ON CERTAIN PROPERTY OF THE NORMS BY MODULARS

By

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Let R be a universally continuous semi-ordered linear space. A functional $m(a)(a\in R)$ is said to be a modular[1] on R if it satisfies the following modular conditions:

(1) $0\leqq m(a)\leqq\infty$ for all $a\in R$;

(2) if $m(\xi a)=0$ for all $\xi>0$, then $a=0$;

(3) for any $a\in R$ there exists $\alpha>0$ such that $m(\alpha a)<\infty$;

(4) for every $a\in R$, $m(\xi a)$ is a convex function of ξ;

(5) $|a|\leqq|b|$ implies $m(a)\leqq m(b)$;

(6) $a\frown b=0$ implies $m(a+b)=m(a)+m(b)$;

(7) $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}a$ implies $m(a)=\sup_{\lambda\in\Lambda}m(a_\lambda)$.

In R, we define functionals $\|a\|$, $|||a|||$ $(a\in R)$ as follows

$$\|a\|=\inf_{\xi>0}\frac{1+m(\xi a)}{\xi},\qquad |||a|||=\inf_{m(\xi a)\leqq 1}\frac{1}{|\xi|}.$$

Then it is easily seen that both $\|a\|$ and $|||a|||$ are norms on R and $|||a|||\leqq\|a\|\leqq 2|||a|||$ for all $a\in R$. $\|a\|$ is said to be the first norm by m and $|||a|||$ is said to be the second norm by m. Let $\overline{R}^m$ be the modular conjugate space of R and $\overline{m}$ be the conjugate modular of m[2] then we can introduce the norms by $\overline{m}$ as above. It is known that if R is semi-regular, the first norm by the conjugate modular $\overline{m}$ is the conjugate norm of the second norm by m and the second norm by the conjugate modular $\overline{m}$ is the conjugate norm of the first norm by m. Since $\|a\|$ and $|||a|||$ are semi-continuous by (7), they are reflexive norms (cf. [7]).

If a modular m is of L_p-type, i.e., $m(\xi x)=\xi^p m(x)$ for all $x\in R$, $\xi\geqq 0$,

1) We owe the notations and the terminologies using here to the book: H. NAKANO [3].

2) The conjugate modular $\overline{m}$ is defined as $\overline{m}(\bar{a})=\sup_{x\in R}\{\bar{a}(x)-m(x)\}$ for every $\bar{a}\in\overline{R}^m$, where $\overline{R}^m$ is the space of the modular bounded universally continuous linear functionals on R.

then we have $\frac{\|x\|}{|||x|||}=p^{\frac{1}{p}}q^{\frac{1}{q}}$ for all $0\neq x\in R$, where $1\leqq p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$ (In the case of $p=1$, we have $\frac{\|x\|}{|||x|||}=1$). The converse of this is studied by S. YAMAMURO [5] and I. AMEMIYA [1]. They proved that if the ratios of two norms are constant for all $0\neq x\in R$, it is of L_p-type essentially. So in the general case, the ratios of two norms are not constant.

A modular m is said to be bounded if there exist real numbers $1<p_1\leqslant p_2<\infty$, such that

$$\xi^{p_1}m(x)\leqq m(\xi x)\leqq \xi^{p_2}m(x)$$

for all $\xi\geqq 1$ and $x\in R$. In [6], S. YAMAMURO obtained that if a modular m on R is bounded then we have

$$\|x\|\geqq \gamma\, |||x|||$$

for all $x\in R$, where $\gamma>1$ is a fixed constant.

In this paper we investigate the case when the two norms by a modular m satisfy

$$\inf_{0\neq x\in R}\frac{\|x\|}{|||x|||}=\gamma>1 \qquad (*)\,.$$

(In this case we say that the norms have property $(*)$ throughout this paper).

As showed above, a bounded modular m has that property $(*)$, but the converse of this is not true in general.

