MODULARS ON SEMI-ORDERED LINEAR SPACES I

By

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In an earlier paper [1], one of the authors defined modulars on linear spaces and discussed their properties: a functional $m(x)$ on a linear space R is said to be a *modular* on R, if

1) $m(0) = 0$;
2) $m(-a) = m(a)$ for every $a \in R$;
3) for any $a \in R$ we can find a positive number α such that
$$m(\alpha a) < +\infty;$$
4) $m(\xi a) = 0$ for every positive number ξ implies $a = 0$;
5) $\alpha + \beta = 1$, $\alpha, \beta \geqq 0$ implies for every $a, b \in R$
$$m(\alpha a + \beta b) \leqq \alpha m(a) + \beta m(b);$$
6) $m(a) = \sup_{0 \leqq \xi < 1} m(\xi a)$ for every $a \in R$.

For universally continuous semi-ordered linear spaces R, modulars were considered with adding conditions: 7) $|a| \leqq |b|$ implies $m(a) \leqq m(b)$, 8) $|a| \frown |b| = 0$ implies $m(a+b) = m(a) + m(b)$, and 9) $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda} a$ implies $m(a) = \sup_{\lambda \in \Lambda} m(a_\lambda)$. (cf. [2])

In this paper we shall discuss modulars on lattice ordered linear spaces only with adding condition 7).

§ 1. Modulars on linear spaces

Firstly we shall give a rough sketch of the properties of modulars on linear spaces which are obtained in [1] and [3], and will be used in this paper. Let $m(x)$ $(x \in R)$ be a modular on a linear space R. A linear functional $\bar{x}(x)$ $(x \in R)$ on R is said to be *modular bounded*, if we can find positive numbers α, β such that

$$\alpha \bar{x}(x) \leqq \beta + m(x) \qquad \text{for every } x \in R.$$

The totality of modular bounded linear functionals on R also builds a linear space which will be called the *modular associated space* of R and denoted by $\bar{R}$. For each $\bar{a} \in \bar{R}$, putting

$$\bar{m}(\bar{a}) = \sup_{x \in R} \{\bar{a}(x) - m(x)\}$$

we obtain a modular $\bar{m}$ on $\bar{R}$, which will be called the *conjugate modular* of m. Then we have the reflexive relation:

$$m(a) = \sup_{\bar{x} \in \bar{R}} \{\bar{x}(a) - \bar{m}(\bar{x})\} \qquad (a \in R)$$

Putting

$$(1) \qquad |||x||| = \inf_{m(\xi x) \leqq 1} \frac{1}{\xi} \qquad (x \in R)$$

we obtain a norm on R, which will be called the *second norm* of m. Concerning the second norm, we have

$$m(x) \leqq |||x||| \qquad \text{if} \quad |||x||| \leqq 1,$$

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Putting

$$\|a\| = \sup_{\bar{m}(\bar{x}) \leqq 1} |\bar{x}(a)| \qquad (a \in R),$$

we also obtain another norm on R, which will be called the *first norm* of m. Between the first and the second norms there is the relation:

$$|||x||| \leqq \|x\| \leqq 2\,|||x||| \qquad (x \in R).$$

The first norm also may be defined as

$$(2) \qquad \|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \qquad (x \in R).$$

For the first and the second norm of the conjugate modular $\bar{m}$ we have

$$\begin{aligned} \|x\| &= \sup_{|||\bar{x}||| \leqq 1} |\bar{x}(x)|, \qquad & |||x||| &= \sup_{\|\bar{x}\| \leqq 1} |\bar{x}(x)| \\ \|\bar{x}\| &= \sup_{|||x||| \leqq 1} |\bar{x}(x)|, \qquad & |||\bar{x}||| &= \sup_{\|x\| \leqq 1} |\bar{x}(x)| \end{aligned} \qquad (x \in R,\ \bar{x} \in \bar{R})).$$

A linear functional $\bar{x}$ on R is modular bounded if and only if $\bar{x}$ is norm bounded, taht is,

$$\sup_{m(x) \leqq 1} |\bar{x}(x)| < +\infty \qquad (x \in R).$$

A sequence $x_\nu \in R$ $(\nu = 1, 2, \cdots)$ is said to be *modular convergent* to $x \in R$, if

$$\lim_{\nu \to \infty} m(\xi(x_\nu - x)) = 0 \qquad \text{for every } \xi > 0.$$

