\Big(-C^m_{\cdot kj}v_m - 2C^m_{\cdot sj}\eta^s\frac{\partial v_m}{\partial \eta^k}\Big) .
\end{aligned}
$$

Taking $\eta^i \equiv x'^i$, $v_i \equiv A_i$ in the above expressions we have

$$
\begin{aligned}
\overset{W'}{\nabla}_j x'^i &= g^{ki}\{A_{kj} - (\Gamma^h_{(k)(j)} + C^h_{\cdot kl}\Gamma^l_{(j)})A_h - 2(\Gamma^l_{(h)(j)} + C^l_{\cdot hs}\Gamma^s_{(j)})x'^h A_{l(k)}\} \\
&\qquad - \{\delta^i_h + g^{ki}(-C^l_{\cdot kh}A_l - 2C^l_{\cdot sh}x'^s A_{l(k)})\}\Gamma^h_{(j)} \\
&= g^{ki}\{A_{kj} - (\Gamma^h_{(k)(j)} + C^h_{\cdot kl}\Gamma^l_{(j)})A_h - 2\Gamma^l_{(j)}A_{l(k)}\} - \Gamma^i_{(j)} + g^{ki}C^l_{\cdot kh}A_l\Gamma^l_{(j)} \\
&= g^{ki}(A_{kj} - A_h\Gamma^h_{(k)(j)} - 2\Gamma^l_{(j)}A_{l(k)} - A_h C^h_{\cdot kl}\Gamma^l_{(j)}) - \Gamma^i_{(j)} + g^{ki}C^l_{\cdot kh}A_l\Gamma^h_{(j)}
\end{aligned}
$$

$$= g^{ki}\overset{C'}{\nabla}_j A_k + \Gamma^i_{(j)} - g^{ki}A_h C^h_{\cdot kl}\Gamma^l_{(j)} - \Gamma^i_{(j)} + g^{ki}C^l_{\cdot kh}A_l\Gamma^h_{(j)}$$

$$= g^{ki}\overset{C'}{\nabla}_j A_k ,$$

$$\overset{W'}{\nabla}_j x'^i = \delta^i_j + g^{ki}(-C^l_{\cdot kj}A_l - 2C^l_{\cdot sj}x'^s A_{l(k)}) = \delta^i_j - g^{ki}C^l_{\cdot kj}A_l ,$$

$$\begin{aligned}\overset{W'}{\nabla}_j A_i &= A_{i(l)}g^{kl}\{A_{kj} - (\Gamma^h_{(k)(j)} + C^h_{\cdot kl}\Gamma^l_{(j)})A_h - 2(\Gamma^m_{(h)(j)} + C^m_{\cdot hs}\Gamma^s_{(j)})x'^h A_{m(k)}\} \\ &\quad - \{A_{i(h)} + A_{i(l)}g^{kl}(-C^m_{\cdot kh}A_m - 2C^m_{\cdot sh}x'^s A_{m(k)})\}\Gamma^h_{(j)} \\ &= A_{i(l)}g^{kl}(A_{kj} - A_h\Gamma^h_{(k)(j)} - 2A_{m(k)}\Gamma^m_{(j)}) - A_{i(h)}\Gamma^h_{(j)} \\ &= A_{i(l)}g^{kl}(\overset{C'}{\nabla}_j A_k + A_{k(h)}\Gamma^h_{(j)} - 2A_{h(k)}\Gamma^h_{(j)}) - A_{i(h)}\Gamma^h_{(j)} = A_{i(l)}g^{kl}\overset{C'}{\nabla}_j A_k ,\end{aligned}$$

$$\begin{aligned}\overset{W'}{\nabla}_j A_i &= A_{i(j)} - A_{i(l)}g^{kl}C^m_{\cdot kj}A_m = A_{i(l)}(\delta^l_j - g^{kl}C^m_{\cdot kj}A_m) \\ &= A_{i(l)}\{\delta^l_j - g^{kl}(A_{k(j)} - \overset{C'}{\nabla}_j A_k)\} = A_{i(l)}(\delta^l_j - g^{kl}A_{k(j)}) + A_{i(l)}g^{kl}\overset{C'}{\nabla}_j A_k .\end{aligned}$$

Summarizing these results we get

$$(5.17)\qquad \begin{aligned} &\overset{W'}{\nabla}_j x'^i = g^{ki}\overset{C'}{\nabla}_j A_k , && \overset{W'}{\nabla}_j x'^i = \delta^i_j - g^{ki}C^l_{\cdot kj}A_l , \\ &\overset{W'}{\nabla}_j A_i = A_{i(l)}g^{kl}\overset{C'}{\nabla}_j A_k , && \overset{W'}{\nabla}_j A_i = A_{i(l)}(\delta^l_j - g^{kl}A_{k(j)} + g^{kl}\overset{C'}{\nabla}_j A_k) . \end{aligned}$$

(5.17) yields at once

$$\overset{W'}{\nabla}_j A_i = A_{i(l)}g^{kl}\overset{C'}{\nabla}_j A_k = A_{i(l)}\overset{W'}{\nabla}_j x'^l ,$$

and similarly we have

$$\overset{W'}{\nabla}_j x'^i = \overset{C'}{\nabla}_j x'^i - g^{ki}A_{k(j)} + g^{ki}\overset{C'}{\nabla}_j A_k .$$

Accordingly one obtains

$$\begin{aligned}\overset{W'}{\nabla}_j A_i &= A_{k(j)}(\delta^k_i - A_{i(l)}g^{kl}) + A_{i(l)}g^{kl}\overset{C'}{\nabla}_j A_k \\ &= A_{k(j)}(\delta^k_i - A_{i(l)}g^{kl}) + A_{i(l)}(\overset{W'}{\nabla}_j x'^l - \overset{C'}{\nabla}_j x'^l + g^{kl}A_{k(j)}) \\ &= A_{i(l)}\overset{W'}{\nabla}_j x'^l + A_{i(l)}(\delta^l_j - \overset{C'}{\nabla}_j x'^l) = A_{i(l)}\overset{W'}{\nabla}_j x'^l ,\end{aligned}$$

namely there exists

$$(5.18)\qquad \overset{W'}{\nabla}_j A_i = A_{i(k)}\overset{W'}{\nabla}_j x'^k , \qquad \overset{W'}{\nabla}_j A_i = A_{i(k)}\overset{W'}{\nabla}_j x'^k .$$

Similar relations appear in the case of the connection C making use of (5.8) and (5.9), that is

$$(5.19)\qquad \overset{W}{\nabla}_j A_i = A_{i(k)}\overset{W}{\nabla}_j x'^k , \qquad \overset{W}{\nabla}_j A_i = A_{i(k)}\overset{W}{\nabla}_j x'^k .$$

Moreover the following relations

$$(5.20)\qquad \overset{W}{\tilde{\nabla}}_j A_i = A_{(i(k)}\overset{W}{\tilde{\nabla}}_j x'^k\,, \qquad \overset{W}{\tilde{\nabla}}'_j A_i = A_{i(k)}\overset{W}{\tilde{\nabla}}'_j x'^k$$

come into existence, but in fact $\overset{W}{\tilde{\nabla}}_j A_i$ and $\overset{W}{\tilde{\nabla}}_j x'^i$ are both zero. Apparently (5.17) corresponds to the WIRTINGER's covariant derivatives given by (5.8) and (5.9).

Now we are going to define covariant derivatives when we take the WIRTINGER's differential in place of the base connection $\delta^{(C')}x'^i$. From (3.24) we have

$$\begin{aligned} D^{(C')}x'^i &= \delta x'^i + g^{ji}(\nabla_k A_j dx^k - A_k C^k_{\cdot jl}\delta x'^l) \\ &= (\delta^i_l - g^{ji}A_k C^k_{\cdot jl})\,\delta x'^l + g^{ji}\nabla_k A_j dx^k = \Lambda^i_l \delta x'^l + g^{ji}\nabla_l A_j dx^l\,, \end{aligned}$$

where we put

$$\begin{aligned} \Lambda^i_l &\equiv \delta^i_l - g^{ji}A_k C^k_{\cdot jl}\,, \\ \delta^{(C)}x'^i &= \delta^{(C')}x'^i \equiv \delta x'^i\,, \\ \overset{C}{\nabla}_k A_j &= \overset{C'}{\nabla}_k A_j \equiv \nabla_k A_j\,. \end{aligned}$$

Since the determinant $|\Lambda^i_l|$ does not vanish in general, there exists the conjugate tensor Δ^j_i, and $\Lambda^i_l \Delta^j_i = \delta^j_l$ holds good.
Thus we have

$$(5.20)'\qquad \delta x'^i = \Delta^i_k D^{(C')}x'^k - \Delta^i_k g^{jk}\nabla_l A_j dx^l\,,$$

and (5.16)′ brings into existence

$$\begin{aligned} D^{(C')}\eta^i &= \overset{W'}{\nabla}_j \eta^i dx^j + \overset{W'}{\nabla}'_j \eta^i \delta x'^j = \overset{W'}{\nabla}_j \eta^i dx^j + \overset{W'}{\nabla}'_j \eta^i (\Delta^j_k D^{(C')}x'^k - \Delta^j_k g^{lk}\nabla_m A_l dx^m) \\ &= (\overset{W'}{\nabla}_j \eta^i - \overset{W'}{\nabla}'_h \eta^i \Delta^h_k g^{lk}\nabla_j A_l)\, dx^j + \overset{W'}{\nabla}'_k \eta^i \Delta^k_j D^{(C')}x'^j \\ &= \overset{W'}{\tilde{\nabla}}_j \eta^i dx^j + \overset{W'}{\tilde{\nabla}}'_j \eta^i D^{(C')}x'^j\,, \\ D^{(C')}\upsilon_i &= \overset{W'}{\nabla}_j \upsilon_i dx^j + \overset{W'}{\nabla}'_j \upsilon_i \delta x'^j = \overset{W'}{\nabla}_j \upsilon_i dx^j + \overset{W'}{\nabla}'_j \upsilon_i (\Delta^j_k D^{(C')}x'^k - \Delta^j_k g^{lk}\nabla_m A_l dx^m) \\ &= (\overset{W'}{\nabla}_j \upsilon_i - \overset{W'}{\nabla}'_h \upsilon_i \Delta^h_k g^{lk}\nabla_j A_l)\, dx^j + \overset{W'}{\nabla}_k \upsilon_i \Delta^k_j D^{(C')}x'^j \\ &= \overset{W'}{\tilde{\nabla}}_j \upsilon_i dx^j + \overset{W'}{\tilde{\nabla}}'_j \upsilon_i D^{(C')}x'^j\,, \end{aligned}$$

where we put

$$(5.21)\qquad \begin{aligned} \overset{W'}{\tilde{\nabla}}_j \eta^i &\equiv \overset{W'}{\nabla}_j \eta^i - \overset{W'}{\nabla}'_j \eta^i \Delta^h_k g^{lk}\nabla_j A_l\,, & \overset{W'}{\tilde{\nabla}}'_j \eta^i &\equiv \overset{W'}{\nabla}'_k \eta^i \Delta^k_j\,, \\ \overset{W'}{\tilde{\nabla}}_j \upsilon_i &\equiv \overset{W'}{\nabla}_j \upsilon_i - \overset{W'}{\nabla}'_h \upsilon_i \Delta^h_k \nabla_j A_l\,, & \overset{W'}{\tilde{\nabla}}'_j \upsilon_i &\equiv \overset{W'}{\nabla}'_k \upsilon_i \Delta^k_j\,. \end{aligned}$$

