ON THE MAXIMAL SPECTRALITY

By

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Let R be a Hilbert space and $\mathfrak{B}$ a totally additive set class in an abstract space Ω. Asystem of proiection operators $E(\Phi)\,(\Phi\in\mathfrak{B})$ in R is called a *spectrality*[1] in R on $\mathfrak{B}$ if (1) $E(\Phi)+E(\Phi^C)=1$, and (2) $E\left(\sum_{i=1}^{\infty}\Phi_i\right) = \bigcup_{i=1}^{\infty} E(\Phi_i)$. We say a spectrality $E(\Phi)\,(\Phi\in\mathfrak{B})$ is *maximal* (due to Prof. NAKANO's suggestion) if

1) for any finite measure ν on $\mathfrak{B}$ we can find an element $x\in R$ such that $\nu(\Phi)=\|E(\Phi)x\|^2\,(\Phi\in\mathfrak{B})$, and

2) $\mathfrak{R}_E$ is a simple ring, where $\mathfrak{R}_E$ is a closed projection operator ring[2] generated by $\{E(\Phi);\ \Phi\in\mathfrak{B}\}$.

$\mathfrak{R}_E$ is *simple*[3] if and only if for any projection operator P that is commutative to $\mathfrak{R}_E$ we have $P\in\mathfrak{R}_E$.

In this paper we shall show that for any given Ω and $\mathfrak{B}$ we can construct a Hilbert space R and a maximal spectrality $E(\Phi)\,(\Phi\in\mathfrak{B})$ in R on $\mathfrak{B}$, and moreover R and $E(\Phi)\ (\Phi\in\mathfrak{B})$ are determined uniquely within an unitary isomorphism (Theorem 1). Conversely for any given R we can find Ω and $\mathfrak{B}$ for which there exists a *discrete* maximal spectrality in R on $\mathfrak{B}$. But it is known in WECKEN [1] that if the dimension of R is cotinuum, there exists in R a maximal spectrality on the Borel sets in the real line. If R is separable, we can prove that there is no maximal spectrality other than a discrete one (Theorem 2).

Theorem 1. *For any given Ω and $\mathfrak{B}$ we can construct a Hilbert space R and a maximal spectrality $E(\Phi)\,(\Phi\in\mathfrak{B})$ in R. Furthermore such R and $E(\Phi)\,(\Phi\in\mathfrak{B})$ are determined uniquely within unitary isomorphism.*

The method of the proof is essentialy same as in [1], so we give an outline only, about details refer [1] or NAKANO [2] Chap. V.

Let $\mathfrak{M}_{\mathfrak{B}}$ be the totality of finite measures on $\mathfrak{B}$. From the property of $\mathfrak{M}_{\mathfrak{B}}$ as a Boolian lattice and Maximal theorem we can find a maximal

1) cf. [2] §28.
2) cf. [2] §14.
3) cf. [2] §20.

system $m_\lambda \in \mathfrak{M}_{\mathfrak{B}}(\lambda \in \Lambda)$ such that (i) $m_\lambda \perp m_{\lambda'} (\lambda \neq \lambda')$, that is, if $\nu \in \mathfrak{M}_{\mathfrak{B}}$ is absolutely continuous about m_λ and $m_{\lambda'}$ then $\nu = 0$, (ii) for any $\nu \in \mathfrak{M}_{\mathfrak{B}}$ we can write $\nu(\Phi) = \sum_{i=1}^{\infty} \mu_{\lambda_i}(\Phi) (\Phi \in \mathfrak{B})$ for suitable countable measures $\mu_{\lambda_i} \in \mathfrak{M}_B$, $\mu_{\lambda_i} \prec m_{\lambda_i}$, $i = 1, 2, \cdots (\mu_{\lambda_i}$ is absolutely continuous about $m_{\lambda_i})$. Putting $R = \sum_{\lambda \in \Lambda} \oplus L^2(m_\lambda)$ and $E(\Phi)\{f_\lambda\} = \{\chi_\Phi f_\lambda\}$, where $\{f_\lambda\}$ is an element of R, $f_\lambda \in L^2(m_\lambda)(\lambda \in \Lambda)$, and χ_Φ a characteristic function of Φ, we obtain a spectrality $E(\Phi)(\Phi \in \mathfrak{B})$ in R. This spectrality is maximal. Because by the definition of $m_\lambda (\lambda \in \Lambda)$ it is almost evident that the condition 1) of the maximality is satisfied. The condition 2) and the uniqueness of R and $E(\Phi)(\Phi \in \mathfrak{B})$ are obtained by the following facts. Denoting by $[x]_{\mathfrak{R}_E}(x \in R)$ the projection operator of the subspace generated by $\{E(\Phi)x;\ \Phi \in \mathfrak{B}\}$, then $[x]_{\mathfrak{R}_E} R$ is unitary isomorphic to $L^2(m_x)$, where $m_x \in \mathfrak{M}_{\mathfrak{B}}$, $m_x(\Phi) = \|E(\Phi)x\|^2 (\Phi \in \mathfrak{B})$. Let $C_{[x]_{\mathfrak{R}_E}}$ be the cover[4] of $[x]_{\mathfrak{R}_E}$, then $C_{[x]_{\mathfrak{R}_E}} \geqq C_{[y]_{\mathfrak{R}_E}}$ is equivalent to $m_x \succ m_y$. And $\mathfrak{R}_E$ is simple if and only if $[x]_{\mathfrak{R}_E} [y]_{\mathfrak{R}_E} = 0$ and $C_{[x]_{\mathfrak{R}_E}} = C_{[y]_{\mathfrak{R}_E}}$ imply $x = y = 0$.