With this definition we have that a sequence $x_\nu \in R$ $(\nu=1, 2, \cdots)$ is modular convergent to $x \in R$ if and only if it is *norm convergent*, that is,

$$\lim_{\nu \to \infty} |||x_\nu - x||| = 0 .$$

A modular m on R is said to be *complete*, if

$$\lim_{\nu, \mu \to \infty} m(\xi(x_\nu - x_\mu)) = 0 \qquad \text{for every } \xi > 0$$

implies the modular convergence of the sequence $x_\nu \in R$ $(\nu = 1, 2, \cdots)$. With this definition, a modular m on R is complete if and only if the first or second norm of m is complete. The conjugate modular $\overline{m}$ of any modular m on R is always complete on $\overline{R}$.

From the postulate 5) we conclude easily for $0 < \varepsilon \leqq 1$

$$(3) \qquad m(x) \leqq m(y) + \frac{\varepsilon}{1+\varepsilon} m((1+\varepsilon)y) + \frac{\varepsilon^2}{1+\varepsilon} m\left(\frac{1+\varepsilon}{\varepsilon^2}(x-y)\right) .$$

§ 2. Monotone modulars

Let R be a lattice ordered linear space. A modular m on R is said to be *monotone* if $|x| \leqq |y|$ implies $m(x) \leqq m(y)$. With this definition we have obviously by the formulas (1) and (2) in §1 that if a modular m on R is monotone, then both the first and the second norm of m are monotone too, that is, $|x| \leqq |y|$ implies $\|x\| \leqq \|y\|$ and $|||x||| \leqq |||y|||$.

A modular m on R is said to be *upper semi-continuous*, if m is monotone and $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies

$$m(x) = \sup_{\lambda \in \Lambda} m(x_\lambda) .$$

Theorem 2.1. *If a modular m on R is upper semi-continuous, then the second norm of m is semi-continuous, that is,* $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ *implies* $\sup_{\lambda \in \Lambda} |||x_\lambda||| = |||x|||$.

Proof. If $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ and $\sup_{\lambda \in \Lambda} |||x_\lambda||| < |||x|||$, then we can find a positive number α such that

$$\sup_{\lambda \in \Lambda} |||\alpha x_\lambda||| < 1 < |||\alpha x||| .$$

Thus we have for such α

$$\sup_{\lambda \in \Lambda} m(\alpha x_\lambda) \leqq 1 < m(\alpha x) , \qquad 0 \leqq \alpha x_\lambda \uparrow_{\lambda \in \Lambda} \alpha x .$$

Therefore we obtain our assertion.

A modular m on R is said to be *lower semi-continuous*, if m is monotone and $x_\lambda \downarrow_{\lambda \in \Lambda} 0$, $m(x_\lambda) < +\infty$ for every $\lambda \in \Lambda$ implies $\inf_{\lambda \in \Lambda} m(x_\lambda) = 0$. If a modular m on R is upper and lower semi-continuous simultaneously, then m is said to be *semi-continuous.*

A modular m on R is said to be *continuous,* if m is monotone and $x_\lambda \downarrow_{\lambda\in\Lambda} 0$ implies always $\inf_{\lambda\in\Lambda} m(x_\lambda) = 0$.

Theorem 2.2. *Every continuous modular is semi-continuous.*

Proof. If a modular m on R is continuous, then m is obviously lower semi-continuous by definition. Since m is monotone, we have for $0 \leqq x_\lambda \uparrow_{\lambda\in\Lambda} x$

$$\sup_{\lambda\in\Lambda} m(x_\lambda) \leqq m(x).$$

On the other hand we have by the formula (3) for $0 < \varepsilon \leqq 1$

$$m\left(\frac{1}{1+\varepsilon} x\right) \leqq m\left(\frac{1}{1+\varepsilon} x_\lambda\right) + \frac{\varepsilon}{1+\varepsilon} m(x_\lambda) + \frac{\varepsilon^2}{1+\varepsilon} m\left(\frac{1}{\varepsilon^2}(x - x_\lambda)\right)$$
$$\leqq \frac{1+2\varepsilon}{1+\varepsilon} \sup_{\lambda\in\Lambda} m(x_\lambda) + \frac{\varepsilon^2}{1+\varepsilon} m\left(\frac{1}{\varepsilon^2}(x - x_\lambda)\right).$$

Since $\frac{1}{\varepsilon^2}(x - x_\lambda) \downarrow_{\lambda\in\Lambda} 0$, we obtain by assumption

$$m\left(\frac{1}{1+\varepsilon} x\right) \leqq \frac{1+2\varepsilon}{1+\varepsilon} \sup_{\lambda\in\Lambda} m(x_\lambda).$$

This relation yields $m(x) \leqq \sup_{\lambda\in\Lambda} m(x)$, because $\sup_{\varepsilon>0} m\left(\frac{1}{1+\varepsilon} x\right) = m(x)$ by the postulate 6). Therefore m is upper semi-continuous too.