Making use of

$$\Delta^h_k g^{lk} \nabla_j A_l \equiv W^h_{\cdot j}$$

in (5.21), then we are led to

$$\overset{W'}{\tilde{\nabla}}_j x'^i = \overset{W'}{\nabla}_j x'^i - W^h_{\cdot j} \overset{W'}{\nabla}'_h x'^i , \qquad \overset{W'}{\tilde{\nabla}}'_j x'^i = \Delta^k_j \overset{W'}{\nabla}'_k x'^i ,$$

$$\overset{W'}{\nabla}_j A_i = \overset{W'}{\nabla}_j A_i - W^h_{\cdot j} \overset{W'}{\nabla}'_h A_i , \qquad \overset{W'}{\tilde{\nabla}}'_j A_i = \Delta^k_j \overset{W'}{\nabla}'_k A_i ,$$

therefore in consideration of (5.17) the relations between them and KAWAGUCHI's covariant derivatives are expressed by

$$(5.22) \qquad \begin{aligned} \overset{W'}{\tilde{\nabla}}_j x'^i &= g^{ki} \nabla_j A_k - W^h_{\cdot j} (\delta^i_h - g^{ki} C^l_{\cdot kh} A_l) , \\ \overset{W'}{\tilde{\nabla}}'_j x'^i &= \Delta^k_j (\delta^i_k - g^{hi} C^l_{\cdot hk} A_l) , \\ \overset{W'}{\tilde{\nabla}}_j A_i &= A_{i(l)} g^{kl} \nabla_j A_k - W^h_{\cdot j} A_{i(l)} (\delta^l_h - g^{kl} A_{k(h)} + g^{kl} \overset{C'}{\nabla}'_h A_k) , \\ \overset{W'}{\tilde{\nabla}}'_j A_i &= \Delta^h_j A_{i(l)} (\delta^l_h - g^{kl} A_{k(h)} + g^{kl} \overset{C'}{\nabla}'_h A_k) . \end{aligned}$$

The covariant derivatives which correspond to (5.16) are obtained with the aid of (5.2)′ and (5.20)′, for which we put

$$\delta^{(C')} \eta^i = \overset{C'}{\nabla}_j \eta^i dx^j + \overset{C'}{\nabla}'_j \eta^i \delta x'^j = \overset{C'}{\tilde{\nabla}}_j \eta^i dx^j + \overset{C'}{\tilde{\nabla}}'_j \eta^i D^{(C')} x'^j ,$$

$$\delta^{(C')} \upsilon_i = \overset{C'}{\nabla}_j \upsilon_i dx^j + \overset{C'}{\nabla}'_j \upsilon_i \delta x'^j = \overset{C'}{\tilde{\nabla}}_j \upsilon_i dx^j + \overset{C'}{\tilde{\nabla}}'_j \upsilon_i D^{(C')} x'^j ,$$

where

$$\overset{C'}{\tilde{\nabla}}_j \eta^i \equiv \overset{C'}{\nabla}_j \eta^i - \overset{C'}{\nabla}'_l \eta^i \Delta^l_k g^{hk} \nabla_j A_h , \qquad \overset{C'}{\tilde{\nabla}}'_j \eta^i \equiv \overset{C'}{\nabla}'_k \eta^i \Delta^k_j ,$$

$$\overset{C'}{\tilde{\nabla}}_j \eta^i \equiv \overset{C'}{\nabla}_j \upsilon_i - \overset{C'}{\nabla}'_l \upsilon_i \Delta^l_k g^{hk} \nabla_j A_h , \qquad \overset{C'}{\tilde{\nabla}}'_j \upsilon_i \equiv \overset{C'}{\nabla}'_k \upsilon_i \Delta^k_j .$$

Accordingly we have at once

$$(5.23) \qquad \begin{aligned} \overset{C'}{\tilde{\nabla}}_j x'^i &= \overset{C'}{\nabla}_j x'^i - \overset{C'}{\nabla}'_l x'^i W^l_{\cdot j} = -W^i_{\cdot j} , & \overset{C'}{\tilde{\nabla}}'_j x'^i &= \overset{C'}{\nabla}'_k x'^i \Delta^k_j = \Delta^i_j , \\ \overset{C'}{\tilde{\nabla}}_j A_i &= \overset{C'}{\nabla}_j A_i - \overset{C'}{\nabla}'_l A_i W^l_{\cdot j} , & \overset{C'}{\tilde{\nabla}}'_j A_i &= \overset{C'}{\nabla}'_k A_i \Delta^k_j . \end{aligned}$$

The following relations can be easily derived from (5.23)

$$(5.24) \qquad \overset{C'}{\tilde{\nabla}}_j A_i = \overset{C'}{\nabla}_j A_i + \overset{C'}{\nabla}'_l A_i \overset{C'}{\tilde{\nabla}}_j x'^l , \qquad \overset{C'}{\tilde{\nabla}}'_j A_i = \overset{C'}{\nabla}'_k A_i \overset{C'}{\tilde{\nabla}}'_j x'^k ,$$

where (5.24) corresponds to (5.18) or (5.19).

Various kinds of covariant derivatives are defined in the special KAWAGUCHI space, but it is possible to define other covariant derivatives which are not involved in stated cases. For examle, we can define a

covariant differential taking the WIRTINGER's differential as the base connection

$$(5.25)\qquad \begin{aligned} D^{(W)}v^i &= dv^i + \Gamma^i_{(j)(k)} v^j dx^k + C^i_{\cdot jk} v^j D^{(C')} x'^k \,, \\ D^{(W)}v_i &= dv_i - \Gamma^j_{(i)(k)} v_j dx^k - C^j_{\cdot ik} v_j D^{(C')} x'^k \,, \end{aligned}$$

so we have

$$(5.25)'\qquad D^{(W)}v^i = \frac{\partial v^i}{\partial x^k} dx^k + \frac{\partial v^i}{\partial x'^k} dx'^k + \Gamma^i_{(j)(k)} v^j dx^k + C^i_{\cdot jk} v^j D^{(C')} x'^k \,.$$

On the other hand, multiplyng

$$D^{(C')} x'^i = \Delta^i_j \delta x'^j + g^{ki} \nabla_l A_k dx^l \,, \quad \text{by } \Delta^h_i$$

we have

$$dx'^h = \Delta^h_j D^{(C')} x'^j - (\Gamma^h_{(k)} + g^{lj} \Delta^h_j \nabla_k A_l)\, dx^k \,,$$

by the use of

$$\Delta^h_i D^{(C')} x'^i = \delta x'^h + g^{ki} \Delta^h_i \nabla_l A_k dx^l = dx'^h + \Gamma^h_{(l)} dx^l + g^{ki} \Delta^h_i \nabla_l A_k dx^l \,.$$

Substituting the above result in (5.25)′ and putting in order, then

$$\begin{aligned} D^{(W)} v^i = &\left\{ \frac{\partial v^i}{\partial x^k} + \Gamma^i_{(j)(k)} v^j - \frac{\partial v^i}{\partial x'^h} (\Gamma^h_{(k)} + g^{lj} \Delta^h_j \nabla_k A_l) \right\} dx^k \\ &+ \left(\frac{\partial v^i}{\partial x'^j} \Delta^j_k + C^i_{\cdot jk} v^j \right) D^{(C')} x'^k \end{aligned}$$

is obtained.

Then, after some calculations we have

$$\begin{aligned} D^{(W)} v_i &= \frac{\partial v_i}{\partial x^k} dx^k + \frac{\partial v_i}{\partial x'^h} dx'^h - \Gamma^j_{(i)(k)} v_j dx^k - C^j_{\cdot ik} v_j D^{(C')} x'^k \\ &= \left\{ \frac{\partial v_i}{\partial x^k} - \Gamma^j_{(i)(k)} v_j - \frac{\partial v_i}{\partial x'^h} (\Gamma^h_{(k)} + g^{lj} \Delta^h_j \nabla_k A_l) \right\} dx^k \\ &\quad + \left(\frac{\partial v_i}{\partial x'^j} \Delta^j_k - C^j_{\cdot ik} v_j \right) D^{(C')} x'^k \,. \end{aligned}$$

After substitution of

$$D^{(C')} x'^i = \Delta^i_l \delta x'^l + g^{ji} \nabla_l A_j dx^l \,, \qquad dx'^i = \delta x'^i - \Gamma^i_{(j)} dx^j \,,$$

in (5.25) we get

$$\begin{aligned} D^{(W)} v^i &= \frac{\partial v^i}{\partial x^j} dx^j + \frac{\partial v^i}{\partial x'^j} dx'^j + \Gamma^i_{(j)(k)} v^j dx^k + C^i_{\cdot jk} v^j D^{(C')} x'^k \\ &= \frac{\partial v^i}{\partial x^j} dx^j + \frac{\partial v^i}{\partial x'^j} (\delta x'^j - \Gamma^j_{(l)} dx^l) + \Gamma^i_{(j)(k)} v^j dx^k \end{aligned}$$

$$+ C^i_{\cdot jk} v^j (\Lambda^k_l \delta x'^l + g^{jk} \nabla_l A_j dx^l)$$

$$= \left(\frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^l} \Gamma^l_{(j)} + \Gamma^i_{(k)(j)} v^k + C^i_{\cdot lk} v^l g^{hk} \nabla_j A_h \right) dx^j$$

$$+ \left(\frac{\partial v^i}{\partial x'^j} + \Lambda^k_j C^i_{\cdot lk} v^l \right) \delta x'^j$$

$$= (\nabla_j v^i + C^i_{\cdot lk} v^l g^{hk} \nabla_j A_h) dx^j + \left(\frac{\partial v^i}{\partial x'^j} + \Lambda^k_j C^l_{\cdot ik} v_l \right) \delta x'^j ,$$

similarly

$$D^{(W)} v_i = (\nabla_j v_i - C^l_{\cdot ik} v_l g^{hk} \nabla_j A_h) dx^j + \left(\frac{\partial v_i}{\partial x'^j} - \Lambda^k_j C^l_{\cdot ik} v_l \right) \delta x'^j$$

is obtained.