Let R be any Hilbert space and $a_\lambda (\lambda \in \Lambda)$ a complete orthonormal system in R.

Putting $\Omega = \Lambda$ and

$\mathfrak{B} = \{\Phi;\ \Phi \not\ni \lambda_0$ and at most countable or $\Phi \ni \lambda_0$ and Φ^C at most countable$\}$, where λ_0 is a fixed index, and for $\Phi \in \mathfrak{B}$ $E(\Phi)$ is the projection operator of the subspace generated by $\{a_\lambda;\ \lambda \in \Phi\}$, then obviously $\mathfrak{B}$ is a totally additive set class and $E(\Phi)(\Phi \in \mathfrak{B})$ is a spectrality in R on $\mathfrak{B}$. We remark that if R is separable, $\mathfrak{B} = 2^\Lambda$. A projection operator P belongs to $\mathfrak{R}_E$ if and only if P is the projection operator of the subspace generated by $\{a_\lambda;\ \lambda \in S\}$ for a subset $S \subseteqq \Lambda$. For $x \in R$ $[x]_{\mathfrak{R}_E}$ is the projection operator of the subspace generated by $\{a_\lambda;\ (x, a_\lambda) \neq 0\}$, hence $[x]_{\mathfrak{R}_E} \in \mathfrak{R}_E$. Therefore $\mathfrak{R}_E$ is simple ([2]th. 20. 3). Next for any $\nu \in \mathfrak{M}_{\mathfrak{B}}$ we put $\Omega_1 = \{\lambda;\ \nu(\{\lambda\}) \neq 0,\ \lambda \neq \lambda_0\}$ $\Omega_2 = \Omega - \Omega_1$, then $\Omega_1 \not\ni \lambda_0$ and at most countable, and we put $x = \sum_{\lambda \in \Omega_1} \sqrt{\nu(\{\lambda\})}\, a_\lambda + \sqrt{\nu(\Omega_2)}\, a_{\lambda_0}$, then for $\Phi \in \mathfrak{B}$

$$\|E(\Phi)x\|^2 = \sum_{\lambda \in \Omega_1} \nu(\{\lambda\}) \|E(\Phi)a_\lambda\|^2 + \nu(\Omega_2)\|E(\Phi)a_{\lambda_0}\|^2$$

$$= \sum_{\lambda \in \Phi \frown \Omega_1} \nu(\{\lambda\}) + \nu(\Omega_2)\|E(\Phi)a_{\lambda_0}\|^2 = \nu(\Phi \frown \Omega_1) + \nu(\Phi \frown \Omega_2) = \nu(\Phi),$$

because $\nu(\Omega_2 \frown \Phi) = \nu(\Omega_2)$ or 0 according to $\Phi \ni \lambda_0$ or $\Phi \not\ni \lambda_0$, hence $m_x = \nu$. Therefore this spectrality is maximal. We call this spectrality *discrete.*

Theorem 2. *In a separable Hilbert space a spectrality is maximal if*

4) cf. [2] § 14.

and only if it is discrete.