Theorem 2.3. *A monotone modular m on R is continuous, if and only if the first or the second norm of m is continuous*: $x_\lambda \downarrow_{\lambda\in\Lambda} 0$ *implies*

$$\inf_{\lambda\in\Lambda} \|x\| = 0 \quad \textit{or} \quad \inf_{\lambda\in\Lambda} |||x||| = 0.$$

Proof. It is obvious that $\inf_{\lambda\in\Lambda} \|x_\lambda\| = 0$ is equivalent to $\inf_{\lambda\in\Lambda} |||x_\lambda||| = 0$. If m is continuous, then for $x_\lambda \downarrow_{\lambda\in\Lambda} 0$ we have $\nu x_\lambda \downarrow_{\lambda\in\Lambda} 0$ for every $\nu = 1, 2, \cdots$, and hence we can find $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots)$ such that $m(\nu x_{\lambda_\nu}) \leqq 1$ $(\nu = 1, 2, \cdots)$. Then we have $|||\nu x_{\lambda_\nu}||| \leqq 1$, namely $|||x_{\lambda_\nu}||| \leqq \frac{1}{\nu}$ for every $\nu = 1, 2, \cdots$, and this relation yields $\inf_{\lambda\in\Lambda} |||x_\lambda||| = 0$. Thus the second norm of m is continuous.

Conversely, if the second norm of m is continuous, then for $x_\lambda \downarrow_{\lambda\in\Lambda} 0$ we can find $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots)$ such that $|||\nu x_{\lambda_\nu}||| \leqq 1$, and hence

$$m(x_{\lambda_\nu}) \leqq \frac{1}{\nu} m(\nu x_{\lambda_\nu}) \leqq \frac{1}{\nu}$$

for every $\nu = 1, 2, \cdots$. This relation yields $\inf_{\lambda \in \Lambda} m(x_\lambda) = 0$. Thus m is continuous by definition.

A monotone modular m on R is said to be *monotone complete*, if

$$0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda}, \qquad \sup_{\lambda \in \Lambda} m(x_\lambda) < +\infty$$

implies the existence of $\bigcup_{\lambda \in \Lambda} x_\lambda$. If m is monotone complete, then R must be universally continuous, because $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda}$, $x_\lambda \leqq x\ (\lambda \in \Lambda)$ implies $\sup_{\lambda \in \Lambda} m(\alpha x_\lambda) < +\infty$ for some positive number α such that $m(\alpha x) < +\infty$.

Theorem 2.4. *A monotone modular m on R is monotone complete if and only if the first or the second norm of m is monotone complete.*

Proof. If $\sup_{\lambda \in \Lambda} m(x_\lambda) \leqq \alpha$ for some $\alpha > 1$, then we have

$$m\left(\frac{1}{\alpha} x_\lambda\right) \leqq \frac{1}{\alpha} m(x_\lambda) \leqq 1 \qquad \text{for every } \lambda \in \Lambda$$

and hence $\sup_{\lambda \in \Lambda} \left\| \left\| \frac{1}{\alpha} x_\lambda \right\| \right\| \leqq 1$, that is, $\sup_{\lambda \in \Lambda} \|| x_\lambda \|| \leqq \alpha$. Conversely if $\sup_{\lambda \in \Lambda} \| x_\lambda \| \leqq \alpha$ for some $\alpha > 0$, then we have

$$\sup_{\lambda \in \Lambda} m\left(\frac{1}{\alpha} x_\lambda\right) \leqq 1 .$$

Therefore we can conclude our assertion.