The covariant derivatives in this case can be defined as usual, namely

$$D^{(W)} v^i = \overset{*}{\nabla}_j v^i dx^j + \overset{*}{\nabla}'_j v^i \delta x'^j = \overset{*}{\nabla}_j v^i dx^j + \overset{*}{\tilde{\nabla}}'_j v^i D^{(C')} x'^j ,$$

$$D^{(W)} v_i = \overset{*}{\nabla}_j v_i dx^j + \overset{*}{\nabla}'_j v_i \delta x'^j = \overset{*}{\tilde{\nabla}}_j v_i dx^j + \overset{*}{\tilde{\nabla}}'_j v_i D^{(C')} x'^j ,$$

where we put

$$\overset{*}{\nabla}_j v^i \equiv \nabla_j v^i + C^i_{\cdot lk} v^l g^{hk} \nabla_j A_h , \qquad \overset{*}{\nabla}'_j v^i \equiv \frac{\partial v^i}{\partial x'^j} + \Lambda^k_j C^i_{\cdot lk} v^l ,$$

$$\overset{*}{\nabla}'_j v_i \equiv \nabla_j v_i - C^l_{\cdot ik} v_l g^{hk} \nabla_j A_h , \qquad \overset{*}{\nabla}'_j v_i \equiv \frac{\partial v_i}{\partial x'^j} - \Lambda^k_j C^l_{\cdot ik} v_l ,$$

$$\overset{*}{\tilde{\nabla}}_j v^i \equiv \nabla_j v^i - \frac{\partial v^i}{\partial x'^h} g^{lm} \Delta^h_m \nabla_j A_l , \qquad \overset{*}{\tilde{\nabla}}'_j v^i \equiv \frac{\partial v^i}{\partial x'^k} \Delta^k_j + C^i_{\cdot kj} v^k ,$$

$$\overset{*}{\tilde{\nabla}} v_i \equiv \nabla_j v_i - \frac{\partial v_i}{\partial x'^h} g^{lm} \Delta^h_m \nabla_j A_l , \qquad \overset{*}{\nabla}'_j v_i \equiv \frac{\partial v_i}{\partial x'^k} \Delta^k_j - C^k_{\cdot ij} v_k .$$

Taking $v^i \equiv x'^i$, $v_i \equiv A_i$ in the above expressions we have

$$\overset{*}{\nabla}_j x'^i = \nabla_j x'^i = 0 , \qquad \overset{*}{\nabla}'_j x'^i = \delta^i_j ,$$

$$\overset{*}{\nabla}_j A_i = \nabla_j A_i - C^l_{\cdot ik} A_l g^{hk} \nabla_j A_h , \qquad \overset{*}{\nabla}'_j A_i = A_{i(j)} - \Lambda^k_j C^l_{\cdot ik} A_l ,$$

$$\overset{*}{\tilde{\nabla}}_j x'^i = -g^{lm} \Delta^i_m \nabla_j A_l = -W^i_{\cdot j} , \qquad \overset{*}{\tilde{\nabla}}'_j x'^i = \Delta^i_j ,$$

$$\overset{*}{\tilde{\nabla}}_j A_i = \nabla_j A_i - A_{i(h)} g^{lm} \Delta^h_m \nabla_j A_l \qquad \overset{*}{\tilde{\nabla}}'_j A_i = A_{i(k)} \Delta^k_j - C^k_{\cdot ij} A_k .$$

$$= \nabla_j A_i - A_{i(h)} W^h_{\cdot j} ,$$

Various covariant derivatives related to the special WIRTINGER's connections in the KAWAGUCHI space have been defined and it goes without saying that various tensors or identities should be derived corresponding to these covariant derivatives.

§ 6. Curvature tensors and identities in a special KAWAGUCHI space

Of the special KAWAGUCHI space we have discussed in §5 concerning its proper covariant derivatives given by KAWAGUCHI and other covariant derivatives based on the WIRTINGER's connection. Also we have treated the relation between them.

The covariant derivatives in the special KAWAGUCHI space introduced with the connection C are expressed by

$$\overset{C}{\nabla}_j v^i = \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^l}\Gamma^l_{(j)} + \Gamma^i_{(k)(j)} v^k , \qquad \overset{C}{\nabla}'_j v^i = \frac{\partial v^i}{\partial x'^j} ,$$

$$\overset{C}{\nabla}_j v_i = \frac{\partial v_i}{\partial x^j} - \frac{\partial v_i}{\partial x'^l}\Gamma^l_{(j)} - \Gamma^k_{(i)(j)} v_k , \qquad \overset{C}{\nabla}'_j v_i = \frac{\partial v_i}{\partial x'^j} ,$$

which are shown already in (5.1) and (5.2).

From the parenthesis of POISSON for the covariant derivatives, we find

$$\begin{aligned} (\overset{C}{\nabla}_j \overset{C}{\nabla}_k - \overset{C}{\nabla}_k \overset{C}{\nabla}_j) v^i &= -\overset{C}{R}{}^{\cdot\cdot\cdot i}_{jkl} v^l + \overset{C}{K}{}^{\cdot\cdot l}_{jk} \overset{C}{\nabla}'_l v^i , \\ (\overset{C}{\nabla}_j \overset{C}{\nabla}'_k - \overset{C}{\nabla}'_k \overset{C}{\nabla}_j) v^i &= -\overset{C}{B}{}^{\cdot\cdot\cdot i}_{jkl} v^l , \end{aligned} \tag{6.1}$$

where

$$\overset{C}{B}{}^{\cdot\cdot\cdot i}_{jkl} \equiv \Gamma^i_{(j)(k)(l)} ,$$

$$\overset{C}{R}{}^{\cdot\cdot\cdot i}_{jkl} \equiv \frac{\partial \Gamma^i_{(l)(j)}}{\partial x^k} - \frac{\partial \Gamma^i_{(l)(k)}}{\partial x^j} + \Gamma^h_{(l)(j)}\Gamma^i_{(k)(h)} - \Gamma^h_{(l)(k)}\Gamma^i_{(j)(h)} + \Gamma^h_{(j)}\Gamma^i_{(l)(k)(h)} - \Gamma^h_{(k)}\Gamma^i_{(l)(j)(h)} ,$$

$$\overset{C}{K}{}^{\cdot\cdot i}_{jk} \equiv \frac{\partial \Gamma^i_{(j)}}{\partial x^k} - \frac{\partial \Gamma^i_{(k)}}{\partial x^j} + \Gamma^h_{(j)}\Gamma^i_{(k)(h)} - \Gamma^h_{(k)}\Gamma^i_{(j)(h)} .$$

are all curvature tensors in our space.

The identities among these curvature tensors can be found easily as follows:

$$
(6.2)\qquad
\begin{aligned}
&\overset{C}{R}{}_{jkl}^{\cdots i} + \overset{C}{R}{}_{kjl}^{\cdots i} = 0 \,, \\
&\overset{C}{R}{}_{jkl}^{\cdots i} + \overset{C}{R}{}_{klj}^{\cdots i} + \overset{C}{R}{}_{ljk}^{\cdots i} = 0 \,, \\
&\overset{C}{R}{}_{[jkl]}^{\cdots i} = 0 \,, \\
&\overset{C}{K}{}_{jk}^{\cdot\cdot i} = \overset{C}{R}{}_{jkl}^{\cdots i} x'^{l} \,, \\
&\overset{C}{B}{}_{jkl}^{\cdots i} x'^{l} = 0 \,, \\
&\overset{C}{R}{}_{jkl}^{\cdots i} = \overset{C}{K}{}_{jk\cdot(l)}^{\cdot\cdot i} = \overset{C}{\nabla}{}'_{l} \overset{C}{K}{}_{jk}^{\cdot\cdot i} \,, \\
&\overset{C}{K}{}_{jk}^{\cdot\cdot i} + \overset{C}{K}{}_{kj}^{\cdot\cdot i} = 0 \qquad \text{etc.}\,.
\end{aligned}
$$

These curvature tensors

$$
\overset{C}{R}{}_{jkl}^{\cdots i}\,, \qquad \overset{C}{B}{}_{jkl}^{\cdots i}\,, \qquad \overset{C}{K}{}_{jk}^{\cdot\cdot i}
$$

are all found in the equations of structure of the connection C

$$
\begin{aligned}
&\Omega^{i} \equiv [dx^{j}\,\omega_{j}^{i}] = 0 \,, \\
&\tilde{\Omega}^{i} \equiv (\omega^{i})' + [\omega_{j}^{i}\,\omega^{j}] = -\frac{1}{2}\,\overset{C}{K}{}_{jk}^{\cdot\cdot i}\,[dx^{j}\,dx^{k}] \,, \\
&\Omega_{j}^{i} \equiv (\omega_{j}^{i})' + [\omega_{\cdot k}^{i}\,\omega_{\cdot j}^{k}] = -\frac{1}{2}\,\overset{C}{R}{}_{klj}^{\cdots i}\,[dx^{k}\,dx^{l}] - \overset{C}{B}{}_{jkl}^{\cdots i}\,[dx^{k}\,\omega^{l}] \,,
\end{aligned}
$$

when we consider the following differential forms

$$
dx^{i}\,, \quad \omega^{i} = dx'^{i} + \Gamma^{i}_{(j)}\,dx^{j}\,, \quad \omega_{i}^{j} = \Gamma^{j}_{(i)(k)}\,dx^{k}\,.
$$

About the connection C' it is the same as in the case of connection C and from (5.4), the covariant derivatives

$$
\begin{aligned}
&\overset{C'}{\nabla}{}_{j} v^{i} = \overset{C}{\nabla}{}_{j} v^{i} \,, \\
&\overset{C'}{\nabla}{}'_{j} v^{i} = \overset{C}{\nabla}{}'_{j} v^{i} + C^{i}_{\cdot kj}\,v^{k} \,, \\
&\overset{C'}{\nabla}{}_{j} v_{i} = \overset{C}{\nabla}{}_{j} v_{i} \,, \\
&\overset{C'}{\nabla}{}'_{j} v_{i} = \overset{C}{\nabla}{}'_{j} v_{i} - C^{k}_{\cdot ij}\,v_{k}
\end{aligned}
$$

are derived.