Proof. Let $E(\Phi)(\Phi\in\mathfrak{B})$ be a maximal spectrality in a separable Hilbert space R. As R is separable, by Maximal theorem there are $e_n\in R$ $n=1,2,\cdots$ such that $[e_n]_{\mathfrak{R}_E}[e_m]_{\mathfrak{R}_E}=0\,(n\neq m)$ and $\bigcup_{n=1}^{\infty}[e_n]_{\mathfrak{R}_E}=I$. If we put $e=\sum_{n=1}^{\infty}\frac{1}{2^n}e_n$, then we have $[e]_{\mathfrak{R}_E}=\bigcup_{n=1}^{\infty}[e_n]_{\mathfrak{R}_E}=I$ ([2]th. 14.5). If $E(\Phi)e=0$, then $E(\Phi)=E(\Phi)[e]_{\mathfrak{R}_E}=[E(\Phi)e]_{\mathfrak{R}_E}=0$, and hence $\Phi=\phi$ from the condition 1) of a maximal spectrality. From the separability of R we can take out $\Phi_n\in\mathfrak{B}$ $n=1,2,\cdots$ such that $\{E(\Phi_n)e\,;\, n=1,2,\cdots\}$ is dense in $\{E(\Phi)e\,;\,\Phi\in\mathfrak{B}\}$. Putting $\Psi_{\{\delta_n\}}=\prod_{n=1}^{\infty}\Phi_n^{\delta_n}\,(\delta_n=1 \text{ or} -1)$, $\Phi_n^{\delta_n}=\Phi_n$ or Φ_n^C according to $\delta_n=1$ or -1, then evidently $\{\delta_n\}\neq\{\delta_n'\}$ implies $\Psi_{\{\delta_n\}}\Psi_{\{\delta_n'\}}=\phi$, and we have $\sum_{\{\delta_n\}}\Psi_{\{\delta_n\}}=\Omega$. Next if $\Psi_{\{\delta_n\}}\neq\phi$, then $\Psi_{\{\delta_n\}}$ is an atomic element in $\mathfrak{B}$, because if $\Psi_{\{\delta_n\}}\supsetneqq\Phi$, $\Phi\in\mathfrak{B}$, then

$$\|E(\Psi_{\{\delta_n\}}-\Phi)e\|^2=\|E(\Psi_{\{\delta_n\}})e-E(\Phi)e\|^2=\delta>0$$

for any positive number $\varepsilon>0$ $(0<\varepsilon<\delta)$ we can find Φ_{n_0} such that $\|E(\Phi)e-E(\Phi_{n_0})e\|^2<\varepsilon$. For such n_0 we have $\delta_{n_0}=-1$, for if $\delta_{n_0}=1$, then $\Phi_{n_0}\geqq\Psi_{\{\delta_n\}}$, therefore

$$\varepsilon>\|E(\Phi_{n_0})e-E(\Phi)e\|^2\geqq\|E(\Psi_{\{\delta_n\}})e-E(\Phi)e\|^2=\delta>\varepsilon,$$

and it is a contradiction. Therefore $\delta_{n_0}=-1$, and hence $\Phi\subseteqq\Phi_{n_0}^C$, so

$$\|E(\Phi_{n_0})e-E(\Phi)e\|^2=\|E(\Phi_{n_0})e\|^2+\|E(\Phi)e\|^2<\varepsilon,$$

hence $\|E(\Phi)e\|^2<\varepsilon$, since ε is arbitrary, $E(\Phi)e=0$, namely $\Phi=\phi$. As R is separable, we have $\mathfrak{R}_E=\{E(\Phi);\,\Phi\in\mathfrak{B}\}$, and hence $E(\Psi_{\{\delta_n\}})$ is atomic in $\mathfrak{R}_E$, and the simplicity of $\mathfrak{R}_E$, we obtain that $E(\Psi_{\{\delta_n\}})\,R$ has one or zero dimension. From the above mentioned $E(\Phi)(\Phi\in\mathfrak{B})$ is a discrete spectrality. q. e. d.

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Refereces

[1] F. J. WECKEN: Unitärinvariant selbstadjungierter operatoren, Math. Ann. (1939).

[2] H. NAKANO: Spectral theory in the Hilbert space, Tokyo Math. Book Series, Vol. IV. (1953).