Theorem 2.5. *For any monotone modular m on R, its conjugate modular $\overline{m}$ is upper semi-continuous and monotone complete.*

Proof. The modular associated space $\overline{R}$ of R is always universally continuous. (cf. [2]) The conjugate modular $\overline{m}$ is obviously monotone by definition. If $0 \leqq \overline{x}_\lambda \uparrow_{\lambda \in \Lambda} \overline{x}$, then we have

$$\begin{aligned} \overline{m}(\overline{x}) &= \sup_{x \in R} \{\overline{x}(x) - m(x)\} = \sup_{0 \leqq x \in R} \{\sup_{\lambda \in \Lambda} \overline{x}_\lambda(x) - m(x)\} \\ &= \sup_{\lambda \in \Lambda} \{\sup_{0 \leqq x \in R} \{\overline{x}_\lambda(x) - m(x)\}\} = \sup_{\lambda \in \Lambda} \overline{m}(\overline{x}_\lambda) . \end{aligned}$$

Thus $\overline{m}$ is upper semi-continuous. The first norm of $\overline{m}$ is the conjugate norm of the second norm of m, and hence monotone complete. (cf. [2]) Thus $\overline{m}$ is monotone complete by Theorem 2.4.

§ 3. Reflexivity of upper semi-continuous modulars

Now we suppose that R is a universally continuous linear space and m is a monotone modular on R. The totality of universally continuous linear functionals on R, which are modular bounded, is called the *modular conjugate space* of R and denoted by $\bar{R}$. $\bar{R}$ is a normal manifold of the modular associated space $\tilde{R}$ of R. If m is continuous, then the second norm of m also is continuous by Theorem 2.3, and hence $\bar{R}=\tilde{R}$.

Theorem 3.1. *If R is semi-regular and m is upper semi-continuous, then m is reflexive, that is, we have for every $x \in R$*

$$m(x) = \sup_{\bar{x}\in\bar{R}} \{\bar{x}(x)-\bar{m}(\bar{x})\}$$

Proof. For any $0 \neq \bar{a} \in \bar{R}$ and $\nu=1,2,\cdots$, putting

$$m_\nu(x) = \inf_{|x|=|y|+|z|} \mathrm{Max}\,\{m(y),\ 2^\nu |\bar{a}|(|z|)\} \qquad \text{for } x \in [\bar{a}]R,$$

we obtain a monotone modular m_ν on $[\bar{a}]R$. Indeed we see easily that m_ν satisfies the all postulates except for 4). If $m_\nu(x)=0$ and $x\in[\bar{a}]R$, then we can find $0 \leqq y_\mu,\ z_\mu \in R(\mu=1,2,\cdots)$ such that

$$|x| = y_\mu + z_\mu, \quad \mathrm{Max}\,\{m(y_\mu),\ 2^\nu|\bar{a}|(z_\mu)\} \leqq \frac{1}{2^\mu}$$

and putting $u_\mu = \bigcup_{\rho\geqq\mu} z_\rho$ $(\mu=1,2,\cdots)$, we have

$$2^\nu|\bar{a}|(u_\mu) \leqq \sum_{\rho\geqq\mu}\frac{1}{2^\rho} \qquad (\mu=1,2,\cdots)$$

and hence $2^\nu|\bar{a}|\left(\bigcap_{\mu=1}^{\infty} u_\mu\right)=0$. This relation yields $\bigcap_{\mu=1}^{\infty} u_\mu = 0$, that is, $u_\mu \downarrow_{\mu=1}^{\infty} 0$. Thus we have $|x|-u_\mu \uparrow_{\mu=1}^{\infty} |x|$ and

$$m(|x|-u_\mu) \leqq m(y_\mu) \leqq \frac{1}{2^\mu} \qquad (\mu=1,2,\cdots).$$

Therefore we obtain $m(x)=0$, because m is upper semi-continuous by assumption, and we conclude that $m_\nu(x)=0$ and $x\in[\bar{a}]R$ implies $m(x)=0$. Consequently the postulate 4) also is satisfied.

