POTENTIALS ON RIEMANN SURFACES

By

Zenjiro KURAMOCHI

The purpose of the present paper is to simplify and to extend the almost all theorems contained in the previous papers.[1] The potential theory has been discussed in euclidean space. Recently it is discussed by many authors under *weaker conditions* of the kernel $K(x, y)$ of the potential $U(x) = \int K(x, y)d\mu(y)$ in more general space S which is *locally compact* and *homogeneous* and $K(x, y)$ has not necessarily symmetry property, superharmonicity. On the other hand, the space in which we shall consider the potential $U(z) = \int N(z, p)\, d\mu(p)$ is a Riemann surface with ideal boundary B, which is locally euclidean in R and locally compact in $R+B$ and $R+B$ is *not homogeneous* by the existence of B. The kernel $N(z, p)$ which will be defined is harmonic in R and superharmonic in some sense in $R+B_1$ (B_1 is the part of B), $N(z, p)$ *has symmetry, lower semicontinuity* in $R+B_1$, $N(p, p) \leqq \infty$. Further there may exist $B_0 = B - B_1$ where we cannot distribute *any true mass.* In the above sense our space is not so restricted. To construct potentials we make some preparations.

1. Let R be a Riemann surface with positive boundary and let R_n $(n=0,1,2,\cdots)$ be its exhaustion with compact relative boundaries ∂R_n. Let $N_n(z, p)$ be a positive function in $R_n - R_0$ such that $N_n(z, p)$ is harmonic in $R_n - R_0$ except one point $p \in R_n - R_0$, $N_n(z, p) = 0$ on ∂R_0, $\frac{\partial}{\partial n} N_n(z, p) = 0$ on ∂R_n and $N_n(z, p) + \log|z-p|$ is harmonic in a neighbourhood of p. We define the $\overset{*}{D}$-integral $D^*(N_{n+i}(z, p), N_n(z, p))$ of $N_{n+i}(z, p)$ and $N_n(z, p)$ over $R_n - R_0$ as follows:

Let $v_r(p)$ be a circular neighbourhood of p with respect to the local parameter at p: $v_r(p) = E[z \in R : |z-p| < r]$. Put

$$D^*_{R_n - R_0 - v_r(p)}(N_{n+i}(z, p), N_n(z, p)) = \int_{\partial R_n + \partial R_0} N_{n+i}(z, p) \frac{\partial}{\partial n} N_n(z, p)\, ds + \int_{\partial v_r(p)} (N_{n+i}(z, p)$$

1) Z. Kuramochi: Mass distributions on the ideal boundaries of abstract Riemann surfaces I, II, III. Osaka Math. Journ. 1956, 1957, 1958.

$$+\log|z-p|)\frac{\partial}{\partial n}N_n(z,p)\,ds=\int_{\partial v_r(p)}(N_{n+i}(z,p)+\log|z-p|)\frac{\partial}{\partial n}N_n(z,p)\,ds.$$

We define $D^*_{R_n-R_0}(N_{n+i}(z,p),\ N_n(z,p))$ as

$$\lim_{r\to 0} D^*_{R_n-R_0-v_r(p)}(N_{n+i}(z,p),\ N_n(z,p))=2\pi U_{n+i}(p), \tag{1}$$

where $U_{n+i}(p)=\lim_{r\to 0}(N_{n+i}(z,p)+\log|z-p|)$. Similarly

$$D^*_{R_n-R_0}(N_n(z,p))=D^*_{R_n-R_0}(N_n(z,p),\ N_n(z,p))=2\pi U_n(p). \tag{2}$$

By the Green's formula

$$\begin{aligned}
&D^*_{R_n-R_0-v_r(p)}(N_n(z,p),\ N_{n+i}(z,p))=\int_{\partial R_0+\partial R_n}N_n(z,p)\frac{\partial}{\partial n}N_{n+i}(z,p)\,ds\\
&+\int_{\partial v_r(p)}(N_n(z,p)+\log|z-p|)\frac{\partial}{\partial n}N_{n+i}(z,p)\,ds=\int_{\partial R_0+\partial R_n+\partial v_r(p)}N_n(z,p)\frac{\partial}{\partial n}N_{n+i}(z,p)\,ds\\
&+\int_{\partial v_r(p)}\log|z-p|\frac{\partial}{\partial n}N_{n+i}(z,p)\,ds=\int_{\partial R_0+\partial R_n+\partial v_r(p)}N_{n+i}(z,p)\frac{\partial}{\partial n}N_n(z,p)\,ds\\
&+\int_{\partial v_r(p)}\log|z-p|\frac{\partial}{\partial n}N_{n+i}(z,p)\,ds=\int_{\partial R_n}N_{n+i}(z,p)\frac{\partial}{\partial n}N_n(z,p)\,ds\\
&+\int_{\partial v_r(p)}(N_{n+i}(z,p)+\log|z-p|)\frac{\partial}{\partial n}N_n(z,p)\,ds\\
&+\int_{\partial v_r(p)}\log|z-p|\frac{\partial}{\partial n}(N_{n+i}(z,p)-N_n(z,p))\,ds=D^*_{R_n-R_0-v_r(p)}(N_{n+i}(z,p),N_n(z,p))\\
&+\int_{\partial v_r(p)}\log|z-p|\frac{\partial}{\partial n}(N_{n+i}(z,p)-N_n(z,p))\,ds.
\end{aligned}$$

Since $N_n(z,p)-N_{n+i}(z,p)$ is harmonic at p, $\int_{\partial v_r(p)}\log|z-p|\frac{\partial}{\partial n}(N_n(z,p)-N_{n+i}(z,p))\,ds\to 0$ as $r\to 0$. Hence

$$\begin{aligned}
D^*_{R_n-R_0}(N_n(z,p),\ N_{n+i}(z,p))&=\lim_{r\to 0}D^*_{R_n-R_0-v_r(p)}(N_n(z,p),\ N_{n+i}(z,p))\\
&=D^*_{R_n-R_0}(N_{n+i}(z,p),\ N_n(z,p)).
\end{aligned} \tag{3}$$