In §1 we prove that if the norms by a modular m satisfy the property $(*)$ then it is uniformly finite and uniformly increasing, provided that R has no atomic element (Theorem 1.1). And we obtain conversely that if a modular m is uniformly finite and uniformly increasing then the norms by m have the property $(*)$ (Theorem 1.4). Thus, we can see that if R has no atomic element, then the property $(*)$ is equivalent to uniform finiteness and uniform increasingness of modular m. Theorem 1.2 shows that uniform simpleness of a modular m implies uniform finiteness, in the case when R has no atomic element. Finally some special cases, where the property $(*)$ is equivalent to boundedness of modular are discussed.

In §2 we define uniform p-properties, that is, uniformly p-finite, p-increasing, p-simple and p-monotone modulars, to determine the degrees of uniform finiteness, increasingness and etc.. Theorems 2.1 and 2.2

show that there exist the conjugate relations between uniformly p-finite modular and uniformly q-increasing modular, where $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. On the other hand, Theorems 2.3 and 2.4 show the similar relations between uniformly p-simple modular and uniformly q-monotone modular. In the case when R has no atomic element, we have more preceisely than in §1, that if a modular m is uniformly p-simple it is uniformly p-finite (Theorem 2.5). There is a modular which is uniformly finite but not uniformly p-finite for any $1\leqq p<\infty$.

In §3 we prove that if the norms by modular m have the property $(*)$ then γ (which appears in $(*)$) determines the degrees of uniform finiteness and uniform increasingness of m. Truely, in the case when R has no atomic element, we obtain that if the norms by a modular m have the property $(*)$, m is uniformly p-increasing and uniformly q-finite, where p, q are positive numbers such that $r=p^{\frac{1}{p}}q^{\frac{1}{q}}$, $\frac{1}{p}+\frac{1}{q}=1$, $p\leqq q$ (Theorem 3.1). The converse of this is not true in general. We show an example of this fact at the end of this paper.

§1. Let R be a modulared semi-ordered linear space with a modular m.

A modular m is said to be uniformly finite, if

$$\sup_{m(x)\leqq 1} m(\xi x)<\infty \qquad \text{for all } \xi>0 .$$

A modular m is said to be uniformly increasing, if

$$\lim_{\xi\to\infty} \inf_{m(x)\geqq 1} \frac{m(\xi x)}{\xi}=+\infty .$$

In [4; Theorems 5.2, 5.3] it is shown that if a modular m is uniformly finite, then the conjugate modular $\overline{m}$ of m is uniformly increasing and if a modular m is uniformly increasing then the conjugate modular $\overline{m}$ is uniformly finite.

Now we shall prove the following

Theorem 1.1. *Suppose R has no atomic element. If the norms by a modular m have the property $(*)$, then m is uniformly finite and uniformly increasing.*

Proof. 1). Let γ be a number, in the sequel, such that $\gamma=\inf_{0\neq x\in R}\frac{\|x\|}{|||x|||}$. Then we have

$$\inf_{0\neq\bar{x}\in \overline{R}^m}\frac{\|\bar{x}\|}{|||\bar{x}|||}=\gamma \qquad (*').$$

In fact we have for every $\bar{x} \in \bar{R}^m$

$$|||\bar{x}||| = \sup_{\|x\| \leqq 1} |\bar{x}(x)| \leqq \sup_{\gamma |||x||| } |\bar{x}(x)| = \frac{1}{\gamma} \|\bar{x}\| .$$

Since the norms $\|x\|$, $|||x|||$ are reflexive, we obtain $(*')$.