The modular m_ν on $[\bar{a}]R$ is continuous for every $\nu=1,2,\cdots$, because we have obviously

$$m_\nu(x) \leqq 2^\nu|\bar{a}|(|x|) \qquad \text{for every } x\in[\bar{a}]R.$$

Thus the modular associated space $\tilde{R}_\nu$ of $[\bar{a}]R$ by m_ν coincides with the modular conjugate space of $[\bar{a}]R$ by m_ν and hence $\tilde{R}_\nu$ is included in the modular conjugate space $\bar{R}$ of R by m, because we have obviously

$$m_\nu(x) \leqq m(x) \qquad \text{for every } x\in[\bar{a}]R.$$

Therefore we have for every $x\in[\bar{a}]R$

$$m_\nu(x) = \sup_{\bar{x}\in\bar{R}_\nu} \{\bar{x}(x) - \bar{m}_\nu(\bar{x})\} \leqq \sup_{\bar{x}\in\bar{R}} \{\bar{x}(x) - \bar{m}(\bar{x})\} ,$$

because we have for $\bar{x}\in\bar{R}_\nu$

$$\bar{m}(\bar{x}) = \sup_{x\in[\bar{a}]R} \{\bar{x}(x) - m(x)\} \leqq \sup_{x\in[\bar{a}]R} \{\bar{x}(x) - m_\nu(x)\} = \bar{m}_\nu(\bar{x}) .$$

On the other hand we have

$$\lim_{\nu\to\infty} m_\nu(x) = m(x) \qquad \text{for every } x\in[\bar{a}]R .$$

Because, for any $x\in[\bar{a}]R$ we can find $0\leqq y_\nu, z_\nu\in R$ ($\nu=1, 2, \cdots$) such that

$$|x| = y_\nu + z_\nu, \quad m(y_\nu) \leqq m_\nu(x) + \frac{1}{2^\nu}, \quad 2^\nu |\bar{a}|(z_\nu) \leqq m_\nu(x) + \frac{1}{2^\nu} .$$

Then putting $u_\nu = \bigcup_{\rho\geqq\nu} z_\rho$ ($\nu=1, 2, \cdots$), we conclude $u_\nu \downarrow_{\nu=1}^\infty 0$ and

$$m(|x| - u_\nu) \leqq m(y_\nu) \leqq m_\nu(x) + \frac{1}{2^\nu} \leqq m(x) + \frac{1}{2^\nu} ,$$

as obtained above. This relation yields $m(x) = \lim_{\nu\to\infty} m_\nu(x)$. Therefore we conclude

$$m(x) \leqq \sup_{\bar{x}\in\bar{R}} \{\bar{x}(x) - \bar{m}(\bar{x})\} \qquad \text{for every } x\in[\bar{a}]R .$$

Since R is semi-regular by assumption, we have $[\bar{a}]x \uparrow_{\bar{a}\in\bar{R}} x$, and hence we obtain furthermore

$$m(x) \leqq \sup_{\bar{x}\in\bar{R}} \{\bar{x}(x) - \bar{m}(\bar{x})\} \qquad \text{for every } x\in R .$$

On the other hand it is obvious by definition

$$m(x) \geqq \sup_{\bar{x}\in\bar{R}} \{\bar{x}(x) - \bar{m}(\bar{x})\} \qquad \text{for every } x\in R .$$

Thus we conclude our assertion.

Recalling Theorem 2.4, we obtain immediately

Theorem 3.2. *If R is semi-regular and m is upper semi-continuous and monotone complete, then R is reflexive and the modular conjugate space of R by m coincides with the conjugate space of R.*

§ 4. Semi-additive modulars

A modular m on a lattice ordered linear space R is said to be *upper semi-additive*, if m is monotone and

$$m(a+b) \geqq m(a) + m(b) \qquad \text{for } 0\leqq a, b\in R .$$

Theorem 4.1. *If an upper semi-additive modular m is upper semi-*

continuous, then m *is lower semi-continuous, and hence semi-continuous.*

Proof. For $x_\lambda \downarrow_{\lambda\in\Lambda} 0$, $m(x_\lambda) < +\infty$ $(\lambda\in\Lambda)$ we have

$$m(x_\lambda) \leqq m(x_{\lambda_0}) - m(x_{\lambda_0} - x_\lambda) \qquad \text{for } x_\lambda \leqq x_{\lambda_0},$$

because m is upper semi-additive by assumption. Since

$$x_{\lambda_0} - x_\lambda \uparrow_{x_\lambda \leqq x_{\lambda_0}} x_{\lambda_0}$$

and m is upper semi-continuous by assumption, we have

$$\sup_{x_\lambda \leqq x_{\lambda_0}} m(x_{\lambda_0} - x_\lambda) = m(x_{\lambda_0}) .$$

Thus we obtain $m(x_\lambda) \downarrow_{\lambda\in\Lambda} 0$.