By (1), (2), (3) and by $D^*_{R_{n+i}-R_0}(N_{n+i}(z,p))\geqq D^*_{R_n-R_0}(N_{n+i}(z,p))$ the $\overset{*}{D}$-integral $D^*_{R_n-R_0}(N_n(z,p)-N_{n+i}(z,p))$ is given as follows:

$$\begin{aligned}
0\leqq D_{R_n-R_0}(N_n(z,p)-N_{n+i}(z,p))&=D^*_{R_n-R_0}(N_n(z,p))-2D^*_{R_n-R_0}(N_n(z,p),N_{n+i}(z,p))\\
&+D^*_{R_n-R_0}(N_{n+i}(z,p))<D^*_{R_n-R_0}(N_n(z,p))-2D^*_{R_n-R_0}(N_n(z,p),N_{n+i}(z,p))\\
&+D^*_{R_{n+i}-R_0}(N_{n+i}(z,p))=2\pi(U_n(p)-U_{n+i}(p)).
\end{aligned}$$

On the other hand, let $G_n(z,p)$ be the Green's function of R_n-R_0 with pole at $p\in R_{n_0}-R_0$. Then

$$G_{n_0}(z,p)<N_{n+j}(z,p)\quad (j=0,1,2,\cdots)\ \text{for}\ n\geqq n_0.$$

This implies

$$\varliminf_{j=\infty} U_{n+j}(p) \geqq \lim_{z\to p}(G_{n_0}(z, p)+\log|z-p|)=L>-\infty \quad (j=0,1,2,\cdots).$$

Hence $U_n(p)$ is decreasing with respect to n and $\varliminf_{n} U_n(p) \geqq L$. Whence $\{U_n(p)\}$ coverges. Therefore $D(N_{n+i}(z, p)-N_n(z, p))\to 0$, if n and $i\to\infty$ or only $n\to\infty$, which implies that $N_n(z, p)$ coverges in mean. Further $N_n(z, p)=0$ on ∂R_0 yields that $\{N_n(z, p)\}$ converges uniformly to a function $N(z, p)$ as $n\to\infty$. Clearly by the compactness of ∂R_0, $\int_{\partial R_0}\frac{\partial}{\partial n}N(z, p)\,ds=2\pi$. We call $N(z, p)$ *the N-Green's function of* $R-R_0$ *with pole* at p.

Remark. If R is a Riemann surface with null-boundary, we see that $N(z, p)$ reduces to be the Green's function of $R-R_0$.

After R. S. Martin we shall define the ideal boundary points as follows. Let $N(z, p)$ be the N-Green's function of $R-R_0$ with pole at p. Consider now a sequence of points $\{p_i\}$ of $R-R_0$ having no points of accumulation in $R-R_0+\partial R_0$. Since the function $N(z, p_i)$ $(i=1,2,\cdots)$ forms, from some i on, a bounded sequence of harmonic functions—thus a normal family. A sequence of these functions, therefore is covergent in every compact part of $R-R_0$ to a positive harmonic function. A sequence $\{p_i\}$ of $R-R_0$ having no point of accumulation in $R-R_0+\partial R_0$, for which the corresponding $N(z, p)$'s have the property just mentioned, that is, $\{N(z,p)\}$ coverges to a harmonic function—will be called fundamental. If two fundamental sequences determine the same limit function $N(z, p)$, we say that they are equivalent. Two fundamental sequences equivalent to a given one determine an ideal boundary point of R. The set of all the ideal boundary points of R will be denoted by B and the set $R-R_0+B$ by $\overline{R}-R_0$. The domain of definition of $N(z, p)$ may now be extended by writing $N(z, p)=\lim_{i=\infty} N(z, p_i)(z\in R-R_0, p\in B)$, where $\{p_i\}$ is any fundamental sequence determining p. The fnnction $N(z, p)$ is characteristic of the point p of their corresponding $N(z, p)$ as a function of z. The function $\delta(p_1, p_2)$ of two points p_1 and p_2 in $\overline{R}-R_0$ is defined as

$$\delta(p_1, p_2)=\sup_{z\in R_1-R_0}\left|\frac{N(z, p_1)}{1+N(z, p_1)}-\frac{N(z,p_2)}{1+N(z,p_2)}\right|.$$

Evidently, $\delta(p_1, p_2)=0$ is equivalent to $N(z, p_1)=N(z, p_2)$ for all points z in R_1-R_0. Therefore we have $N(z, p)=N(z, p_2)$ in $R-R_0$, i.e. $\delta(p_1, p_2)=0$ implies $p_1=p_2$ and it is clear that $\delta(p_1, p_2)$ satisfies the axioms of distance. Therefore $\delta(p_1, p_2)$ can be considered as the distance between two points p_1 and p_2 of $\overline{R}-R_0$.

The topology (we call N-Martin's topology) induced by this metric is homeomorphic to the original topology, when it is restricted in $R-R_0$. Since $N(z,p):p\in\overline{R}-R_0$ is also a normal family, both $(R-R_1)+\partial R_1+B$ and B are closed and compact. For a fixed point z, $N(z,p)$ is continuous with respect to this metric (we denote it shortly by δ-continuous) as a function of p in $\overline{R}-R_0$ except at p.

2. Properties of the function N(z, p).

Lemma 1. *a). Let G_1 be a compact or non compact domain in $R-R_0$*[2] *containing another domain G_2. Let $U(z)$ be a function of C_1-class*[3] *such that $D(U(z))$ is finitely minimal Dirichlet integral (we abbreviate it by* M.D.I.) *among all functions of C_1-class with the same boundary value on ∂G_1. Then $U(z)$ is also* M.D.I. *function in G_2 among all functions with the same boundary value as $U(z)$ on ∂G_2.*

b). Let G be a domain as a) and let $U(z)$ be a harmonic function with M.D.I. *over G with the boundary value $\varphi(z)$ on ∂G. Then $U(z)$ is uniquely determined and $U_n(z)\Rightarrow U(z)$ (we denote by $\Rightarrow$ that $U_n(z)$ converges and converges in mean to $U(z)$), where $U_n(z)$ is a harmonic function in $R_n\frown G$ such that $U_n(z)=U(z)$ on $\partial G\frown R_n$ and $\frac{\partial}{\partial n}U_n(z)=0$ on ∂R_n. Whence $\inf_{z\in\partial G}U(z)\leqq U(z)\leqq\sup_{z\in\partial G}U(z)$. If $U(z)\neq const$, $\inf_{z\in\partial G}U(z)<U(z)<\sup_{z\in\partial G}U(z)$.*

c). Let G be a domain. The necessary and sufficient condition for a harmonic function $U(z)$ to have M.D.I. *over G among all functions with the value $U(z)$ on ∂G is that $D(U(z),C(z))=0$ for every harmonic function $C(z)$ such that $C(z)=0$ on ∂G and $D(C(z))<\infty$.*

d). Let $U_n(z)$ $(n=1,2,\cdots)$ be a harmonic function in $G\frown R_n$ with boundary value $\varphi_n(z)$ on $\partial G\frown R_n$ such that $U_n(z)$ has M.D.I.$<M$ *over $R_n\frown G$. If $U_n(z)\Rightarrow U(z)$, then $U(z)$ has* M.D.I. *over G with $\varphi(z)=\lim_n\varphi_n(z)$ on ∂G. Similarly let $U_n(z)$ be a harmonic function in G with boundary value $\varphi_n(z)$ on ∂G such that $U_n(z)$ has* M.D.I. *over G. If $U_n(z)\Rightarrow U(z)$, then $U(z)$ has* M.D.I. *over G with boundary value $\varphi(z)=\lim_n\varphi_n(z)$ on ∂G.*