2). If m is not uniformly finite, then there exists a number $\xi_0 \geqq 1$ such that

$$\sup_{m(x) \leqq 1} m(\xi x) < +\infty \qquad \text{for all } \xi < \xi_0 ,$$

$$\sup_{m(x) \leqq 1} m(\eta x) = +\infty \qquad \text{for all } \eta > \xi_0 .$$

Since $\gamma > 1$, we obtain a number α such that $1 > \alpha > 0$ and $\alpha\gamma - 1 > 0$, and we can find also $\varepsilon > 0$ such that $\alpha(\xi_0 + \varepsilon) < \xi_0$.
Then by the definition of ξ_0, we can find a sequence of elements $\{x_n\}$ $(n = 1, 2, \cdots)$ such that

$$m(x_n) \leqq 1 , \quad m(\alpha(\xi_0 + \varepsilon) x_n) \leqq k , \quad m((\xi_0 + \varepsilon) x_n) \geqq n \qquad (n = 1, 2, \cdots) ,$$

where k is a fixed positive number.

Since R has no atomic element, we can obtain also a sequence of projectors $\{[p_n]\}$ $(n = 1, 2, \cdots)$ such that

$$m(\alpha(\xi_0 + \varepsilon)[p_n] x_n) \leqq \frac{k}{n} , \quad m((\xi_0 + \varepsilon)[p_n] x_n) \geqq 1 .$$

Putting $y_n = (\xi_0 + \varepsilon)[p_n] x_n$, we have

$$m(y_n) \geqq 1 , \quad m(\alpha y_n) \leqq \frac{k}{n} \qquad (n = 1, 2, \cdots) .$$

This implies $\lim_{n \to \infty} \frac{1 + m(\alpha y_n)}{\alpha} = \frac{1}{\alpha} < \gamma$ and contradicts $(*)$, because on the other hand, we have $|||y_n||| \geqq 1$ and $\|y_n\| \leqq \frac{1 + m(\alpha y_n)}{\alpha}$ for all $n \geqq 1$.

Then by 1) $\bar{m}$ is also uniformly finite, thus m is uniformly increasing[3]. This completes the proof.

In the proof of the theorem above, we have shown that if a modular m is not uniformly finite, then there exists a sequence of elements y_n such that

3) We note here that $\bar{\bar{m}}(x) = \sup_{\bar{x} \in \bar{R}^m} \{\bar{x}(x) - m(x)\} \leqq m(x)$ for all $x \in R$ by virtue of the definition of conjugate modular. If R is semi-regular, then modular m is reflexive; i.e. $m(x) = \bar{\bar{m}}(x) = \sup_{\bar{x} \in \bar{R}^m} \{\bar{x}(x) - m(x)\}$ for all $x \in R$ ([3]; §39).

$$m(y_n) \geqq 1, \quad \lim_{n\to\infty} m(\xi y_n) = 0 \qquad (n=1,2,\cdots)$$

for some $\xi > 0$. Then the sequence $\{y_n\}$ $(n=1,2,\cdots)$ is conditionally modular convergent to 0, but it is not modular convergent. A modular m is said to be uniformly simple if conditionally modular convergence coincides with modular convergence, i.e., $\lim_{n\to\infty} m(x_n)=0$ implies $\lim_{n\to\infty} m(\xi x_n) = 0$ for every $\xi \geqq 0$.

Thus we have

Theorem 1.2. *Suppose that R has no atomic element. If a modular m is uniformly simple, then it is uniformly finite.*

The conjugate property to uniform simpleness of modular is uniform monotoneness.[4] Therefore we obtain also

Theorem 1.3. *Suppose that R has no atomic element. If a modular m is uniformly monotone, then it is uniformly increasing.*

The converse part of Theorem 1.1 is always true (without the assumption that R has no atomic element). That is, we obtain

Theorem 1.4. *If a modular m is uniformly finite and uniformly increasing, then the norms by m have the property $(*)$.*

Proof. If the property $(*)$ is not satisfied, then we can find $x_n \geqq 0$ $(n=1,2,\cdots)$ such that

$$1 \leqq \|x_n\| < 1 + \frac{1}{n}, \quad \|\!|x|\!\| = m(x_n) = 1 \qquad (n=1,2,\cdots).$$

And we can find also $\xi_n > 0$ such that

$$1 + m(\xi_n x_n) < \left(1 + \frac{1}{n}\right)\xi_n$$

for all $n \geqq 1$ by the definition of the first norm.