A modular m on R is said to be *lower semi-additive,* if m is monotone and

$$m(a \smile b) \leqq m(a) + m(b) \qquad \text{for } 0 \leqq a, b \in R .$$

A modular m on R is said to be *additive,* if m is upper and lower semi-additive simultaneously. Additive modulars are discussed in detail already in [2]. When R is universally continuous, if a modular m on R is upper semi-continuous and

$$m(a+b) = m(a) + m(b) \qquad \text{for } a \frown b = 0 ,$$

then m is additive. (cf. [2])

Theorem 4.2. *The conjugate modulars of the upper semi-additive modulars are lower semi-additive, and the conjugate modulars of the lower semi-additive modulars are upper semi-additive.*

Proof. If a modular m on R is upper semi-additive, then for the conjugate modular $\overline{m}$ of m and the modular associated space $\overline{R}$ of R we have by definition for $0 \leqq \overline{x}, \overline{y} \in \overline{R}$

$$\begin{aligned}\overline{m}(\overline{x}) + \overline{m}(\overline{y}) &= \sup_{x, y \in R} \{\overline{x}(x) + \overline{y}(y) - m(x) - m(y)\} \\ &\geqq \sup_{0 \leqq z \in R} \{ \sup_{\substack{z = x+y, \\ x, y \geqq 0}} \{\overline{x}(x) + \overline{y}(y)\} - m(z)\} \\ &= \sup_{z \in R} \{\overline{x} \smile \overline{y}(z) - m(z)\} = \overline{m}(\overline{x} \smile \overline{y}) ,\end{aligned}$$

and hence $\overline{m}$ is lower semi-additive by definition. If m is lower semi-additive, then we have by definition for $0 \leqq \overline{x}, \overline{y} \in \overline{R}$

$$\begin{aligned}\overline{m}(\overline{x}) + \overline{m}(\overline{y}) &= \sup_{0 \leqq x, y \in R} \{\overline{x}(x) + \overline{y}(y) - m(x) - m(y)\} \\ &\leqq \sup_{0 \leqq x, y \in R} \{\overline{x}(x \smile y) + \overline{y}(x \smile y) - m(x \smile y)\}\end{aligned}$$

$$\leqq \sup_{z\in R} \{\bar{x}(z)+\bar{y}(z)-m(z)\} = \bar{m}(\bar{x}+\bar{y}),$$

and hence $\bar{m}$ is upper semi-additive by definition.

§ 5. Bimodulars

Let R, S be two lattice ordered linear spaces. A functional $M(x, y)$ ($x\in R$, $y\in S$) is said to be a *bimodular*, if $M(x, y)$ is an additive upper semi-continuous modular on R for every fixed $0 \neq y\in S$,

$$M(x, |y_1|+|y_2|) = M(x, y_1)+M(x, y_2),$$

$$M(x, \beta y) = |\beta| M(x, y),$$

and for any $x\in R$ we can find a positive number α such that

$$M(\alpha x, y)< +\infty \qquad \text{for every } y\in S.$$

A bimodular $M(x, y)$ ($x\in R$, $y\in S$) is said to be *finite*, if

$$M(x, y)< +\infty \qquad \text{for every } x\in R, y\in S.$$

If S is a normed space and complete, then putting

$$m(x) = \sup_{\|y\|\leqq 1} M(x, y) \qquad (x\in R, \ y\in S),$$

we obtain a modular m on R. This modular m is said to be a *norm-modular* of M by the norm of S.

Theorem 5.1. *Every norm-modular of a bimodular* $M(x, y)$ ($x\in R$, $y\in S$) *is lower semi-additive and upper semi-continuous.*

Proof. For $0\leqq x_1, x_2\in R$ we have by definition

$$m(x_1\smile x_2) = \sup_{\|y\|\leqq 1} M(x_1\smile x_2, y)$$
$$\leqq \sup_{\|y\|\leqq 1} M(x_1, y)+\sup_{\|y\|\leqq 1} M(x_2, y)= m(x_1)+m(x_2),$$

because $M(x_1\smile x_2, y)\leqq M(x_1, y)+M(x_2, y)$. Thus the norm-modular m is lower semi-additive. For $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda} x$ we have by definition

$$m(x) = \sup_{\|y\|\leqq 1} M(x, y) = \sup_{\|y\|\leqq 1} \{\sup_{\lambda\in\Lambda} M(x_\lambda, y)\}$$
$$= \sup_{\lambda\in\Lambda} \{\sup_{\|y\|\leqq 1} M(x_\lambda, y)\} = \sup_{\lambda\in\Lambda} m(x_\lambda).$$

Thus the norm-modular m is upper semi-continuous by definition.