The parentheses of Poisson are

$$
\begin{aligned}
&(\overset{C'}{\nabla}{}_{j}\overset{C'}{\nabla}{}_{k} - \overset{C'}{\nabla}{}_{k}\overset{C'}{\nabla}{}_{j})\,v^{i} = -\overset{C'}{R}{}_{jkl}^{\cdots i}\,v^{l} + \overset{C'}{K}{}_{jk}^{\cdot\cdot l}\,\overset{C'}{\nabla}{}'_{l} v^{i} \,, \\
&(\overset{C'}{\nabla}{}_{j}\overset{C'}{\nabla}{}'_{k} - \overset{C'}{\nabla}{}'_{k}\overset{C'}{\nabla}{}_{j})\,v^{i} = -\overset{C'}{B}{}_{jkl}^{\cdots i}\,v^{l} + C^{l}_{\cdot jk}\,\nabla_{l} v^{i} \,,
\end{aligned}
$$

$$(\overset{C'}{\nabla}{}'_j \overset{C'}{\nabla}{}'_k - \overset{C'}{\nabla}{}'_k \overset{C'}{\nabla}{}'_j)\, v^i = -\overset{C'}{P}{}^{\cdots i}_{jkl} v^l - 2C^h_{\cdot[kj]} \overset{C'}{\nabla}{}'_h v^i ,$$

where

$$\overset{C'}{R}{}^{\cdots i}_{jkl} = \overset{C}{R}{}^{\cdots i}_{jkl} + \overset{C}{K}{}^{\cdot\cdot h}_{jk} C^i_{\cdot lh} ,$$

$$\overset{C'}{B}{}^{\cdots i}_{jkl} = \Gamma^i_{(l)(j)(k)} + C^i_{\cdot hk}\Gamma^h_{(l)(j)} - \frac{\partial C^i_{\cdot lk}}{\partial x^j} + C^i_{\cdot lk(h)}\Gamma^h_{(j)} - \Gamma^i_{(h)(j)} C^h_{\cdot lk} + \Gamma^h_{(j)(k)} C^i_{\cdot lh}$$

$$= \Gamma^i_{(l)(j)(k)} - \overset{C'}{\nabla}_j C^i_{\cdot lk} ,$$

$$\overset{C'}{P}{}^{\cdots i}_{jkl} = C^i_{\cdot lj(k)} - C^i_{\cdot lk(j)} + C^i_{\cdot hk} C^h_{\cdot lj} - C^i_{\cdot hj} C^h_{\cdot lk}$$

are curvature tensors and identities and

$$\begin{aligned}
&\overset{C'}{R}{}^{\cdots i}_{jkl} + \overset{C'}{R}{}^{\cdots i}_{kjl} = 0 , \\
&\overset{C'}{R}{}^{\cdots i}_{jkl} + \overset{C'}{R}{}^{\cdots i}_{kij} + \overset{C'}{R}{}^{\cdots i}_{ijk} = \overset{C}{K}{}^{\cdot\cdot h}_{jk} C^i_{\cdot lh} + \overset{C}{K}{}^{\cdot\cdot h}_{kl} C^i_{\cdot jh} + \overset{C}{K}{}^{\cdot\cdot h}_{ij} C^i_{\cdot kh} , \\
(6.3)\qquad &\overset{C'}{R}{}^{\cdots i}_{(jkl)} = 0 , \\
&\overset{C'}{P}{}^{\cdots i}_{jkl} + \overset{C'}{P}{}^{\cdots i}_{kjl} = 0 , \\
&\overset{C'}{P}{}^{\cdots i}_{(jkl)} = 0 \qquad \text{etc.}
\end{aligned}$$

can be easily verified.

The equations of structure in our case are

$$\Omega^i \equiv [dx^j \omega^i_j] = C^i_{\cdot jk}[dx^j \omega^k] ,$$

$$\tilde{\Omega}^i \equiv (\omega^i)' + [\omega^i_j \omega^j] = -\frac{1}{2}\overset{C}{K}{}^{\cdot\cdot i}_{jk}[dx^j dx^k] - C^i_{\cdot[jk]}[\omega^j \omega^k] ,$$

$$\Omega^i_j \equiv (\omega^i_j)' + [\omega^i_k \omega^k_j] = -\frac{1}{2}\overset{C'}{R}{}^{\cdots i}_{kij}[dx^k dx^l] - \overset{C'}{B}{}^{\cdots i}_{kij}[dx^k \omega^l] ,$$

$$-\frac{1}{2}\overset{C'}{P}{}^{\cdots i}_{kij}[\omega^k \omega^l]$$

under the differential forms

$$dx^i , \qquad \omega^i = dx'^i + \Gamma^i_{(j)} dx^j , \qquad \omega^i_j = \Gamma^i_{(j)(k)} dx^k + C^i_{\cdot jk}\omega^k .$$

Based upon these preliminary considerations we are going to discuss the cases WIRTINGER's connections.

Since

$$\delta^{(C)} v^i = \overset{C}{\tilde{\nabla}}_j v^i dx^j + \overset{C}{\tilde{\nabla}}{}'_j v^i D^{(C)} x'^j ,$$

and the relation between $\overset{C}{\tilde{\nabla}}_j v^i$ and $\overset{C}{\nabla}_j v^i$ is

$$\overset{C}{\tilde{\nabla}}_j v^i = \overset{C}{\nabla}_j v^i - \overset{C}{\nabla}'_l v^i g^{kl} \overset{C}{\nabla}_j A_k = \overset{C}{\nabla}_j v^i + S^l_{\cdot j} \overset{C}{\nabla}'_l v^i ,$$

where we put

$$S^l_{\cdot j} \equiv -g^{kl} \overset{C}{\nabla}_j A_k ,$$

so that we have

$$\overset{C}{\tilde{\nabla}}_j \overset{C}{\tilde{\nabla}}_k v^i = \overset{C}{\nabla}_j \overset{C}{\nabla}_k v^i + S^l_{\cdot j} \overset{C}{\nabla}'_l (\overset{C}{\nabla}_k v^i) + \overset{C}{\nabla}_j U^i_{\cdot k} + S^l_{\cdot j} \overset{C}{\nabla}'_l U^i_{\cdot k} ,$$

the parenthesis of Poisson is expressed as

$$\begin{aligned}(\overset{C}{\tilde{\nabla}}_j \overset{C}{\tilde{\nabla}}_k - \overset{C}{\tilde{\nabla}}_k \overset{C}{\tilde{\nabla}}_j) v^i &= (\overset{C}{\nabla}_j \overset{C}{\nabla}_k - \overset{C}{\nabla}_k \overset{C}{\nabla}_j) v^i + S^l_{\cdot j} \overset{C}{\nabla}'_l \overset{C}{\nabla}_k v^i - S^l_{\cdot k} \overset{C}{\nabla}'_l \overset{C}{\nabla}_j v^i \\ &\quad + \overset{C}{\nabla}_j U^i_{\cdot k} - \overset{C}{\nabla}_k U^i_{\cdot j} + S^l_{\cdot j} \overset{C}{\nabla}'_l U^i_{\cdot k} - S^l_{\cdot k} \overset{C}{\nabla}'_l U^i_{\cdot j} ,\end{aligned}$$

where we put

$$U^i_{\cdot k} \equiv S^l_{\cdot k} \overset{C}{\nabla}'_l v^i .$$

From (6.1) and the following expressions

$$\overset{C}{\nabla}'_l \overset{C}{\nabla}_k v^i = \frac{\partial^2 v^i}{\partial x^k \partial x'^l} - \frac{\partial^2 v^i}{\partial x'^n \partial x'^l} \Gamma^n_{(k)} - \frac{\partial v^i}{\partial x'^n} \Gamma^n_{(k)(l)} + \Gamma^i_{(n)(k)(l)} v^n + \Gamma^i_{(n)(k)} \frac{\partial v^n}{\partial x'^l} ,$$

$$\overset{C}{\nabla}_j U^i_{\cdot k} = \frac{\partial U^i_{\cdot k}}{\partial x^j} - \frac{\partial U^i_{\cdot k}}{\partial x'^l} \Gamma^l_{(j)} + \Gamma^i_{(l)(j)} U^l_{\cdot k} - \Gamma^l_{(k)(j)} U^i_{\cdot l} ,$$

$$\overset{C}{\nabla}'_l U^i_{\cdot k} = \frac{\partial S^n_{\cdot k}}{\partial x'^l} \frac{\partial v^i}{\partial x'^n} + S^n_{\cdot k} \frac{\partial^2 v^i}{\partial x'^n \partial x'^l} ,$$

the parenthesis of Poisson is led to

$$\begin{aligned}(6.4) \qquad (\overset{C}{\tilde{\nabla}}_j \overset{C}{\tilde{\nabla}}_k - \overset{C}{\tilde{\nabla}}_k \overset{C}{\tilde{\nabla}}_j) v^i &= (-\overset{C}{R}{}^{\cdot\cdot\cdot i}_{jkl} + \Gamma^i_{(l)(k)(n)} S^n_{\cdot j} - \Gamma^i_{(l)(j)(n)} S^n_{\cdot k}) v^l \\ &\quad + \Big(\overset{C}{K}{}^{\cdot\cdot l}_{jk} + \Gamma^l_{(j)(n)} S^n_{\cdot k} - \Gamma^l_{(k)(n)} S^n_{\cdot j} + \frac{\partial S^l_{\cdot k}}{\partial x^j} - \frac{\partial S^l_{\cdot j}}{\partial x^k} \\ &\quad + \frac{\partial S^l_{\cdot j}}{\partial x'^m} \Gamma^m_{(k)} - \frac{\partial S^l_{\cdot k}}{\partial x'^m} \Gamma^m_{(j)} + S^n_{\cdot j} \frac{\partial S^l_{\cdot k}}{\partial x'^n} - S^n_{\cdot k} \frac{\partial S^l_{\cdot j}}{\partial x'^n} \Big) \overset{C}{\nabla}'_l v^i \\ &\equiv -\overset{C}{\tilde{R}}{}^{\cdot\cdot\cdot i}_{jkl} v^l + \overset{C}{\tilde{K}}{}^{\cdot\cdot l}_{jk} \overset{C}{\tilde{\nabla}}'_l v^i ,\end{aligned}$$

where we put

$$-\overset{C}{\tilde{R}}{}^{\cdot\cdot\cdot i}_{jkl} \equiv \overset{C}{R}{}^{\cdot\cdot\cdot i}_{jkl} + 2\Gamma^i_{(l)[(k)|(n)|} S^n_{\cdot j]} ,$$

$$\overset{C}{\tilde{K}}{}^{\cdot\cdot l}_{jk} \equiv \overset{C}{K}{}^{\cdot\cdot l}_{jk} + 2\Gamma^l_{[(j)|(n)|} S^n_{\cdot k]} + 2 \frac{\partial S^l_{\cdot [k}}{\partial x^{j]}} + 2 \frac{\partial S^l_{\cdot [j}}{\partial x'^m} \Gamma^m_{(k)]} + 2 S^n_{\cdot [j} \frac{\partial S^l_{\cdot k]}}{\partial x'^n} .$$

Since

$$\overset{C}{\widetilde{\nabla}}{}'_k \overset{C}{\widetilde{\nabla}}_j v^i = \frac{\partial^2 v^i}{\partial x^j \partial x'^k} - \frac{\partial^2 v^i}{\partial x'^n \partial x'^k} \Gamma^n_{(j)} - \frac{\partial v^i}{\partial x'^n} \Gamma^n_{(j)(k)} + \Gamma^i_{(n)(j)(k)} v^n + \Gamma^i_{(n)(j)} \frac{\partial v^n}{\partial x'^k}$$
$$+ \frac{\partial S^l_{\cdot j}}{\partial x'^k} \frac{\partial v^i}{\partial x'^l} + S^l_{\cdot j} \frac{\partial^2 v^i}{\partial x'^l \partial x'^k} ,$$
$$\overset{C}{\widetilde{\nabla}}_j \overset{C}{\widetilde{\nabla}}{}'_k v^i = \frac{\partial^2 v^i}{\partial x'^k \partial x^j} - \frac{\partial^2 v^i}{\partial x'^k \partial x'^l} \Gamma^l_{(j)} + \Gamma^i_{(l)(j)} \frac{\partial v^l}{\partial x'^k} - \Gamma^l_{(k)(j)} \frac{\partial v^i}{\partial x'^l} + S^l_{\cdot j} \frac{\partial^2 v^i}{\partial x'^k \partial x'^l}$$

another parenthesis of Poisson is computed as

$$(6.4)' \qquad (\overset{C}{\widetilde{\nabla}}_j \overset{C}{\widetilde{\nabla}}{}'_k - \overset{C}{\widetilde{\nabla}}{}'_k \overset{C}{\widetilde{\nabla}}_j) v^i = -\overset{C}{\widetilde{B}}{}_{ljk}^{\cdot\cdot\cdot i} v^l - \overset{C}{\widetilde{H}}{}^l_{\cdot jk} \overset{C}{\widetilde{\nabla}}{}'_l v^i ,$$

where we put

$$\overset{C}{\widetilde{B}}{}_{lkj}^{\cdot\cdot\cdot i} \equiv \overset{C}{B}{}_{lkj}^{\cdot\cdot\cdot i} , \qquad \overset{C}{\widetilde{H}}{}^l_{\cdot jk} \equiv \frac{\partial S^l_{\cdot j}}{\partial x'^k} .$$