Considering a subsequence of $\{\xi_n\}$, it is sufficient for us to investigate only the following cases.

1) In this case, $\{\xi_n\}$ satisfies $0 < \xi_n \leqq 1$ for all $n \geqq 1$. If $\xi_n \leqq \xi_0 < 1$ $(n=1,2,\cdots)$ for some $\xi_0 < 1$, then we obtain

$$\left(1 + \frac{1}{n}\right) > \frac{1 + m(\xi_n x_n)}{\xi_n} \geqq \frac{1}{\xi_0} > 1 \qquad (n=1,2,\cdots).$$

This is a contradiction. Now without a loss of a generality, we may

4) A modular m is said to be uniformly monotone, if $\lim_{\xi\to 0} \frac{1}{\xi} \sup_{m(x)\leqq 1} m(\xi x) = 0$.

assume that

$$\xi_n \uparrow 1, \quad 1-\xi_n < \frac{1}{n} \qquad (n=1,2,\cdots).$$

Since we have

$$m(\xi_n x_n) < \left(1+\frac{1}{n}\right)\xi_n - 1 \leqq \frac{1}{n}$$

and $m(\xi x)$ is a non-decreasing convex function of $\xi \geqq 0$, we obtain

$$m((1+(1-\xi_n))x_n) \geqq 1+\frac{n-1}{n} \qquad (n=1,2,\cdots),$$

and furthermore

$$m((1+n(1-\xi_n))x_n) \geqq 1+(n-1) \qquad (n=1,2,\cdots).$$

This implies

$$\sup_{m(x)\leqq 1} m(2x) \geqq \sup_{n=1,2,\cdots} m(2x_n) \geqq \sup_{n=1,2,\cdots} (1+(n-1)) = +\infty,$$

which contradicts that m is uniformly finite.

2). In this case, $\{\xi_n\}$ $(n=1,2,\cdots)$ satisfies $1 \leqq \xi_n$ for all $n \geqq 1$. By definition of $\{\xi_n\}$, we have

$$1+\frac{1}{n} \geqq \frac{1+m(\xi_n x_n)}{\xi_n} \geqq \frac{1}{\xi_n}+1 \qquad \text{for all} \quad n \geqq 1.$$

This implies $n \leqq \xi_n$ for all $n \geqq 1$. Therefore we may assume $\xi_n \uparrow +\infty$ $(n=1,2,\cdots)$, so we obtain

$$\liminf_{\xi \to \infty \, m(x) \geqq 1} \frac{m(\xi x)}{\xi} \leqq \lim_{n\to\infty} \frac{m(\xi_n x_n)}{\xi_n} \leqq \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1,$$

which contradicts that m is uniformly increasing. This completes the proof.

In the case when a modular m on R is of unique spectra ([3]; §54), the property $(*)$ implies boundedness of m. In fact we have

Theorem 1.5. *If a modular m on R is of unique spectra, then boundedness of m is equivalent to the property* $(*)$.

The proof is easily obtained by simple calculations, so it is omitted.

In the case of the constant modular ([3]; §55), the property $(*)$ dose not imply simpleness of m, and even in the case of the simple constant modular it dose not generally imply the boundedness of m (the examples are easily obtained). Only in the particular case, we have

Theorem 1.6. *If a modular m on R is constant, monotone complete and R has neither complete constant element nor atomic element, then the property $(*)$ is equivalent to boundedness of m.*

Proof. By Theorem 1.1 m is finite, then m is upper bounded by Theorem 55.10 in [3]. Since $\bar{m}$ is constant and has no complete constant element [3; §55], $\bar{m}$ is also upper bounded, that is, m is lower bounded. Thus m is a bounded modular on R.

§ 2. In this section we investigate the degrees of uniform properties of modulars.

Set for $\xi \geqq 1$

$$f(\xi) = \sup_{m(x)\leqq 1} m(\xi x) \quad \text{and} \quad g(\xi) = \inf_{m(x)\geqq 1} m(\xi x),$$

then $f(\xi)$ and $g(\xi)$ are defined in $[1, \infty)$ and non-decreasing functions. In the following, let p be a number such that $1<p<\infty$.