Theorem 5.2. *If a bimodular* $M(x, y)$ ($x\in R$, $y\in S$) *is finite, then the norm-modular of* M *is finite.*

Proof. For each $x\in R$, since $M(x, y)< +\infty$ by assumption, putting

$$x(y) = M(x, y^+) - M(x, y^-) \qquad (y \in S)$$

we obtain a positive linear functional x on S. Since the norm of S is complete by assumption, this linear functional x on S is norm bounded, and hence

$$\sup_{\|y\| \leqq 1} M(x, y) < +\infty \qquad \text{for every } x \in R.$$

Thus the norm-modular of M is finite by definition.

For an additive complete modular m_S on S, putting

$$m(x) = \sup_{y \in S} \{M(x, y) - m_S(y)\} \qquad (x \in R)$$

we obtain a monotone modular m on R. This modular m on R is said to be the *double-modular* of M by m_S.

Theorem 5.3. *Every double-modular of a bimodular $M(x, y)$ $(x \in R, y \in S)$ is upper semi-additive and semi-continuous.*

Proof. For $0 \leqq x_1, x_2 \in R$ we have by definition

$$\begin{aligned} m(x_1 + x_2) &= \sup_{y \in S} \{M(x_1 + x_2, y) - m_S(y)\} \\ &\geqq \sup_{y \in S} \{M(x_1, y) + M(x_2, y) - m_S(y)\} \\ &\geqq \sup_{0 \leqq y_1, y_2 \in S} \{M(x_1, y_1) + M(x_2, y_2) - m_S(y_1 \smile y_2)\} \\ &\geqq \sup_{0 \leqq y_1, y_2 \in S} \{M(x_1, y_1) + M(x_2, y_2) - m_S(y_1) - m_S(y_2)\} = m(x_1) + m(x_2), \end{aligned}$$

because

$$M(x_1 + x_2, y) \geqq M(x_1, y) + M(x_2, y),$$

$$m_S(y_1 \smile y_2) \leqq m_S(y_1) + m_S(y_2).$$

Thus the double-modular m is upper semi-additive. For $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ we have by definition

$$m(x) = \sup_{y \in S} \left\{ \sup_{\lambda \in \Lambda} \{M(x_\lambda, y) - m_S(y)\} \right\} = \sup_{\lambda \in \Lambda} m(x_\lambda).$$

Thus m is upper semi-continuous. Recalling Theorem 4.1, we conclude therefore that m is semi-continuous.

Theorem 5.4. *Let m_S be a complete, additive modular on S. For a bimodular $M(x, y)$ $(x \in R, y \in S)$, denoting by m_d the double-modular of M by m_S and by m_n the norm-modular of M by the first norm of m_S, then we have*

$$m_d(x) \leqq m_n(x) \qquad \textit{for} \quad m_n(x) \leqq 1,$$

$$m_d(x) \geqq m_n(x) \qquad \textit{for} \quad m_n(x) \geqq 1,$$

and the second norm of m_d coincides with that of m_n.

Proof. If $M(x, y) < +\infty$ for every $y \in S$, then putting

$$x(y) = M(x, y^+) - M(x, y^-) \qquad (y \in S)$$

we obtain a positive linear functional x on S. Since the modular m_S is complete by assumption, this linear functional x is modular bounded. Thus we have by definition

$$m_d(x) = \overline{m}_S(x), \qquad m_n(x) = |||x|||$$

for the conjugate modular $\overline{m}_S$ of m_S and the second norm $|||x|||$ of $\overline{m}_S$. If $M(x, y) = +\infty$ for some $y \in S$, then we have obviously by definition

$$m_d(x) = m_n(x) = +\infty .$$

Therefore we conclude that $m_n(x) \leqq 1$ implies $m_d(x) \leqq m_n(x)$, and that $m_n(x) \geqq 1$ implies $m_d(x) \geqq m_n(x)$. Consequently the second norm of m_d coincides with that of m_n.