The identities which correspond to (6.2) or (6.3) are

$$\overset{C}{\widetilde{R}}{}_{jkl}^{\cdot\cdot\cdot i} + \overset{C}{\widetilde{R}}{}_{kjl}^{\cdot\cdot\cdot i} = 0 ,$$
$$\overset{C}{\widetilde{R}}{}_{jkl}^{\cdot\cdot\cdot i} + \overset{C}{\widetilde{R}}{}_{klj}^{\cdot\cdot\cdot i} + \overset{C}{\widetilde{R}}{}_{ljk}^{\cdot\cdot\cdot i} = 0 ,$$
$$\overset{C}{\widetilde{R}}{}_{jkl}^{\cdot\cdot\cdot i} x'^l = \overset{C}{K}{}_{jk}^{\cdot\cdot i}$$
$$\overset{C}{\widetilde{B}}{}_{lkj}^{\cdot\cdot\cdot i} x'^l = 0, \qquad \overset{C}{\widetilde{K}}{}_{jk}^{\cdot\cdot i} + \overset{C}{\widetilde{K}}{}_{kj}^{\cdot\cdot i} = 0 , \quad \text{etc..}$$

Similar treatment is allowed in the case of the connection C'. As we have

$$\overset{C'}{\widetilde{\nabla}}_j v^i = \overset{C'}{\nabla}_j v^i - \Delta^k_l S^l_{\cdot j} \overset{C'}{\nabla}{}'_k v^i ,$$
$$\overset{C'}{\widetilde{\nabla}}{}'_j v^i = \Delta^k_j \overset{C'}{\nabla}{}'_k v^i ,$$
$$\overset{C'}{\widetilde{\nabla}}_j \overset{C'}{\widetilde{\nabla}}_k v^i = \overset{C'}{\nabla}_j \overset{C'}{\nabla}_k v^i - \overset{C'}{\nabla}_j \widetilde{S}^l_{\cdot k} \overset{C'}{\nabla}{}'_l v^i - \widetilde{S}^l_{\cdot k} \overset{C'}{\nabla}_j \overset{C'}{\nabla}{}'_l v^i - \widetilde{S}^l_{\cdot j} \overset{C'}{\nabla}{}'_l \overset{C'}{\nabla}_k v^i$$
$$+ \widetilde{S}^l_{\cdot j} \overset{C'}{\nabla}{}'_l \widetilde{S}^m_{\cdot k} \overset{C'}{\nabla}_m v^i + \widetilde{S}^l_{\cdot j} \widetilde{S}^m_{\cdot k} \overset{C'}{\nabla}{}'_l \overset{C'}{\nabla}{}'_m v^i ,$$

where we put

$$\widetilde{S}^k_{\cdot j} \equiv \Delta^k_l S^l_{\cdot j} ,$$

the parenthesis of Poisson is computed as follows:

$$(\overset{C'}{\widetilde{\nabla}}_j \overset{C'}{\widetilde{\nabla}}_k - \overset{C'}{\widetilde{\nabla}}_k \overset{C'}{\widetilde{\nabla}}_j) v^i = (\overset{C'}{\nabla}_j \overset{C'}{\nabla}_k - \overset{C'}{\nabla}_k \overset{C'}{\nabla}_j) v^i + (\overset{C'}{\nabla}_k \widetilde{S}^l_{\cdot j} - \overset{C'}{\nabla}_j \widetilde{S}^l_{\cdot k}) \overset{C'}{\nabla}{}'_l v^i + \widetilde{S}^l_{\cdot j} (\overset{C'}{\nabla}_k \overset{C'}{\nabla}{}'_l - \overset{C'}{\nabla}{}'_l \overset{C'}{\nabla}_k) v^i$$
$$- \widetilde{S}^l_{\cdot k} (\overset{C'}{\nabla}_j \overset{C'}{\nabla}{}'_l - \overset{C'}{\nabla}{}'_l \overset{C'}{\nabla}_j) v^i + (\widetilde{S}^l_{\cdot j} \overset{C'}{\nabla}{}'_l \widetilde{S}^m_{\cdot k} - \widetilde{S}^l_{\cdot k} \overset{C'}{\nabla}{}'_l \widetilde{S}^m_{\cdot j}) \overset{C'}{\nabla}{}'_m v^i$$
$$+ \widetilde{S}^l_{\cdot j} \widetilde{S}^m_{\cdot k} (\overset{C'}{\nabla}{}'_l \overset{C'}{\nabla}{}'_m - \overset{C'}{\nabla}{}'_m \overset{C'}{\nabla}{}'_l) v^i$$

$$
\begin{aligned}
&= -\overset{C'}{R}_{jkl}{}^{\cdot\cdot\cdot i} v^l + \overset{C}{K}_{jk}{}^{\cdot\cdot l} \overset{C'}{\nabla}{}'_l v^i + (\overset{C'}{\nabla}_k \tilde{S}^l_{\cdot j} - \overset{C'}{\nabla}_j \tilde{S}^l_{\cdot k}) \overset{C'}{\nabla}{}'_l v^i + \tilde{S}^l_{\cdot j}(-\overset{C'}{B}_{klm}{}^{\cdot\cdot\cdot i} v^m + C^m_{\cdot kl} v^i) \\
&\quad - \tilde{S}^l_{\cdot k}(-\overset{C'}{B}_{jlm}{}^{\cdot\cdot\cdot i} v^i + C^m_{\cdot jl} \overset{C'}{\nabla}{}'_m v^i) + (\tilde{S}^m_{\cdot j} \overset{C'}{\nabla}{}'_m \tilde{S}^l_{\cdot k} - \tilde{S}^m_{\cdot k} \overset{C'}{\nabla}{}'_m \tilde{S}^l_{\cdot j}) \overset{C'}{\nabla}{}'_l v^i \\
&\quad + \tilde{S}^l_{\cdot j} \tilde{S}^m_{\cdot k}(-\overset{C'}{P}_{lmh}{}^{\cdot\cdot\cdot i} v^h - 2C^h_{\cdot [lm]} \overset{C'}{\nabla}{}'_h v^i) \\
&= -(\overset{C'}{R}_{jkl}{}^{\cdot\cdot\cdot i} + \tilde{S}^m_{\cdot j} \overset{C'}{B}_{kml}{}^{\cdot\cdot\cdot i} - \tilde{S}^m_{\cdot k} \overset{C'}{B}_{jml}{}^{\cdot\cdot\cdot i} + \tilde{S}^h_{\cdot j} \tilde{S}^m_{\cdot k} \overset{C'}{P}_{hml}{}^{\cdot\cdot\cdot i}) v^l \\
&\quad + (\tilde{S}^m_{\cdot j} C^l_{\cdot km} - \tilde{S}^m_{\cdot k} C^l_{\cdot jm}) \overset{C'}{\nabla}_l v^i \\
&\quad + (\overset{C}{K}_{jk}{}^{\cdot\cdot l} + \overset{C'}{\nabla}_k \tilde{S}^l_{\cdot j} - \overset{C'}{\nabla}_j \tilde{S}^l_{\cdot k} + \tilde{S}^m_{\cdot j} \overset{C'}{\nabla}{}'_m \tilde{S}^l_{\cdot k} - \tilde{S}^m_{\cdot k} \overset{C'}{\nabla}{}'_m \tilde{S}^l_{\cdot j} - 2\tilde{S}^h_{\cdot j} \tilde{S}^m_{\cdot k} C^l_{\cdot [hm]}) \overset{C'}{\nabla}{}'_l v^i ,
\end{aligned}
$$

therefore the above expressions give rise to

$$
(6.5) \qquad (\overset{C'}{\tilde{\nabla}}_j \overset{C'}{\tilde{\nabla}}_k - \overset{C}{\tilde{\nabla}}_k \overset{C}{\tilde{\nabla}}_j) v^i = -\overset{C'}{\tilde{R}}_{jkl}{}^{\cdot\cdot\cdot i} v^l + J_{jk}{}^{\cdot\cdot l} \overset{C'}{\nabla}{}'_l v^i + \tilde{K}_{jk}{}^{\cdot\cdot l} \overset{C'}{\nabla}{}'_l v^i ,
$$

where we put

$$
\begin{aligned}
\overset{C'}{\tilde{R}}_{jkl}{}^{\cdot\cdot\cdot i} &\equiv \overset{C'}{R}_{jkl}{}^{\cdot\cdot\cdot i} + 2\tilde{S}^m_{\cdot [j} \overset{C'}{B}_{k]ml}{}^{\cdot\cdot\cdot i} + \tilde{S}^h_{\cdot j} \tilde{S}^m_{\cdot k} \overset{C'}{P}_{hml}{}^{\cdot\cdot\cdot i} , \\
\tilde{K}_{jk}{}^{\cdot\cdot l} &\equiv \overset{C}{K}_{jk}{}^{\cdot\cdot l} + 2\overset{C'}{\nabla}_{[k} \tilde{S}^l_{\cdot j]} + 2\tilde{S}^m_{[j} \overset{C'}{\nabla}{}'_{|m|} \tilde{S}^l_{\cdot k]} - 2\tilde{S}^h_{\cdot j} \tilde{S}^m_{\cdot k} C^l_{\cdot [hm]} , \\
J_{jk}{}^{\cdot\cdot l} &\equiv 2\tilde{S}^m_{\cdot [j} C^l_{\cdot k]m} .
\end{aligned}
$$

Making use of

$$
\begin{aligned}
\overset{C'}{\tilde{\nabla}}{}'_l v^i &= \Delta_l^h \overset{C'}{\nabla}{}'_h v^i , \\
\overset{C'}{\tilde{\nabla}}_j v^i &= \overset{C'}{\nabla}_j v^i - A_l^k S^l_{\cdot j} \overset{C'}{\nabla}{}'_k v^i = \overset{C'}{\nabla}_j v^i - S^m_{\cdot j} \overset{C'}{\tilde{\nabla}}{}'_m v^i ,
\end{aligned}
$$

we substitute $\overset{C'}{\nabla}_l v^i$, $\overset{C'}{\nabla}{}'_l v^i$ in (6.5). Then we have

$$
(6.6) \qquad (\overset{C'}{\tilde{\nabla}}_j \overset{C'}{\tilde{\nabla}}_k - \overset{C'}{\tilde{\nabla}}_k \overset{C'}{\tilde{\nabla}}_j) v^i = -\overset{C'}{\tilde{R}}_{jkl}{}^{\cdot\cdot\cdot i} v^l + J_{jk}{}^{\cdot\cdot l} \overset{C'}{\tilde{\nabla}}_l v^i + \overset{C'}{\tilde{K}}_{jk}{}^{\cdot\cdot l} \overset{C'}{\tilde{\nabla}}_l v^i ,
$$

where we put

$$
\overset{C'}{\tilde{K}}_{jk}{}^{\cdot\cdot l} \equiv \Lambda_m{}^l \tilde{K}_{jk}{}^{\cdot\cdot m} + S^l_{\cdot m} J_{jk}{}^{\cdot\cdot m} .
$$

The parenthesis of Poisson (6.6) corresponds to (6.4) and what corresponds to (6.4)′ is obtained as follows.