Definition 2.1. A modular m on R is said to be *uniformly p-finite* if there exist $\gamma>0$ and $\xi_0\geqq 1$ such that

$$f(\xi) \leqq \gamma\xi^p \qquad \text{for all} \quad \xi \geqq \xi_0 .$$

Definition 2.2. A modular m on R is said to be *uniformly p-increasing,* if there exist $\gamma>0$ and $\xi\geqq 1$ such that

$$g(\xi) \geqq \gamma\xi^p \qquad \text{for all} \quad \xi \geqq \xi_0 .$$

It is easily seen that if m is uniformly p-finite, it is also uniformly p'-finite for $p\leqq p'$, and if m is uniformly p-increasing it is also uniformly p''-increasing for $1\leqq p''\leqq p$.

In the sequel, we set $q = \dfrac{p}{p-1}$. Now we have

Theorem 2.1. *If a modular m is uniformly p-finite, then the conjugate modular $\bar{m}$ of m is uniformly q-increasing.*

Proof. We have by the assumption for some $\rho_0\geqq 1$, $\gamma>0$,

$$f(\xi) \leqq \gamma\xi^p \qquad (\xi \geqq \rho_0 \geqq 1).$$

If $\bar{m}(\bar{x})\geqq 1$, $\bar{x}\in\bar{R}^m$ and $0<\alpha<1$, we can find x_0 such that $\bar{x}(x_0)>\alpha$, $m(x_0)\leqq 1$. For such x_0, we have by the definition of conjugate modular

$$\bar{m}(\lambda\bar{x}) \geqq \lambda\bar{x}(\rho x_0) - m(\rho x_0) \geqq \alpha\lambda\rho - \gamma\rho^p$$

for all $\rho\geqq\rho_0$. This implies

$$\bar{m}(\lambda\bar{x}) \geqq \sup_{\rho\geqq\rho_0} \{\alpha\lambda\rho - \gamma\rho^p\}$$

for all $\bar{x}\in\bar{R}^m$ such that $\bar{m}(\bar{x})\geqq 1$.

Then we have for $\lambda\geqq\lambda_0=\frac{\gamma p}{\alpha}\rho_0^{\frac{p}{q}}$,

$$\bar{m}(\lambda\bar{x})\geqq\frac{\gamma p}{q}\left(\frac{\alpha}{p\gamma}\right)^q\lambda^q.$$

Hence the conjugate modular $\bar{m}$ is uniformly q-increasing modular by definition.

Theorem 2.2. *If a modular m is uniformly p-increasing, then the conjugate modular $\bar{m}$ of m is uniformly q-finite.*

Proof. By the assumption we have for some γ and ρ_0

$$m(x)\geqq 1 \text{ implies } m(\rho x)\geqq\gamma\rho^p \qquad \text{for } \rho\geqq\rho_0 .$$

Set $\lambda_0=\mathrm{Max}\left(\frac{\gamma}{2}\rho_0^{p-1}, 1\right)$ and for $\lambda\geqq\lambda_0$ we define $\rho=\rho(\lambda)$ such that $\rho(\lambda)=\left(\frac{2}{\gamma}\lambda\right)^{\frac{q}{p}}$. Then we have $\rho\geqq\rho_0$. Thus we obtain $\frac{m(\rho x)}{\rho}\geqq\gamma\rho^{p-1}=2\lambda$. If $\bar{x}\in\bar{R}^m$, $\bar{m}(\bar{x})\leqq 1$ and $1\leqq m(x)<+\infty$, then there is $\xi>0$ such that

$$m\left(\frac{1}{\xi}x\right)=1, \quad 0<\frac{1}{\xi}<1$$

and hence by the definition of the conjugate modular $\bar{m}(\bar{x})$ we obtain

$$\bar{x}\left(\frac{1}{\xi}x\right)\leqq\bar{m}(\bar{x})+m\left(\frac{1}{\xi}x\right)\leqq 2 .$$