Proof of a). Assume that there exists another function $B(z)$ of C_1-class such that $B(z)=U(z)$ on ∂G_2 and $D_{G_2}(B(z))<D_{G_2}(U(z))$. Put $U^*(z)=B(z)$

2) In the present paper, we suppose that ∂G of a domain G consists of at most enumerably infinite number of analytic curves clustering nowhere in R.

3) If $U(z)$ is continuous and has partial derivatives almost everywhere, we say that $U(z)\in C_1$-class.

in G_2 and $U^*(z)=U(z)$ in G_1-G_2. Then $D(U^*(z))<D(U(z))$. This contradicts that $U(z)$ has M.D.I. Hence we have a).

Proof of b). Let $U_i(z)$ $(i=1,2)$ be a function of C_1-class such that $U_i(z)=\varphi(z)$ on ∂G and has M.D.I. Then

$$D(U_i(z)+\varepsilon(U_1(z)-U_2(z)))\geqq D(U_i(z)) \quad \text{for any } \varepsilon.$$

By considering $\pm\varepsilon$ such that $|\varepsilon|$ is sufficiently small, we have

$$D(U_i(z),\, U_1(z)-U_2(z))=0 : i=1,2. \quad \text{Hence } D(U_1(z)-U_2(z))=0, \text{ i.e.}$$
$$U_1(z)=U_2(z).$$

Let $U_n(z)$ $(n=1,2,\cdots)$ be a harmonic function in b). Then

$$D_{G\frown R_n}(U(z)-U_n(z),\ U_n(z))=0,$$

whence

$$D_{G\frown R_n}(U(z))-D_{G\frown R_n}(U_n(z))=D_{G\frown R_n}(U(z)-U_n(z))\geqq 0$$
$$\text{and } D_{G\frown R_n}(U_n(z))\uparrow L\leqq D(U(z)). \tag{4}$$

Similarly

$$0\leqq D_{G\frown R_n}(U_{n+i}(z)-U_n(z))=D_{G\frown R_n}(U_{n+i}(z))-D_{G\frown R_n}(U_n(z))$$
$$\leqq D_{G\frown R_{n+i}}(U_{n+i}(z))-D_{G\frown R_n}(U_n(z))\to 0$$

as $n\to\infty$ and $i\to\infty$ by (4). This implies $U_n(z)\Rightarrow U^*(z)$.

Now by (4) and by Fatou's lemma

$$D_{G\frown R_m}(U^*(z))\leqq \lim_{n=\infty} D_{G\frown R_n}(U_n(z))\leqq D_G(U(z)), \text{ for } n\geqq m.$$

Let $m\to\infty$. Then $D_G(U^*(z))\leqq D_G(U(z))$. Thus $U^*(z)$ has M.D.I. By the assumption that $U(z)$ has M.D.I., we have $U^*(z)=U(z)$.

Let $U(z)$ be the function in b). Then by $\frac{\partial}{\partial n}U_n(z)=0$ on ∂R_n, $\inf\limits_{z\in\partial G} U_n(z)\leqq U_n(z)\leqq \sup\limits_{z\in\partial G} U_n(z)$ is clear by the maximum principle, whence $\inf\limits_{z\in\partial G} U(z)\leqq U(z)\leqq \sup\limits_{z\in\partial G} U(z)$. Suppose $U(z)\neq$ const.

Then also by the maximum principle we have

$$\inf_{z\in\partial G} U(z)<U(z)<\sup_{z\in\partial G} U(z).$$

Proof of c). Suppose, $U(z)$ has M.D.I. Since $U(z)+\varepsilon C(z)=U(z)$ on ∂G,

$$D(U(z)+\varepsilon C(z))=D(U(z))+2\varepsilon D(U(z),C(z))+\varepsilon^2 D(C(z))\geqq D(U(z))$$

for any ε. We see that $D(U(z),C(z))=0$ in considering $\varepsilon=\pm\gamma$ for ε such that $|\gamma|$ is sufficiently small.

Conversely, assume $D(C(z),\,U(z))=0$. Let $U'(z)$ be a harmonic function such that $U'(z)=U(z)$ on ∂G and $D(U'(z))<\infty$. Then by putting $C(z)=U(z)-U'(z)$, we have $D(U(z))=D(U(z),\,U'(z))$ and $D(U'(z))\geqq D(U(z))$. Now $U'(z)$

is any function. Hence $U(z)$ has M.D.I. Thus we have c).

Proof of d). At first we remark, by $U_n(z) \Rightarrow U(z)$,

$$D_G(U(z)) = \lim_m D_{G \frown R_m}(U(z)) \leqq \lim_m (\underline{\lim}_n D_{R_m \frown G}(U_n(z))) \leqq M.$$

Let $C(z)$ be a harmonic function in G such that $D(C(z)) < \infty$ and $C(z)=0$ on ∂G. Then

$$|D_{G\frown(R_n - R_m)}(U_n(z), C(z))| \leqq \sqrt{D_{G\frown(R_n-R_m)}(U_n(z)) D_{G\frown(R_n-R_m)}(C(z))}$$
$$\leqq \sqrt{M}\sqrt{D_{G\frown(R-R_m)}(C(z))} \to 0 \text{ as } m \to \infty. \tag{5}$$

Hence for any given positive number ε, there exists a number m_0 such that

$$|D_{G\frown(R_n-R_m)}(U_n(z), C(z))| < \varepsilon \quad \text{for } m \geqq m_0,\ n \geqq m_0. \tag{6}$$