Since

$$
\begin{aligned}
\overset{C'}{\tilde{\nabla}}_k \overset{C'}{\tilde{\nabla}}{}'_l v^i &= \overset{C'}{\nabla}_k \overset{C'}{\tilde{\nabla}}{}'_l v^i - \tilde{S}^j_{\cdot k} \overset{C'}{\nabla}{}'_j \overset{C'}{\tilde{\nabla}}{}'_l v^i = \overset{C'}{\nabla}_k (\Delta_l^j \overset{C'}{\nabla}{}'_j v^i) - \tilde{S}^j_{\cdot k} \overset{C'}{\nabla}{}'_j (\Delta_l^s \overset{C'}{\nabla}{}'_s v^i) \\
&= \overset{C'}{\nabla}_k \Delta_l^j \overset{C'}{\nabla}{}'_j v^i + \Delta_l^j \overset{C'}{\nabla}_k \overset{C'}{\nabla}{}'_j v^i - \tilde{S}^j_{\cdot k} \overset{C'}{\nabla}{}'_j \Delta_l^s \overset{C'}{\nabla}{}'_s v^i - \tilde{S}^j_{\cdot k} \Delta_l^s \overset{C'}{\nabla}{}'_j \overset{C'}{\nabla}{}'_s v^i ,
\end{aligned}
$$

accordingly

$$
\begin{aligned}
(\overset{C'}{\widetilde{\nabla}}_k \overset{C'}{\widetilde{\nabla}}'_l - \overset{C'}{\widetilde{\nabla}}'_l \overset{C'}{\widetilde{\nabla}}_k) v^i &= \Delta_l^j (\overset{C'}{\nabla}_k \overset{C'}{\nabla}_j - \overset{C'}{\nabla}_j \overset{C'}{\nabla}_k) v^i + \Delta_l^j \widetilde{S}^s_{\cdot k} (\overset{C'}{\nabla}_j \overset{C'}{\nabla}_s - \overset{C'}{\nabla}_s \overset{C'}{\nabla}_j) v^i \\
&\quad + (\overset{C'}{\nabla}_k \Delta_l^s - \widetilde{S}^j_{\cdot k} \overset{C'}{\nabla}'_j \Delta_l^s - \Delta_l^j \overset{C'}{\nabla}'_j \widetilde{S}^s_{\cdot k}) \overset{C'}{\widetilde{\nabla}}'_s v^i \\
&= -\Delta_l^j (\overset{C'}{B}{}^{\cdots i}_{kjh} + \widetilde{S}^s_{\cdot k} \overset{C'}{P}{}^{\cdots i}_{jsh}) v^i + (\Delta_l^j C^h_{\cdot kj}) \overset{C'}{\nabla}_h v^i + (-2\Delta_l^j \widetilde{S}^s_{\cdot k} C^h_{\cdot [sj]} \\
&\quad + \overset{C'}{\nabla}_k \widetilde{S}^h_{\cdot l} - \widetilde{S}^j_{\cdot k} \overset{C'}{\nabla}'_j \Delta_j^h - \Delta_l^j \overset{C'}{\nabla}'_j \widetilde{S}^h_{\cdot k}) \overset{C'}{\widetilde{\nabla}}'_h v^i \\
&= -\overset{C'}{\widetilde{B}}{}^{\cdots i}_{klh} v^h + \widetilde{C}^h_{\cdot kl} \overset{C'}{\nabla}_h v^i + E^h_{\cdot kl} \overset{C'}{\nabla}'_h v^i ,
\end{aligned}
$$

where we put

$$
\begin{aligned}
\overset{C'}{\widetilde{B}}{}^{\cdots i}_{klh} &\equiv \Delta_l^j (\overset{C'}{B}{}^{\cdots i}_{kjh} + \widetilde{S}^s_{\cdot k} \overset{C'}{P}{}^{\cdots i}_{jsh}) , \\
\widetilde{C}^h_{\cdot kl} &\equiv \Delta_l^j C^h_{\cdot kj} , \\
E^h_{\cdot kl} &\equiv -2\Delta_l^j \widetilde{S}^s_{\cdot k} C^h_{\cdot [sj]} + \overset{C'}{\nabla}_k \widetilde{S}^h_{\cdot l} - \widetilde{S}^j_{\cdot k} \overset{C'}{\nabla}'_j \Delta_j^h - \Delta_l^j \overset{C'}{\nabla}'_j S^h_{\cdot k} .
\end{aligned}
$$

So that, from

$$
(\overset{C'}{\widetilde{\nabla}}_k \overset{C'}{\widetilde{\nabla}}'_l - \overset{C'}{\widetilde{\nabla}}'_l \overset{C'}{\widetilde{\nabla}}_k) v^i \equiv -\overset{C'}{\widetilde{B}}{}^{\cdots i}_{klh} v^h + \widetilde{C}^h_{\cdot kl} (\overset{C'}{\widetilde{\nabla}}_h v^i + \widetilde{S}^m_{\cdot h} \overset{C'}{\widetilde{\nabla}}'_m v^i) + E^h_{\cdot kl} \Lambda_h^m \overset{C'}{\widetilde{\nabla}}'_m v^i ,
$$

we get at once

$$
(\overset{C'}{\widetilde{\nabla}}_k \overset{C'}{\widetilde{\nabla}}_l - \overset{C'}{\widetilde{\nabla}}_l \overset{C'}{\widetilde{\nabla}}_k) v^i = -\overset{C'}{\widetilde{B}}{}^{\cdots i}_{klh} v^h + \widetilde{C}^h_{\cdot kl} \overset{C'}{\widetilde{\nabla}}_h v^i + \overset{C'}{\widetilde{E}}{}^h_{\cdot kl} \overset{C'}{\widetilde{\nabla}}'_h v^i ,
$$

where we put

$$
\overset{C'}{\widetilde{E}}{}^m_{\cdot kl} \equiv \widetilde{C}^h_{\cdot kl} S^m_{\cdot h} + E^h_{\cdot kl} \Lambda_h^m .
$$

In the same manner one obtains

$$
(\overset{C'}{\widetilde{\nabla}}'_k \overset{C'}{\widetilde{\nabla}}'_l - \overset{C'}{\widetilde{\nabla}}'_l \overset{C'}{\widetilde{\nabla}}'_k) v^i = -\overset{C'}{\widetilde{P}}{}^{\cdots i}_{klh} v^h - \overset{C'}{\widetilde{C}}{}^m_{\cdot kl} \overset{C'}{\widetilde{\nabla}}'_m v^i ,
$$

where we put

$$
\begin{aligned}
\overset{C'}{\widetilde{P}}{}^{\cdots i}_{klh} &\equiv \Delta_k^j \Delta_l^s \overset{C'}{P}{}^{\cdots i}_{jsh} , \\
\overset{C'}{\widetilde{C}}{}^m_{\cdot kl} &\equiv (-\Delta_k^j \overset{C'}{\nabla}'_j \Delta_l^h + \Delta_l^j \overset{C'}{\nabla}'_j \Delta_k^h + 2\Delta_k^j \Delta_l^s C^h_{\cdot [sj]}) \Lambda_h^m .
\end{aligned}
$$

From the KAWAGUCHI's differentials $\delta^{(C)} v^i$, $\delta^{(C')} v^i$ based on the connection C and the connection C' respectively, two kinds of covariant derivatives are introduced according to the selection of base connections, i. e. the one represents KAWAGUCHI's proper base connections and the other WIRTINGER's base connections.

In these cases the relations between KAWAGUCHI's curvature tensors

and WIRTINGER's have been already developed, moreover the same consideration is applicable to the WIRTINGER's covariant derivatives derived from $D^{(W)}v^i$.

But we should devote our attention to the cases of $D^{(C)}v^i$ and $D^{(C')}v^i$, in which the vector v^i is not a general vector but a vector included in the double vector (v^i, v_i), therefore it is impossible to continue our discussion as was done above.

In the case of $D^{(W)}v^i$ we have

$$D^{(W)}v^i = \overset{*}{\tilde{\nabla}}_j v^i dx^j + \overset{*}{\tilde{\nabla}}'_j v^i D^{(C')}x'^j$$

where we put

$$\overset{*}{\tilde{\nabla}}_j v^i = \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^l}(\Gamma^l_{(j)} - \tilde{S}^l_{\cdot j}) + \Gamma^i_{(l)(j)} v^l ,$$

$$\overset{*}{\tilde{\nabla}}'_j v^i = \frac{\partial v^i}{\partial x'^l} \Delta^l_j + C^i_{\cdot lj} v^l ,$$

accordingly taking

$$\tilde{\Gamma}^i_{\cdot j} \equiv \Gamma^i_{(j)} - \tilde{S}^i_{\cdot j}$$

we are led to

$$\overset{*}{\tilde{\nabla}}_j \overset{*}{\tilde{\nabla}}_k v^i = \frac{\partial \overset{*}{\tilde{\nabla}}_k v^i}{\partial x^j} - \frac{\partial \overset{*}{\tilde{\nabla}}_k v^i}{\partial x'^l} \tilde{\Gamma}^l_{\cdot j} + \Gamma^i_{(l)(j)} \overset{*}{\tilde{\nabla}}_k v^l - \Gamma^l_{(k)(j)} \overset{*}{\tilde{\nabla}}_l v^i$$

and the parenthesis of POISSON is expressed by

$$\begin{aligned}
\overset{*}{\tilde{\nabla}}_{[j} \overset{*}{\tilde{\nabla}}_{k]} v^i &= \frac{\partial \overset{*}{\tilde{\nabla}}_{[k} v^i}{\partial x^{j]}} - \frac{\partial \overset{*}{\tilde{\nabla}}_{[k} v^i}{\partial x'^l} \Gamma^l_{j]} + \Gamma^i_{(l)[(j)} \overset{*}{\tilde{\nabla}}_{k]} v^l \\
&= -\left(\frac{\partial \Gamma^i_{(l)[(j)}}{\partial x^{k]}} + \Gamma^i_{(l)[(k)|(h)|} \tilde{\Gamma}^h_{j]} + \Gamma^i_{(h)[(k)} \Gamma^h_{|(l)|(j)]} \right) v^l \\
&\quad + \left(\frac{\partial \tilde{\Gamma}^l_{[j}}{\partial x^{k]}} + \tilde{\Gamma}^l_{[k|(h)|} \tilde{\Gamma}^h_{j]} \right) \frac{\partial v^i}{\partial x'^l} .
\end{aligned}$$