For such ξ, if $\xi\geqq\rho(\lambda)$, then we have

$$\lambda\bar{x}(x)-m(x)=\xi\left\{\lambda\bar{x}\left(\frac{1}{\xi}x\right)-\frac{1}{\xi}m\left(\xi\frac{1}{\xi}x\right)\right\}\leqq 0 ,$$

and if $0<\xi\leqq\rho(\lambda)$, then we have

$$\lambda\bar{x}(x)-m(x)\leqq\xi\lambda\bar{x}\left(\frac{1}{\xi}x\right)\leqq 2\rho\lambda=2\lambda\left(\frac{2}{\gamma}\lambda\right)^{\frac{q}{p}} .$$

If $\bar{m}(\bar{x})\leqq 1$, $m(x)\leqq 1$, we have also

$$\lambda\bar{x}(x)-m(x)\leqq\lambda(\bar{m}(\bar{x})+m(x))-m(x)\leqq 2\lambda .$$

Therefore we obtain consequently

$$\bar{m}(\lambda\bar{x})\leqq 2\lambda\rho=\gamma_0\lambda^q \qquad \text{for all } \lambda\geqq\lambda_0$$

where $\gamma_0=2^q\left(\frac{1}{\gamma}\right)^{\frac{q}{p}}$. Hence the conjugate modular $\bar{m}$ is uniformly q-

finite modular.

As similarly as uniformly p-finite modulars, we can define uniformly p-simple and uniformly p-monotone modular. In order to define them, we set for $0 \leqq \xi \leqq 1$

$$\varphi(\xi) = \sup_{m(x) \leqq 1} m(\xi x), \quad \psi(\xi) = \inf_{m(x) \geqq 1} m(\xi x).$$

Then $\varphi(\xi)$, $\psi(\xi)$ are defined in $[0, 1]$ and finite non-decreasing functions.

Definition 2.3. A modular m on R is said to be *uniformly p-simple* if there exist $\gamma > 0$, and $0 < \xi_0 \leqq 1$, such that

$$\psi(\xi) \geqq \gamma \xi^p \qquad \text{for all } 0 \leqq \xi \leqq \xi_0.$$

Definition 2.4. A modular m on R is said to be *uniformly p-monotone,* if there exist $\gamma > 0$ and $0 < \xi_0 \leqq 1$, such that

$$\varphi(\xi) \leqq \gamma \xi^p \qquad \text{for all } 0 \leqq \xi \leqq \xi_0.$$

It is easily seen that if m is uniformly p-simple, it is also uniformly p'-simple for $p \leqq p'$, and if m is uniformly p-monotone, it is also uniformly p''-monotone for $1 \leqq p'' \leqq p$.

Concerning uniformly p-simple and uniformly q-monotone modulars there exist the conjugate relations, in fact we have

Theorem 2.3. *If a modular m on R is uniformly p-monotone, then the conjugate modular $\bar{m}$ of m is uniformly q-simple.*

Theorem 2.4. *If a modular m on R is uniformly p-simple, then the conjugate modular $\bar{m}$ of m is uniformly q-monotone.*

The proofs of these theorems are analogous to those of Theorems 4.9, 4.10 in [4] and of Theorems 2.1, 2.2, so it is omitted.