Since $U_n(z)$ has M.D.I. over $G \frown R_n$, by c) $D_{G\frown R_n}(U_n(z), C(z))=0$, we have by (6)

$$|D_{G\frown R_m}(U_n(z), \mathrm{C}(z))| < \varepsilon. \tag{7}$$

On the other hand, by $U_n(z) \Rightarrow U(z)$, for the same ε and the above number m, there exists a number $n_0 = n_0(m)$ such that

$$|D_{G\frown R_m}(U(z) - U_n(z), C(z))| < \varepsilon, \quad \text{for } n \geqq n_0 \tag{8}$$

because

$$|D_{G\frown R_m}(U(z)-U_n(z), C(z))| \leqq \sqrt{D_{G\frown R_m}(U(z)-U_n(z)) D(C(z))}.$$

Thus

$$|D_{G\frown R_m}(U(z), C(z))| \leqq |D_{G\frown R_m}(U(z)-U_n(z), C(z))|$$
$$+ |D_{G\frown R_m}(U_n(z), C(z))| < 2\varepsilon,\ m > m_0,\ n > n_0(m).$$

Hence

$$D_{G\frown R_m}(U(z), C(z)) \to 0 \text{ as } m \to \infty.$$

Whence by c) $U(z)$ has M.D.I. over G with value $\varphi(z) = \lim_n \varphi_n(z)$ on ∂G. The latter part is proved similarly.

Theorem 1. *a*). *Let $N(z, p)$ be the N-Green's function of $R-R_0$ with pole p in $R-R_0$. Let G be a compact or non compact domain containing p. Then $N(z, p)$ has* M.D.I. *over $R-R_0-G$ among all functions with the same value as $N(z, p)$ on $\partial G+\partial R_0$, whence by b) of Lemma 1* $\underset{z \in (R-G)}{N(z, p)} < \sup_{z \in \partial G} N(z, p)$ *and* $\lim_{M=\infty} V_M(p) = p$, *where* $V_M(p) = E[z \in R\colon N(z, p) > M]$.

b). *$N(z, p)$ satisfies*

$$D(\min(M, N(z, p))) \leqq 2\pi M \textit{ for } p \in \overline{R}-R_0.$$

Proof of a). Let G' be a compact domain with smooth boundary such that $G \supset G' \ni p$ and that $\partial G'$ is rectifiable. Since $N(z, p)$ is harmonic in $R-R_0$

$-G$, $\max_{z\in\partial G'} N(z,p)\leqq L<\infty$. Let $N_n(z,p)$ be a harmonic function in R_n-R_0 such that $N_n(z,p)=0$ on ∂R_0, $\frac{\partial}{\partial n}N_n(z,p)=0$ on ∂R_n and $N_n(z,p)$ has a logarithmic singularity at p. Then $N_n(z,p)\Rightarrow N(z,p)$. Hence by the compactness of $\partial G'$ there exists a number n_0 such that $N_n(z,p)<L+\varepsilon$ on $\partial G'$ for $n\geqq n_0$, for any given positive number ε. Let $V^n_{L+\varepsilon}(p)=E[z\in R: N_n(z,p)>L+\varepsilon]$. Then by the maximum principle $G'\supset V^n_{L+\varepsilon}(p)$ for $n\geqq n_0$, because $\frac{\partial}{\partial n}N_n(z,p)=0$ on ∂R_n. Since $N_n(z,p)$ is harmonic in $R-R_0-V^n_{L+\varepsilon}(p)$ with continuous normal derivative on $\partial V^n_{L+\varepsilon}(p)$, Dirichlet integral of $N_n(z,p)$ over $R_n-R_0-V^n_{L+\varepsilon}(p)$ is finite. Hence there exists at least one harmonic function $A(z)$ such that the Dirichlet integral of $A(z)$ over $R-R_0-V^n_{L+\varepsilon}(p)$ is finite with $A(z)=N_n(z,p)$ on $\partial R_0+\partial V^n_{L+\varepsilon}(p)+\partial R_n$. Let $U(z)$ be a harmonic function in $R_n-R_0-V^n_{L+\varepsilon}(p)$ such that $U(z)=N_n(z,p)$ on $\partial R_0+\partial V^n_{L+\varepsilon}(p)+\partial R_n$ and the Dirichlet integral of $U(z)$ is finite. Now

$$D_{R_n-R_0-V^n_{L+\varepsilon}(p)}(N_n(z,p),\ N_n(z,p)-U(z))=0.$$

But $U(z)$ is arbitrary, hence $N_n(z,p)$ has M.D.I. over $R_n-R_0-V^n_{L+\varepsilon}(p)$ and

$$D_{R_n-R_0-V^n_{L+\varepsilon}(p)}(N_n(z,p))=\int_{\partial V^n_{L+\varepsilon}(p)} N_n(z,p)\frac{\partial}{\partial n}N_n(z,p)\,ds=(L+\varepsilon)\int_{\partial R_0}\frac{\partial}{\partial n}N_n(z,p)\,ds$$
$$=2\pi(L+\varepsilon).$$

Hence by Lemma 1. *a*) $N_n(z,p)$ has M.D.I. ($\leqq 2\pi(L+\varepsilon)$) over R_n-R_0-G $\subset R_n-R_0-G'$ and over $R-R_0-V^n_{L+\varepsilon}(p)$ for $n\geqq n_0$ by $G\supset G'\supset V^n_{L+\varepsilon}(p)$. On the other hand, $N_n(z,p)\Rightarrow N(z,p)$. This implies by Lemma 1. *c*) that $N(z,p)$ has also M.D.I. over $R-R_0-G'$ and over $R-R_0-G$, which is clearly $\leqq 2\pi(L+\varepsilon)$. Next by Lemma 1. *b*)

$$N(z,p)_{z\in R-R_0-G'}\leqq \sup_{z\in\partial G'} N(z,p)\leqq L,$$

i.e. $E[z\in R: N(z,p)>L]=V_L(p)\subset G'$. Now G' is arbitrary. Hence

$$\lim_{M=\infty} V_M(p)=p.$$

Proof of b). Case 1. $p\in R-R_0$. Since $R_m-V_M(p)-R_0$ is compact for sufficiently large number M by $\lim_{M=\infty} V_M(p)=p$, for any given positive number ε and a number m, we can find a number $n_0=n_0(\varepsilon,m)$ such that

$$R_m-R_0-V_M(p)\subset E[z\in R: N_n(z,p)<M+\varepsilon]\quad\text{for } n\geqq n_0.$$