§ 6. Proper bimodular

Let m be an additive upper semi-continuous modular on a universally continuous semi-ordered linear space R, and $\mathfrak{S}$ the proper space of R. We denote by D_m the totality of such dilatators T in R that for any $x \in R$ we can find a positive number α for which

$$\int_{\mathfrak{S}} \left(\frac{|T|}{1}, \mathfrak{p} \right) m(\alpha d\mathfrak{p}x) < +\infty .$$

Then, putting

$$M_m(x, T) = \int_{\mathfrak{S}} \left(\frac{|T|}{1}, p \right) m(d\mathfrak{p}x) \qquad (x \in R,\ T \in D_m)$$

we obtain a bimodular M_m. Here we see easily that D_m is a semi-normal manifold of the dilatator space and $1 \in D_m$, because $M_m(x, 1) = m(x)$. This bimodular M_m is said to be the *proper bimodular* of m.

For a semi-normal manifold D of D_m containing 1, and for a complete norm $\|T\|$ $(T \in D)$ on D, putting

$$m_n(x) = \sup_{\|T\| \leqq 1, T \in D} M_m(x, T) \qquad (x \in R),$$

we obtain a norm-modular m_n of M_m.

Theorem 6.1. *If the modular m on R is monotone complete, then every norm-modular of the proper bimodular M_m of m also is monotone complete.*

Proof. If $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda}$, $\sup_{\lambda \in \Lambda} m_n(x_\lambda) < +\infty$, then we have by definition

$$\sup_{\lambda \in \Lambda} m(x_\lambda) = \sup_{\lambda \in \Lambda} M_m(x_\lambda, 1) < +\infty$$

and hence x_λ $(\lambda \in \Lambda)$ is bounded, because m is monotone complete by assumption. Therefore the norm-modular m_n also is monotone complete.

For a complete, additive modular $m_D(T)$ $(T \in D)$ on D, putting

$$m_d(x) = \sup_{T \in D} \{M_m(x, T) - m_D(T)\} \qquad (x \in R),$$

we obtain a double-modular m_d of M_m.

Theorem 6.2. *Every double-modular of the proper bimodular M_m of m also is additive.*

Proof. If $M_m(x, T) < +\infty$ for every $T \in D$, then, putting

$$x(T) = \int_{\mathfrak{S}} \left(\frac{T}{1}, \mathfrak{p}\right) m(d\mathfrak{p}x) \qquad (T \in D),$$

we obtain a positive linear functional $x(T)$ $(T \in D)$ on D. Furthermore if

$$x \frown y = 0, \quad M_m(x, T) < +\infty, \quad M_m(y, T) < +\infty \qquad \text{for every } T \in D,$$

then we also have $x \frown y = 0$ considering both x and y linear functionals on D, and hence

$$\overline{m}_D(x+y) = \overline{m}_D(x) + \overline{m}_D(y)$$

for the conjugate modular $\overline{m}_D$ of m_D, because m_D is additive by assumption. On the other hand we have by definition

$$m_d(x) = \begin{cases} \overline{m}_D(x) & \text{if } M_m(x, T) < +\infty \quad \text{for every } T \in D, \\ +\infty & \text{if } M_m(x, T) = +\infty \quad \text{for some } T \in D. \end{cases}$$

Thus we conclude that $x \frown y = 0$ implies $m_d(x+y) = m_d(x) + m_d(y)$. Therefore the double-modular m_d is additive. (cf. [2])

Theorem 6.3. *If the modular m on R is monotone complete, then every double-modular of the proper bimodular M_m of m also is monotone complete.*

Proof. For an additive complete modular m_D on D, we can find a positive number α such that $m_D(\alpha) < +\infty$ considering α a dilatator in R. If $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda}$ and $\sup_{\lambda \in \Lambda} m_d(x_\lambda) < +\infty$, then we have

$$\sup_{\lambda \in \Lambda} m(x_\lambda) = \frac{1}{\alpha} \sup_{\lambda \in \Lambda} M_m(x_\lambda, \alpha) \leqq \frac{1}{\alpha} \{\sup_{\lambda \in \Lambda} m_d(x_\lambda) + m_D(\alpha)\} < +\infty,$$

and hence x_λ $(\lambda \in \Lambda)$ is bounded, because m is monotone complete by assumption. Therefore the double-modular m_d also is monotone complete by definition.

References

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[2] H. NAKANO: Modulared semi-ordered linear spaces, Tokyo (1950).

[3] H. NAKANO: Modern spectral theory, Tokyo (1950).