Concerning uniform simpleness and uniform finiteness we proved in Theorem 2.2 that uniform simpleness implies uniform finiteness, provided that R has no atomic element. On uniformly p-simple modular we obtain more precisely

Theorem 2.5. *Let R has no atomic element. If a modular m on R is uniformly p-simple, then it is uniformly p-finite.*

Proof. It is known already that m is uniformly finite. If it is not uniformly p-finite, then there exists a sequence of real numbers $\xi_n \geqq 0$ $(n = 1, 2, \cdots)$ such that

$$+\infty > f(\xi_n) > n\xi_n^p, \ \xi_n \uparrow +\infty \qquad (n = 1, 2, \cdots).$$

And by definition of $f(\xi)$, we can choose a sequence of elements $\{x_n\}$ $(n = 1, 2, \cdots)$ such that

$$m(\xi_n x_n) > n\xi_n^p, \quad m(x_n) = 1 \qquad (n=1,2,\cdots).$$

Here, we can assume without a loss of generality that

$$m(\xi_n x_n) = N_n$$

where N_n is a natural number, for every $n \geqq 1$. Because, if there are $\gamma > 0$ and $\xi_0 \geqq 1$ satisfying $m(\xi x) \leqq \gamma \xi^p$ for every $\xi \geqq \xi_0$ such that $m(\xi x)$ is a natural number, then we have $m(\xi x) \leqq (\gamma+1)\xi^p$ for all $\xi \geqq \xi_0$. This shows that m is uniformly p-finite.

Then we can find a sequence of projectors $\{[p_n]\}$ $(n=1,2,\cdots)$ by orthogonal decompositions of x_n $(n=1,2,\cdots)$ such that

$$m([p_n]\xi_n x_n) = 1, \quad m([p_n]x_n) < \frac{1}{n\xi_n^p} \qquad (n=1,2,\cdots),$$

since $m(\xi_n x_n)$ is natural number for all $n \geqq 1$. Set $y_n = [p_n]\xi_n x_n$ and $\eta_n = \frac{1}{\xi_n}$ for every $n \geqq 1$, then we have $m(y_n) = 1$ and $m(\eta_n y_n) < \frac{\eta_n^p}{n}$. Since $\lim_{n\to\infty} \eta_n = 0$, we show that m is not uniformly p-simple. Thus the proof is completed.

Corresponding to Theorem 2.5 we have

Theorem 2.6. *Let R have no atomic element. If a modular m on R is uniformly p-monotone, then it is uniformly p-increasing.*

It will be conjectured that if a modular m is uniformly finite, then it is uniformly p-finite for some $1 < p < +\infty$. But the following example shows that it is not true.

Example. Set $\phi(u) = \begin{cases} \frac{1}{2}u & u \leqslant 2 \\ e^{u-2} & u > 2 \end{cases}$

and consider Orlicz sequence space l_ϕ. Then l_ϕ is uniformly finite as easily seen, but not uniformly p-finite for any $1 < p < \infty$. This example shows at the same time that there exists a modular m which is uniformly increasing but not uniformly p-increasing for any $1 < p < \infty$.

I. Amemiya proved in [2] that if a modular m on R is monotone complete and finite, then m is semi-upper bounded, i.e., $m(2x) \leqslant \gamma m(x)$ for every x such that $m(x) \geqq \varepsilon$ for some fixed $\gamma, \varepsilon > 0$, provided that R has no atomic element. Applying this result, it is seen that the above conjecture is affirmative, in the case when m is monotone complete and R has no atomic element. In fact we have

Theorem 2.7. *Suppose that R has no atomic element and m is monotone complete. If m is uniformly finite (finite) then it is uniformly p-finite for some $p>1$.*

§3. To any γ such that $1<\gamma\leqq 2$, there exist a unique pair of positive numbers (p, q) satisfying the following

1) $\gamma = p^{\frac{1}{p}} q^{\frac{1}{q}}$

2) $\frac{1}{p} + \frac{1}{q} = 1$

3) $1 \lneqq p \leqslant 2 \leqslant q$.

This correspondence is unique and it is easily seen that if γ_n is convergent increasingly to 2, then the corresponding p_n (q_n) is also convergent increasingly (decreasingly) to 2.