On the other hand, by Factou's lemma

$$D_{R_m-R_0}(\min(M,N(z,p)))\leqq \lim_n D(\min(M+\varepsilon,N_n(z,p)))\leqq 2\pi(M+\varepsilon).$$

Let $\varepsilon\to 0$ and then $m\to\infty$. Then

$$D_{R-R_0}(\min(M,N(z,p)))\leqq 2\pi M.$$

Case 2. $p\in B$. Let $\{p_i\}$ be a fundamental sequence determining p and let $V_M(p_i)=E[z\in R: N(z,p_i)>M]$. Then by case 1, $D_{R-R_0-V_M(p_i)}(N(z,p_i))\leqq 2\pi M$ for every M. On the other hand, $N(z,p_i)\to N(z,p)$. Hence by the same manner as in case 1, we have by Fatou's lemma

$$D_{R-R_0}(\min(M, N(z,p)))\leqq 2\pi M.$$

3. Harmonic measure (H.M.) and capacities (C.P.) of the ideal boundary ($B\frown G_2$) determined by a domain G_2 with respect to a domain G_1, $G_2\subset G_1$.

So far as we discuss the ideal boundary, without loss of generality, we can suppose that non compact domains have no intersection with R_0. In the following we assume $G_1\frown R_0=0$. Let $w_{n,n+i}(z)(\omega_{n,n+i}(z))$ be a function in $(G_1\frown R_{n+i})$ such that $w_{n,n+i}(z)=\omega_{n,n+i}(z)=1$ in $G_2\frown(R_{n+i}-R_n)$ and is harmonic in $\Omega_{n,n+i}=(G_1\frown R_{n+i})-(G_2\frown(R_{n+i}-R_n))$, $w_{n,n+i}(z)=\omega_{n,n+i}(z)=0$ on $\partial G_1\frown R_{n+i}$, $w_{n,n+i}(z)=\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\partial R_{n+i}\frown(G_1-G_2)$. Then by the maximum principle $w_{n,n+i}(z)\uparrow w_n(z)$ as $i\to\infty$ and $w_n(z)\downarrow w(z)$ as $n\to\infty$. We call $w(z)$ the *harmonic measure* H.M. *of the ideal boundary* $(G_2\frown B)$ *determined by* G_2 *relative* G_1. We denote it by $w(G_2\frown B, z, G_1)$.

If there exists a constant M and a number n_0 such that

$$D_{\Omega_{n,n+i}}(\omega_{n,n+i}(z))\leqq M$$

for every $n\geqq n_0$ and $i\geqq 0$, then $\omega_{n,n+i}(z)\Rightarrow\omega_n(z)$ as $i\to\infty$ and $\omega_n(z)\Rightarrow\omega(z)$ as $n\to\infty$.

In fact, $D_{\Omega_{n,n+i}}(\omega_{n,n+i}(z), \omega_{n,n+i+j}(z))=\int_{\partial(G_2\frown(R_{n+i}-R_n))}\omega_{n,n+i+j}(z)\frac{\partial}{\partial n}\omega_{n,n+i}(z)\,ds$

$$=\int_{\partial(G_2\frown(R_{n+i}-R_n))}\frac{\partial}{\partial n}\omega_{n,n+i}(z)\,ds=D_{\Omega_{n,n+i}}(\omega_{n,n+i}(z)),$$

whence

$$0\leqq D_{\Omega_{n,n+i}}(\omega_{n,n+i}(z)-\omega_{n,n+i+j}(z))=D_{\Omega_{n,n+i}}(\omega_{n,n+i}(z))-D_{\Omega_{n,n+i}}(\omega_{n,n+i+j}(z))$$
$$\leqq D_{\Omega_{n,n+i+j}}(\omega_{n,n+i+j}(z))-D_{\Omega_{n,n+i}}(\omega_{n,n+i}(z)).$$

Whence $D_{\Omega_{n,n+i}}(\omega_{n,n+i}(z))\uparrow$ as $i\to\infty$. But $\leqq M$. Hence

$$D_{\Omega_{n,n+i}}(\omega_{n,n+i+j}(z)-\omega_{n,n+i}(z))$$
$$\leqq D_{\Omega_{n,n+i+j}}(\omega_{n,n+i+j}(z))-D_{\Omega_{n,n+i}}(\omega_{n,n+i}(z))\downarrow 0 \text{ as } i\to\infty. \quad (9)$$

Thus $\omega_{n,n+i}(z)\Rightarrow\omega_n(z)$ as $i\to\infty$.

Next similarly,

$$D_{\Omega_{n,n+i+j}}(\omega_{n,n+i+j}(z), \omega_{n+i,n+i+j}(z))=\int_{\partial((R_{n+i}-R_n)\frown G_2)}\omega_{n,n+i}(z)\frac{\partial}{\partial n}\omega_{n+i,n+i+j}(z)\,ds$$

$$=\int_{\partial((R_{n+i}-R_n)\frown G_2)}\frac{\partial}{\partial n}\omega_{n+i,\,n+i+j}(z)\,ds=\int_{\partial((R_{n+i+j}-R_{n+i})\frown G_2)}\frac{\partial}{\partial n}\omega_{n+i,\,n+i+j}(z)\,ds$$
$$=D_{\Omega_{n+i,\,n+i+j}}(\omega_{n+i,n+i+j}(z)). \tag{10}$$

Whence $D_{\Omega_{n,\,n+i+j}}(\omega_{n,\,n+i+j}(z)-\omega_{n+i,\,n+i+j}(z))=D_{\Omega_{n,\,n+i+j}}(\omega_{n,\,n+i+j}(z))+D_{\Omega_{n,\,n+i+j}}(\omega_{n+i,\,n+i+j}(z))-2D_{\Omega_{n,\,n+i+j}}(\omega_{n,\,n+i+j}(z),\ \omega_{n+i,\,n+i+j}(z))\leqq D_{\Omega_{n,\,n+i+j}}(\omega_{n,\,n+i+j}(z))+D_{\Omega_{n+i,\,n+i+j}}(\omega_{n+i,\,n+i+j}(z))-2D_{\Omega_{n,\,n+i+j}}(\omega_{n,\,n+i}(z),\ \omega_{n+i,\,n+i+j}(z))$.