Substituting

$$\frac{\partial v^i}{\partial x'^l} = \Delta_l{}^j \overset{*}{\tilde{\nabla}}'_j v^i - \Delta_l{}^j C^i_{\cdot kj} v^k$$

in the above expression, then we have

$$2\overset{*}{\tilde{\nabla}}_{[j} \overset{*}{\tilde{\nabla}}_{k]} v^i = -\overset{*}{\tilde{R}}_{jkl}^{\cdot\cdot\cdot i} v^l + \overset{*}{\tilde{K}}_{jk}^{\cdot\cdot l} \overset{*}{\tilde{\nabla}}'_l v^i ,$$

where we put

$$\overset{*}{\widetilde{R}}{}_{jkl}^{\cdot\cdot\cdot i} \equiv 2\frac{\partial \Gamma^{i}_{(l)[(j)}}{\partial x^{k]}} + 2\Gamma^{i}_{(l)[(k)|(h)|}\widetilde{\Gamma}^{h}_{j]} + 2\Gamma^{i}_{(h)[(k)}\Gamma^{h}_{|(l)|(j)]}$$

$$+2\left(\frac{\partial \widetilde{\Gamma}^{s}_{[j}}{\partial x^{k]}} + \widetilde{\Gamma}^{s}_{[k|(p)|}\widetilde{\Gamma}^{p}_{j]}\right)\Lambda^{m}_{s}C^{i}_{\cdot lm} ,$$

$$\overset{*}{\widetilde{K}}{}_{jk}^{\cdot\cdot l} \equiv 2\frac{\partial \widetilde{\Gamma}^{l}_{[j}}{\partial x^{k]}} + 2\widetilde{\Gamma}^{l}_{[k|(h)|}\widetilde{\Gamma}^{h}_{j]} .$$

Similarly since

$$(\overset{*}{\widetilde{\nabla}}_{j}\overset{*}{\widetilde{\nabla}}'_{k} - \overset{*}{\widetilde{\nabla}}'_{k}\overset{*}{\widetilde{\nabla}}_{j})v^{i} = -(\Delta^{l}_{k}\Gamma^{i}_{(h)(j)(l)} - \Gamma^{i}_{(h)(l)}C^{l}_{\cdot jk} - \overset{*}{\widetilde{\nabla}}_{j}C^{i}_{\cdot hk})v^{h} + C^{l}_{\cdot jk}\frac{\partial v^{i}}{\partial x^{l}}$$

$$+(\overset{*}{\widetilde{\nabla}}_{j}\Delta^{h}_{k} - \Gamma^{h}_{(l)(j)}\Delta^{l}_{k} + \widetilde{\Gamma}^{h}_{j(l)}\Delta^{l}_{k} - C^{l}_{\cdot jk}\widetilde{\Gamma}^{h}_{l})\frac{\partial v^{i}}{\partial x'^{h}}$$

$$= -(\Delta^{l}_{k}\Gamma^{i}_{(h)(j)(l)} - \Gamma^{i}_{(h)(l)}C^{l}_{\cdot jk} - \overset{*}{\widetilde{\nabla}}_{j}C^{i}_{\cdot hk})v^{h} + C^{l}_{\cdot jk}(\overset{*}{\widetilde{\nabla}}_{l}v^{i} + \frac{\partial v^{i}}{\partial x'^{m}}\widetilde{\Gamma}^{m}_{l} - \Gamma^{i}_{(m)(l)}v^{m})$$

$$+(\overset{*}{\widetilde{\nabla}}_{j}\Delta^{h}_{k} - \Gamma^{h}_{(l)(j)}\Delta^{l}_{k} + \widetilde{\Gamma}^{h}_{j(l)}\Delta^{l}_{k} - C^{l}_{\cdot jk}\widetilde{\Gamma}^{h}_{l})(\Lambda^{s}_{h}\overset{*}{\widetilde{\nabla}}'_{s}v^{i} - \Lambda^{s}_{h}C^{i}_{\cdot ts}v^{t})$$

$$= -\{\Delta^{l}_{k}\Gamma^{i}_{(h)(j)(l)} - \Gamma^{i}_{(h)(l)}C^{l}_{\cdot jk} - \overset{*}{\widetilde{\nabla}}_{j}C^{i}_{\cdot hk} + C^{l}_{\cdot jk}\Gamma^{i}_{(h)(l)} + (\overset{*}{\widetilde{\nabla}}_{j}\Delta^{t}_{k} - \Gamma^{t}_{(l)(j)}\Delta^{l}_{k}$$

$$+\widetilde{\Gamma}^{t}_{j(l)}\Delta^{l}_{k})\Lambda^{s}_{t}C^{i}_{\cdot hs}\}v^{h} + C^{h}_{\cdot jk}\overset{*}{\widetilde{\nabla}}_{h}v^{i} + (\overset{*}{\widetilde{\nabla}}_{j}\Delta^{s}_{k} - \Gamma^{s}_{(l)(j)}\Delta^{l}_{k} + \widetilde{\Gamma}^{s}_{j(l)}\Delta^{l}_{k})\Lambda^{h}_{s}\overset{*}{\widetilde{\nabla}}'_{h}v^{i} ,$$

we get

$$(\overset{*}{\widetilde{\nabla}}_{j}\overset{*}{\widetilde{\nabla}}'_{k} - \overset{*}{\widetilde{\nabla}}'_{k}\overset{*}{\widetilde{\nabla}}_{j})v^{i} = -\overset{*}{\widetilde{B}}{}_{jkl}^{\cdot\cdot\cdot i}v^{l} + C^{l}_{\cdot jk}\overset{*}{\widetilde{\nabla}}_{l}v^{i} + A_{jk}^{\cdot\cdot h}\overset{*}{\widetilde{\nabla}}'_{h}v^{i} ,$$

where we put

$$\overset{*}{\widetilde{B}}{}_{jkl}^{\cdot\cdot\cdot i} \equiv \Delta^{l}_{k}\Gamma^{i}_{(h)(j)(l)} - \Gamma^{i}_{(h)(l)}C^{l}_{\cdot jk} - \overset{*}{\widetilde{\nabla}}_{j}C^{i}_{\cdot hk} + C^{l}_{\cdot jk}\Gamma^{i}_{(h)(l)}$$

$$+(\overset{*}{\widetilde{\nabla}}_{j}\Delta^{t}_{k} - \Gamma^{t}_{(l)(j)}\Delta^{l}_{k} + \widetilde{\Gamma}^{t}_{j(l)}\Delta^{l}_{k})\Lambda^{s}_{t}C^{i}_{\cdot hs} ,$$

$$A_{jk}^{\cdot\cdot h} \equiv (\overset{*}{\widetilde{\nabla}}_{j}\Delta^{s}_{k} - \Gamma^{s}_{(l)(j)}\Delta^{l}_{k} + \widetilde{\Gamma}^{s}_{j(l)}\Delta^{l}_{k})\Lambda^{h}_{s} .$$

Another parenthesis of Poisson can be computed as follows: since

$$(\overset{*}{\widetilde{\nabla}}'_{j}\overset{*}{\widetilde{\nabla}}'_{k} - \overset{*}{\widetilde{\nabla}}'_{k}\overset{*}{\widetilde{\nabla}}'_{j})v^{i} = \left(\frac{\partial C^{i}_{\cdot hk}}{\partial x'^{l}}\Delta^{l}_{j} - \frac{\partial C^{i}_{\cdot hj}}{\partial x'^{l}}\Delta^{l}_{k} + C^{i}_{\cdot lj}C^{l}_{\cdot hk} - C^{i}_{\cdot lk}C^{l}_{\cdot hj}\right.$$

$$\left.+C^{l}_{\cdot jk}C^{i}_{\cdot hl} - C^{l}_{\cdot kj}C^{i}_{\cdot hl}\right)v^{h} + (C^{l}_{\cdot jk}\Delta^{h}_{l} - C^{l}_{\cdot kj}\Delta^{h}_{l})\frac{\partial v^{i}}{\partial x'^{h}}$$

$$= \left\{\frac{\partial C^{i}_{\cdot hk}}{\partial x'^{l}}\Delta^{l}_{j} - \frac{\partial C^{i}_{\cdot hj}}{\partial x'^{l}}\Delta^{l}_{k} + C^{i}_{\cdot lj}C^{l}_{\cdot hk} - C^{i}_{\cdot lk}C^{l}_{\cdot hj} + C^{l}_{\cdot jk}C^{i}_{\cdot hl} - C^{l}_{\cdot kj}C^{i}_{\cdot hl}\right.$$

$$\left.-(C^{l}_{\cdot jk}\Delta^{p}_{l} - C^{l}_{\cdot kj}\Delta^{p}_{l})\Lambda^{s}_{p}C^{i}_{\cdot hs}\right\}v^{h} + (C^{l}_{\cdot jk} - C^{l}_{\cdot kj})\Delta^{h}_{l}\Lambda^{s}_{h}\overset{*}{\widetilde{\nabla}}'_{s}v^{i} ,$$

one obtains

$$(\overset{*}{\tilde{\nabla}}{}'_j \overset{*}{\tilde{\nabla}}{}'_k - \overset{*}{\tilde{\nabla}}{}'_k \overset{*}{\tilde{\nabla}}{}'_j)\, v^i = -\overset{*}{\tilde{P}}{}_{jkh}^{\cdot\cdot\cdot i} v^h + 2C^s_{\cdot[jk]} \overset{*}{\tilde{\nabla}}{}'_s v^i ,$$

where we put

$$\overset{*}{\tilde{P}}{}_{jkh}^{\cdot\cdot\cdot i} \equiv \frac{\partial C^i_{\cdot hk}}{\partial x'^l} \Delta_j^l - \frac{\partial C^i_{\cdot hj}}{\partial x'^l} \Delta_k^l + C^i_{\cdot lj} C^l_{\cdot hk} - C^i_{\cdot lk} C^l_{\cdot hj} .$$

Identities which correspond to BIANCHI's identies can be obtained from coefficients of the bracket expressions

$$[dx^i\, dx^j\, dx^k] , \quad [dx^i\, dx^j\, \omega^k] , \quad [dx^i\, \omega^j\, \omega^k] , \quad [\omega^i\, \omega^j\, \omega^k]$$

involved in the PFAFFIAN form

$$(\Omega^i_j)' - [\omega^k_j \Omega^i_k] + [\Omega^k_j \omega^i_k] = 0$$

or can be derived from covariant derivatives.

For example, the BIANCHI's identities in the case of the WIRTINGER's connection based on the connection C can be computed as follows.