If the norms of modular m have the property $(*)$ we can find a pair of numbers such that $\gamma = p^{\frac{1}{p}} q^{\frac{1}{q}}$. It is already seen that m is uniformly finite and uniformly increasing, provided that R has no atomic element. Now we shall show that (p, q) gives the degrees of uniform finiteness and increasingness. In fact we can state

Theorem 3.1. *Suppose that R has no atomic element. If the norms by a modular m have the property $(*)$, then m is uniformly p-increasing and uniformly q-finite.*

Proof. Set $\alpha = \left(\frac{p}{q}\right)^{\frac{1}{q}}$, then $\gamma\alpha - 1 = \alpha^q$.

Thus we obtain by assumption,

$$m(x) = 1 \quad \text{implies} \quad m(\alpha x) \geqq \alpha^q .$$

If $m(x) = 1 + \frac{m}{n}$ (for natural numbers $m<n$), we can decompose orthogonally $x = x_1 + x_2 \cdots + x_{n+m}$ such that

$$m(x_i) = m(x_j) = \frac{1}{n} \qquad (i, j = 1, 2, \cdots, n+m) .$$

The numbers of i such that $m(\alpha x_i) < \alpha^q m(x_i)$ are less than n, because if there exists $(i_1, i_2, \cdots, i_n)$ such that $m(\alpha x_{i_\nu}) > \alpha^q m x_{i_\nu})$ $(\nu = 1, 2, \cdots, n)$, then we have $m\left(\alpha \sum_{\nu=1}^{n} x_{i_\nu}\right) < \alpha^q m\left(\sum_{\nu=1}^{n} x_{i_\nu}\right)$ and $m\left(\sum_{\nu=1}^{n} x_{i_\nu}\right) = 1$. This is a contradiction.

Thus there exists $\{i_k\}$ $(k = 1, 2, \cdots, m)$ such that $m(\alpha x_{i_k}) \geqq \alpha^q m(x_{i_k})$ $(k =$

$1, 2, \cdots, m$). Putting $y = \sum_{k=1}^{m} x_{i_k}$ we have $m(x-y)=1$ and

$$m(\alpha y) \geqq \alpha^q m(y)$$
$$m(\alpha(x-y)) \geqq \alpha^q m(x-y) .$$

Hence we obtain $m(\alpha x) \geqq \alpha^q m(x)$. Generally, if $1 \leqq m(x) < 2$, since $m(\xi x)$ is continuous function of ξ, we have also

$$m(\alpha x) \geqq \alpha^q m(x) .$$

Since $m(x)$ is finite for all $x \in R$ and R has no atomic element, we have for x such that $m(x)=1$

$$m(\alpha \xi x) \geqq \alpha^q m(\xi x) \qquad \text{for all} \quad \xi \geqq 1 .$$

Here, putting $\beta = \frac{1}{\alpha} > 1$, we obtain

$$m(\beta^n x) \leqq \beta^{q \cdot n} m(x) \qquad (n=1, 2, \cdots)$$

for all x such that $m(x)=1$. From this we have

$$m(\xi x) \leqq \beta^q \xi^q \qquad \xi \geqq \beta ,$$

which shows that m is uniformly q-finite. By Theorem 2.1 and ($*'$) we can see m is uniformly p-increasing.

Remark 1. The converse of the theorem is not true. For example, set

$$\phi(u) = \begin{cases} u^{\frac{3}{2}} & u \leqslant 2 \\ \frac{1}{\sqrt{2}} u^2 & u > 2 . \end{cases}$$

Then the ORLICZ space $L[0,1]$ is uniformly 2-finite and uniformly 2-increasing, but it is easily seen that there is an element such that $\frac{\|x\|}{|||x|||} < 2$. And for any $1 < \alpha < 2$, we can get the example of modulared space such that m is uniformly 2-finite and uniformly 2-increasing but the norms by m do not satisfy $\inf_{x \neq 0} \frac{\|x\|}{|||x|||} \geqq \alpha$.

Remark 2. If R is a discrete modulared semi-ordered linear space, the property ($*$) dose not imply finiteness of m, and even if in the case where m is finite, the property ($*$) does not imply uniform finiteness of m. The examples are obtained easily. In this case the equivalent condition to the property $(*)$ is unknown.

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