Hence by (10)

$$0\leqq D_{\Omega_{n,\,n+i+j}}(\omega_{n,\,n+i+j}(z)-\omega_{n+i,\,n+i+j}(z))$$
$$\leqq D_{\Omega_{n,\,n+i+j}}(\omega_{n,\,n+i+j}(z))-D_{\Omega_{n+i,\,n+i+j}}(\omega_{n+i,\,n+i+j}(z)).$$

Let $j\to\infty$. Then by (9)

$$0\leqq D_{\Omega_n}(\omega_n(z)-\omega_{n+i}(z))\leqq D_{\Omega_n}(\omega_n(z))-D_{\Omega_{n+i}}(\omega_{n+i}(z)),$$

where $\Omega_n=\lim_i \Omega_{n,\,n+i}=G_1-(G_2\frown(R-R_n))$.

Hence $D_{\Omega_n}(\omega_n(z))\downarrow\geqq 0$ and

$$D(\omega_n(z)-\omega_{n+i}(z))\to 0 \text{ as } n\to\infty.$$

Thus $\omega_n(z)\Rightarrow\omega(z)$. We call $D(\omega(z))$ and $\omega(z)$ the *capacity and the capacitary potential* C.P. *of* $(B\frown G_2)$ *relative* G_1 and denote it by $\omega(B\frown G_2, z, G_1)$.

Let G_i $(i=3,4,\cdots)$ be non compact domains $\subset G_1$. We consider H.M. and C.P. of $G_{i_1}\frown G_{i_2}\cdots\frown B$. If $G_i\frown G_j=0$, we define $w(G_i\frown G_j\frown B,z,G_1)=\omega(G_i\frown G_j\frown B,z,G_1)=0$ $i\neq j$. We shall prove the following

Theorem 2. *Let G' be a compact or non compact domain such that $G'\subset G_1$, and $G'\frown G_2\frown(R-R_{n_0})=0$ for a certain number n_0.*

P.H.1. *Let $V(z)$ be the non negatively least harmonic function in G' such that $V(z)=w(B\frown G_2,z,G_1)$ on $\partial G'$. Then $V(z)=w(B\frown G_2,z,G_1)$ in G'.*

P.C.1. *Let $V(z)$ be a harmonic function in G' such that $V(z)=\omega(B\frown G_2, z,G_1)$ on $\partial G'$ and $V(z)$ has* M.D.I. *over G'. Then $V(z)=\omega(B\frown G_2,z,G_1)$ in G'.*

P.H.2. $w(B\frown G_2,z,G_1)>0$ *implies* $\sup_{z\in(G_2\frown(R-R_n))} w(B\frown G_2,z,G_1)=1$ *for every* n.

P.C.2. $\omega(B\frown G_2 z,G_1)>0$ *implies* $\sup_{z\in(G_2\frown(R-R_n))}\omega(B\frown G_2,z,G_1)=1$ *for every* n.

P.H.3. $w(B\frown G_2\frown G_\delta,z,G_1)=0$ *for* $G_\delta=E[z\in R: w(B\frown G_2,z,G_1)<1-\delta]$, $1>\delta>0$.

P.C.3. $\omega(B\frown G_2\frown G_\delta,z,G_1)=0$ *for* $G_\delta=E[z\in R: \omega(B\frown G_2,z,G_1)<1-\delta]$, $1>\delta>0$.

We define H.M.(C.P.) *for a set K in G_1 denoted by $w(K,z,G_1)$ $(\omega(K,z,G_1))$ such that $w(K,z,G_1)$ $(\omega(K,z,G_1))$ is harmonic in G_1-K and $w(K,z,G_1)=\omega(K,z,G_1)=0$ on ∂G_1 and $w(K,z,G_1)=\omega(K,z,G_1)=1$ on K except a set of* *capacity zero and non negatively least (has finitely* M.D.I.). *Then*

P.H.4. $(1-\delta)w(G'_\delta, z, G)=w(B\frown G_2, z, G_1)$ *in* $G_1-G'_\delta$ *for* $G'_\delta=E[z\in G_1: w(B\frown G_2, z, G_1)>1-\delta]$.

P.C.4. $(1-\delta)\omega(G'_\delta, z, G_1)=\omega(B\frown G_2, z, G_1)$ *in* $G_1-G'_\delta$ *for* $G'_\delta=E[z\in G_1: \omega(B\frown G_2, z, G_1)>1-\delta]$.

Let G_k $(k=3,4,\cdots)$ *be compact or non compact domains in* G_1. *Then*

P.H.5. $\sum_k w(B\frown G_k, z, G_1)\geqq w(B\frown\sum_k G_k, z, G_1)$.

P.C.5. $\sum_k \omega(B\frown G_k, z, G_1)\geqq \omega(B\frown\sum_k G_k, z, G_1)$.

P.H.6. $w(B\frown G_2\frown G'_\delta, z, G_1)=w(B\frown G_2, z, G_1)$: G'_δ *is the domain in* P.H.4.

P.C.6. $\omega(B\frown G_2\frown G'_\delta, z, G_1)=\omega(B\frown G_2, z, G_1)$; G'_δ *is the domain in* P.C.4.

P.C.7. *If* $w(B\frown G_2, z, G_1)>0$, *there exists an exceptional set* E *of measure zero in the interval* $(0,1)$ *such that if* $L\notin E$, *then the niveau curve*[4] $C_L=E[z\in G_1: \omega(B\frown G_2, z, G_1)=L]$ *has the following property*

$$\int_{C_L}\frac{\partial}{\partial n}\omega(B\frown G_2, z, G_1)\,ds=D_{G_1}(\omega(B\frown G_2, z, G_1)).$$

Proof of P.H.1. Since $w_{n,n+i}(z)=0$ on $G'\frown\partial R_{n+i}$,

$$w_{n,n+i}(z)=\frac{1}{2\pi}\int_{\partial G'\frown R_{n+i}} w_{n,n+i}(\zeta)\frac{\partial}{\partial n}G_{n+i}(\zeta, z)\,ds,$$

where $G_{n+i}(\zeta, z)$ is the Green's function of $G'\frown R_{n+i}$.

Since $0\leqq\frac{\partial}{\partial n}G_{n+i}(\zeta, z)\uparrow\frac{\partial}{\partial n}G(\zeta, z)$ and $w_{n,n+i}(z)\uparrow w_n(z)$ on $\partial G'$, we have by Lebesgue's theorem

$$w_n(z)=\frac{1}{2\pi}\int_{\partial G'} w_n(\zeta)\frac{\partial}{\partial n}G(\zeta, z)\,ds,$$

where $G(\zeta, z)$ is the Green's function of G'.