Since

$$(\overset{C}{\tilde{\nabla}}_h \overset{C}{\tilde{\nabla}}_j - \overset{C}{\tilde{\nabla}}_j \overset{C}{\tilde{R}}_h) \overset{C}{\tilde{\nabla}}_k v^i = -\overset{C}{\tilde{R}}{}_{hjl}^{\cdot\cdot\cdot i} \overset{C}{\tilde{\nabla}}_k v^l + \overset{C}{\tilde{R}}{}_{hjk}^{\cdot\cdot\cdot l} \overset{C}{\tilde{\nabla}}_l v^i + \overset{C}{\tilde{K}}{}_{hj}^{\cdot\cdot l} \overset{C}{\tilde{\nabla}}{}'_l \overset{C}{\tilde{\nabla}}_k v^i ,$$

$$2\overset{C}{\tilde{\nabla}}_h \overset{C}{\tilde{\nabla}}_{[j} \overset{C}{\tilde{\nabla}}_{k]} v^i = -\overset{C}{\tilde{\nabla}}_h \overset{C}{\tilde{R}}{}_{jkl}^{\cdot\cdot\cdot i} v^l - \overset{C}{\tilde{R}}{}_{jkl}^{\cdot\cdot\cdot i} \overset{C}{\tilde{\nabla}}_h v^l + \overset{C}{\tilde{\nabla}} \overset{C}{\tilde{K}}{}_{jk}^{\cdot\cdot l} \overset{C}{\tilde{\nabla}}{}'_l v^i + \overset{C}{\tilde{K}}{}_{jk}^{\cdot\cdot l} \overset{C}{\tilde{\nabla}}_h \overset{C}{\tilde{\nabla}}{}'_l v^i ,$$

one obtains

$$-\overset{C}{\tilde{\nabla}}_{[h} \overset{C}{\tilde{R}}{}_{jk]l}^{\cdot\cdot\cdot i} v^l - \overset{C}{\tilde{R}}{}_{[jk|l|}^{\cdot\cdot\cdot i} \overset{C}{\tilde{\nabla}}_{h]} v^l + \overset{C}{\tilde{\nabla}}_{[k} \overset{C}{\tilde{K}}{}_{jk]}^{\cdot\cdot l} \overset{C}{\tilde{\nabla}}{}'_l v^i + \overset{C}{\tilde{K}}{}_{[jk}^{\cdot\cdot l} \overset{C}{\tilde{\nabla}}_{h]} \overset{C}{\tilde{\nabla}}{}'_l v^i$$
$$= -\overset{C}{\tilde{R}}{}_{[hj|l|}^{\cdot\cdot\cdot i} \overset{C}{\tilde{\nabla}}_{k]} v^l + \overset{C}{\tilde{R}}{}_{[hjk]}^{\cdot\cdot\cdot l} \overset{C}{\tilde{\nabla}}_l v^i + \overset{C}{\tilde{K}}{}_{[hj}^{\cdot\cdot l} \overset{C}{\tilde{\nabla}}_{|l|} \overset{C}{\tilde{\nabla}}_{k]} v^i ,$$

namely

$$-(\overset{C}{\tilde{\nabla}}_{[h} \overset{C}{\tilde{R}}{}_{jk]l}^{\cdot\cdot\cdot i} + \overset{C}{\tilde{K}}{}_{[hj}^{\cdot\cdot r} \overset{C}{\tilde{B}}{}_{k]rl}^{\cdot\cdot\cdot i}) v^l + (\overset{C}{\tilde{\nabla}}_{[h} \overset{C}{\tilde{K}}{}_{jk]}^{\cdot\cdot l} - \overset{C}{\tilde{K}}{}_{[hj}^{\cdot\cdot r} S^l_{\cdot k](r)}) \overset{C}{\tilde{\nabla}}{}'_l v^i = 0 .$$

Then we have at once

$$(6.7) \qquad \overset{C}{\tilde{\nabla}}_{[h} \overset{C}{\tilde{R}}{}_{jk]l}^{\cdot\cdot\cdot i} + \overset{C}{\tilde{K}}{}_{[hj}^{\cdot\cdot r} \overset{C}{\tilde{B}}{}_{k]rl}^{\cdot\cdot\cdot i} = 0 , \qquad \overset{C}{\tilde{\nabla}}_{[h} \overset{C}{\tilde{K}}{}_{jk]}^{\cdot\cdot l} - \overset{C}{\tilde{K}}{}_{[hj}^{\cdot\cdot r} S^l_{\cdot k](r)} = 0 .$$

The former of these identities corresponds formally to the KAWAGUCHI's, while the latter adds the term involving

$$S^l_{\cdot k(r)} \equiv \frac{\partial S^l_{\cdot k}}{\partial x'^r}$$

in the KAWAGUCHI's form.

Similarly, from

$$\overset{C}{\tilde{\nabla}}_h(\overset{C}{\tilde{\nabla}}_j\overset{C}{\tilde{\nabla}}'_k-\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{\nabla}}_j)\,v^i = -\overset{C}{\tilde{\nabla}}_h\overset{C}{\tilde{B}}{}^{\cdots i}_{jkl}v^l-\overset{C}{\tilde{B}}{}^{\cdots i}_{jkl}\overset{C}{\tilde{\nabla}}_h v^l-\overset{C}{\tilde{\nabla}}_h S^l_{\cdot j(k)}\overset{C}{\tilde{\nabla}}'_l v^i-S^l_{\cdot j(k)}\overset{C}{\tilde{\nabla}}_h\overset{C}{\tilde{\nabla}}'_l v^i\,,$$

$$\overset{C}{\tilde{\nabla}}'_k(\overset{C}{\tilde{\nabla}}_h\overset{C}{\tilde{\nabla}}_j-\overset{C}{\tilde{\nabla}}_j\overset{C}{\tilde{\nabla}}_h)\,v^i = -\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{R}}{}^{\cdots i}_{hjl}v^l-\overset{C}{\tilde{R}}{}^{\cdots i}_{hjl}\overset{C}{\tilde{\nabla}}'_k v^l+\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{K}}{}^{\cdot\cdot l}_{hj}\overset{C}{\tilde{\nabla}}'_l v^i+\overset{C}{\tilde{K}}{}^{\cdot\cdot l}_{hj}\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{\nabla}}'_l v^i\,,$$

$$-\overset{C}{\tilde{\nabla}}_j(\overset{C}{\tilde{\nabla}}_h\overset{C}{\tilde{\nabla}}'_k-\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{\nabla}}_h)\,v^i = \overset{C}{\tilde{\nabla}}_j\overset{C}{\tilde{B}}{}^{\cdots i}_{hkl}v^l+\overset{C}{\tilde{B}}{}^{\cdots i}_{hkl}\overset{C}{\tilde{\nabla}}_j v^l+\overset{C}{\tilde{\nabla}}_j S^l_{\cdot h(k)}\overset{C}{\tilde{\nabla}}'_l v^i+\tilde{S}^l_{h(k)}\overset{C}{\tilde{\nabla}}_j\overset{C}{\tilde{\nabla}}'_l v^i\,,$$

$$(\overset{C}{\tilde{\nabla}}_h\overset{C}{\tilde{\nabla}}_j-\overset{C}{\tilde{\nabla}}_j\overset{C}{\tilde{\nabla}}_h)\overset{C}{\tilde{\nabla}}'_k v^i = -\overset{C}{\tilde{R}}{}^{\cdots i}_{hjl}\overset{C}{\tilde{\nabla}}'_k v^l+\overset{C}{\tilde{R}}{}^{\cdots i}_{hjl}\overset{C}{\tilde{\nabla}}'_l v^i+\overset{C}{\tilde{K}}{}^{\cdot\cdot l}_{hj}\overset{C}{\tilde{\nabla}}'_l\overset{C}{\tilde{\nabla}}'_k v^i\,,$$

$$-(\overset{C}{\tilde{\nabla}}_h\overset{C}{\tilde{\nabla}}'_k-\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{\nabla}}_h)\overset{C}{\tilde{\nabla}}_j v^i = \overset{C}{\tilde{B}}{}^{\cdots i}_{hkl}\overset{C}{\tilde{\nabla}}_j v^l-\overset{C}{\tilde{B}}{}^{\cdots l}_{hkj}\overset{C}{\tilde{\nabla}}_l v^i+S^l_{\cdot h(k)}\overset{C}{\tilde{\nabla}}'_l\overset{C}{\tilde{\nabla}}_j v^i\,,$$

$$(\overset{C}{\tilde{\nabla}}_j\overset{C}{\tilde{\nabla}}'_k-\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{\nabla}}_j)\overset{C}{\tilde{\nabla}}_h v^i = -\tilde{B}{}^{\cdots i}_{jkl}\overset{C}{\tilde{\nabla}}_h v^l+\overset{C}{\tilde{B}}{}^{\cdots l}_{jkh}\overset{C}{\tilde{\nabla}}_l v^i-S^l_{\cdot j(k)}\overset{C}{\tilde{\nabla}}'_l\overset{C}{\tilde{\nabla}}_h v^i\,,$$

one obtains

$$-(2\overset{C}{\tilde{\nabla}}_{[h}\overset{C}{\tilde{B}}{}^{\cdots i}_{j]kl}+\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{R}}{}^{\cdots i}_{hjl}-S^r_{\cdot j(k)}\overset{C}{\tilde{B}}{}_{hrl}+S^r_{\cdot h(k)}\overset{C}{\tilde{B}}{}^{\cdots i}_{jrl})\,v^l$$
$$-(2\overset{C}{\tilde{\nabla}}_{[h}S^l_{\cdot j](k)}+\overset{C}{\tilde{R}}{}^{\cdots l}_{hjk}-\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{K}}{}^{\cdot\cdot l}_{hj}-S^r_{\cdot j(k)}S^l_{\cdot h(r)}+S^r_{\cdot h(k)}S^l_{\cdot j(r)})\overset{C}{\tilde{\nabla}}'_l v^i = 0\,,$$

so that the identities

$$(6.8)\qquad \begin{aligned} &2\overset{C}{\tilde{\nabla}}_{[h}\overset{C}{\tilde{B}}{}^{\cdots i}_{j]kl}+\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{R}}{}^{\cdots i}_{hjl}+2S^r_{\cdot[h(k)}\overset{C}{\tilde{B}}{}^{\cdots i}_{j]rl} = 0\,,\\ &2\overset{C}{\tilde{\nabla}}_{[h}S^l_{\cdot j](k)}+\overset{C}{\tilde{R}}{}^{\cdots l}_{hjk}-\overset{C}{\tilde{\nabla}}'_k\overset{C}{\tilde{K}}{}^{\cdot\cdot l}_{hj}+2S^r_{\cdot[h|(k)|}S^l_{\cdot j](r)} = 0 \end{aligned}$$

are the required result.

Identities (6.7) and (6.8) are the BIANCHI's identities in our case, in which the tensor $S^l_{\cdot j}$ characterizes the WIRTINGER's connection based on the connection C and our identities correspond completely to KAWAGUCHI's identities except the tensor $S^l_{\cdot j}$.

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