Next similarly $w_n(z)\downarrow w(z)$ and

$$w(z)=\lim_{n=\infty} w_n(z)=\lim_{n=\infty}\frac{1}{2\pi}\int_{\partial G'} w_n(\zeta)\frac{\partial}{\partial n}G(\zeta, z)\,ds=\frac{1}{2\pi}\int_{\partial G'}\lim_{n=\infty} w_n(\zeta)\frac{\partial}{\partial n}G(\zeta, z)\,ds$$
$$=\frac{1}{2\pi}\int_{\partial G'} w(\zeta)\frac{\partial}{\partial n}G(\zeta, z)\,ds. \quad (11)$$

On the other hand, clearly $V(z)=\lim_{i=\infty} V_i(z)$, where $V_i(z)$ is a harmonic function in $G'\frown R_{n+i}$ such that $V_i(z)=w(z)$ on $\partial G'\frown R_{n+i}$ and $V_i(z)=0$ on $\partial R_{n+i}\frown G'$. Thus as above by (11)

4) We call a niveau curve with the property: $\int_C \frac{\partial}{\partial n}\omega(z)\,ds=D(\omega(z))$ a regular niveau curve.

$$V(z)=\lim_{i=\infty} V_i(z)=\frac{1}{2\pi}\lim_{i=\infty}\int_{\partial G'\frown R_{n+i}} w(\zeta)\frac{\partial}{\partial n}G_{n+i}(\zeta,z)ds=\frac{1}{2\pi}\int_{\partial G'} w(\zeta)\frac{\partial}{\partial n}G(\zeta,z)ds=w(z).$$

Hence we have P.H.1.

Proof of P.C.1. Let $\omega_{n,n+i}(z)$ be the function in the definition of C.P. Now

$$\Omega_{n,n+i}\supset G' \text{ for } n\geqq n_0, \text{ whence } G'\frown\Omega_{n,n+i}=G'\frown R_{n+i} \text{ for } n\geqq n_0.$$

Since $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\partial R_{n+i}\frown G'$ and $D(\omega_{n,n+i}(z))\leqq M$,

$$D_{R'\frown\Omega_{n,n+i}}(\omega_{n,n+i}(z), C(z))=0,$$

for any harmonic function $C(z)$ such that $C(z)=0$ on $\partial G'$ and $D_{G'}(C(z))<\infty$. Hence by Lemma 1. *c*), $\omega_{n,n+i}(z)$ has M.D.I. over $G'\frown R_{n+i}$. Now $\omega_{n,n+i}(z)\Rightarrow\omega_n(z)$ implies by Lemma 1. *d*) that $\omega_n(z)$ has M.D.I. over G'. Also by $\omega_n(z)\Rightarrow\omega(z)$ and by the latter part of Lemma 1. *d*) $\omega(z)$ has M.D.I. over G'. Thus we have P.C.1.

Proof of P.H.2. As G' in the proof of P.H.1 take $G'=G_1-((G_2\frown R_n)$. Then $G'\frown G_2\frown(R-R_n)=0$. Put $w(z)=(G_2\frown B,z,G_1)$. Fix n at present. Then by P.H.1.

$$w(z)=\lim_{i=\infty} V_i(z),$$

where $V_i(z)$ is a harmonic function in $G'\frown R_{n+i}$ such that $V_i(z)=0=w_{n,n+i}(z)$ on $\partial G_1\frown R_{n+i}$, $V_i(z)=w_{n,n+i}(z)=0$ on $\partial R_{n+i}\frown(G_1-G')$ and $V_i(z)=w(z)\leqq w_{n,n+i}(z)=1$ on $\partial((R_{n+i}-R_n)\frown G_2)$.

Assume $\sup w(z)\leqq K<1$ in $G_1\frown(R-R_n)$. Then by the maximum principle $V_i(z)\leqq Kw_{n,n+i}(z)$. Let $i\to\infty$ and then $n\to\infty$. Then

$$w(z)\leqq Kw(z).$$

Hence $w(z)=0$. This is a contradiction. Thus we have P.H.2.

Proof of P.C.2. Put $\omega(z)=\omega(G_2\frown B,z,G_1)$. Assume $\omega(z)\leqq K<1$ in $G_2\frown(R-R_n)$. $\omega(z)$ has M.D.I. over $G'=G_1-(G_2\frown R_n)$. Hence by Lemma 1. *b*)

$$\omega(z)=\lim_{i=\infty} U_i(z),$$

where $U_i(z)$ is a harmonic function in $G'\frown R_{n+i}$ such that $U_i(z)=0=\omega_{n,n+i}(z)$ on $\partial G_1\frown R_{n+i}$, $U_i(z)=\omega(z)<\omega_{n,n+i}(z)=1$ on $\partial((R_{n+i}-R_n)\frown G_2)$ and $\frac{\partial}{\partial n}U_i(z)=0$ $=\frac{\partial}{\partial n}\omega_{n,n+j}(z)$ on $\partial R_{n+i}\frown(G_1-G')$. Hence by the maximum principle and by the assumption $U_i(z)\leqq K\omega_{n,n+i}(z)$.

Let $i\to\infty$ and then $n\to\infty$. Then $\omega(z)\leqq K\omega(z)$.

Hence we have P.C.2.

Proof of P.H.3. (P.C.3). H.M. (C.P.) of $(G_2 \frown B) \uparrow$ as $G_2 \uparrow$. Hence $w(G_2 \frown G_\delta \frown B, z, G_1) \leqq w(G_2 \frown B, z, G_1) (\omega(G_2 \frown G_\delta \frown B, z, G_1) \leqq \omega(G_2 \frown B, z, G_1))$ and

$$\sup_{z \in (G_2 \frown G_\delta)} w(G_2 \frown G_\delta \frown B, z, G_1) \leqq \sup_{z \in G_\delta} w(G_2 \frown B, z, G) \leqq 1-\delta.$$

$$\sup_{z \in (G_2 \frown G_\delta)} \omega(G_2 \frown G_\delta \frown B, z, G_1) \leqq \sup_{z \in G_\delta} \omega(G_2 \frown B, z, G_1) \leqq 1-\delta.$$

Whence by P.H.2(P.C.2)

$$w(G_2 \frown G_\delta \frown B, z, G_1)=0 \text{ and } \omega(G_2 \frown G_\delta \frown B, z, G_1)=0.$$

Proof of P.H.5 (P.C.5). Let $\omega^k_{n,n+i}(z)$ be a harmonic function in $G_1-(G_k \frown (R_{n+i}-R_n))$ such that $\omega^k_{n,n+i}(z)=0$ on ∂G_1, $\omega^k_{n,n+i}(z)=1$ on $\partial((G_1-G_k) \frown (R_{n+i}-R_n))$ and $\frac{\partial}{\partial n}\omega^k_{n,n+i}(z)=0$ on $\partial R_{n+i} \frown (G_1-G_k)$. Let $\omega_{n,n+i}(z)$ be a harmonic function in $G_1-(\sum_k G_k \frown (R_{n+i}-R_n))$ such that $\omega_{n,n+i}(z)=0$ on ∂G_1, $\omega_{n,n+i}(z)=1$ on $\partial(G_1-\sum_k G_k) \frown (R_{n+i}-R_n)$ and $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\partial R_n-(G_1-\sum_k G_k)$. Then by the maximum principle

$$\sum_k \omega^k_{n,n+i}(z) \geqq \omega_{n,n+i}(z).$$

Let $i \to \infty$ and then $n \to \infty$. Then

$$\sum_k \omega(G_k \frown B, z, G_1) \geqq \omega(\sum_k G_k \frown B, z, G_1).$$

Similarly we have P.H.5.

Proof of 6). $G_2 \supset (G_2 \frown G'_\delta)$ and $G_2=(G_2 \frown \bar{G}_\delta)$[5] $+(G_2 \frown G'_\delta)$. Hence by P.C.5 $\omega(B \frown G_2 \frown G'_\delta, z, G_1)+\omega(B \frown G_2 \frown \bar{G}_\delta, z, G_1) \geqq \omega(B \frown G_2, z, G_1) \geqq \omega(B \frown G_2 \frown G'_\delta, z, G_1)$. But by P.C.2

$$\omega(B \frown G_2 \frown \bar{G}_\delta, z, G_1)=0, \text{ whence}$$
$$\omega(B \frown G_2 \frown G'_\delta, z, G_1)=\omega(B \frown G_2, z, G_1).$$

Similarly we have P.H.6.

Proof of P.C.7. Let $\omega_{n,n+i}(z)$ be the function in the definition of $\omega(G_2 \frown B, z, G_1)$. Put $\Omega^L=E[z \in G_1 : \omega(B \frown G_2, z, G_1)<L]$ and $\Omega^L_{n,n+i}=E[z \in G_1 : \omega_{n,n+i}(z)<L] : 0<L<1$ respectively. Let Ω' be a domain completely contained in Ω^L. Since $\omega_{n,n+i}(z) \Rightarrow \omega_n(z)$ and $\omega_n(z) \Rightarrow \omega(B \frown G_2, z, G_1)$, there exist numbers n_0 and i_0 for any given number m such that

$$(R_m \frown \Omega') \subset \Omega^L_{n,n+i} \quad \text{for } n \geqq n_0 \text{ and } i \geqq i_0(i_0(n_0)).$$

Then by Fatou's lemma

$$D_{\Omega' \frown R_m}(\omega(B \frown G_2, z, G_1)) \leqq D_{\Omega' \frown R_m}(\lim_n \lim_i \omega_{n,n+i}(z)) \leqq \lim_n \lim_i D_{\Omega^L_{n,n+i} \frown R_m}(\omega_{n,n+i}(z))$$

5) $\bar{G}$ means the closure of G with respect to N-Martin's topology.

$$\leqq \lim_n \lim_i D_{\Omega^L_{n,n+i}}(\omega_{n,n+i}(z)) = \lim_n \lim_i \int_{\partial \Omega^L_{n,n+i}} \omega_{n,n+i}(z) \frac{\partial}{\partial n}\omega_{n,n+i}(z)\, ds = \lim_n \lim_i$$

$$L \int_{\partial \Omega^L_{n,n+i}} \frac{\partial}{\partial n}\omega_{n,n+i}(z)\, ds = L \lim_n \lim_i \int_{\partial G_1 \frown R_{n+i}} \frac{\partial}{\partial n}\omega_{n,n+i}(z)\, ds = L \lim_n \lim_i D_{\Omega^L_{n,n+i}}(\omega_{n,n+i}(z))$$

$$= L D_{G_1}(\omega(B \frown G_2, z, G_1)),$$

because $\omega_{n,n+i}(z) \Rightarrow \omega_n(z)$ and $\omega_n(z) \Rightarrow \omega(B \frown G_2, z, G_1)$.

Let $m \to \infty$ and then $n \to \infty$. Then let $\Omega' \uparrow \Omega^L$, then

$$D_{\Omega^L}(\omega(B \frown G_2, z, G_1)) \leqq L D_{G_1}(\omega(G_2 \frown B, z, G_1)).$$

Similarly

$$D_{G_1 - \Omega^L}(\omega(B \frown G_2 z, G_1)) \leqq (1-L) D_{G_1}(\omega(B \frown G_2, z, G_1)).$$

On the other hand,

$$D_{G_1}(\omega(B \frown G_2, z, G_1)) = D_{\Omega^L}(\omega(B \frown G_2, z, G_1)) + D_{G_1 - \Omega^L}(\omega(B \frown G_2, z, G_1)).$$

Hence

$$D_{\Omega^L}(\omega(B \frown G_2, z, G_1)) = L D_{G_1}(\omega(B \frown G_2, z, G_1)) \quad \text{for } 1 > L > 0. \tag{12}$$

Let $\omega'_n(z)$ be a harmonic function in $\Omega^L \frown R_n$ such that $\omega'_n(z) = L$ on $C_L = E[z \in G_1 : \omega(B \frown G_2, z, G_1) = L]$, $\frac{\partial}{\partial n}\omega'_n(z) = 0$ on $\Omega^L \frown \partial R_n$ and $\omega'_n(z) = 0$ on ∂G_1. Since $\omega(B \frown G_2, z, G_1)$ has M.D.I. over Ω^L, by lemma 1. *b*) $\omega'_n(z) \Rightarrow \omega(B \frown G_2, z, G_1)$ and by (12)

$$\lim_n D_{\Omega^L}(\omega'_n(z)) = D_{\Omega^L}(\omega(B \frown G_2, z, G_1)) = L D_{G_1}(\omega(B \frown G_2, z, G_1)).$$

Since $\frac{\partial}{\partial n}\omega'_n(z) \geqq 0$ on C_L and $\frac{\partial}{\partial n}\omega'_n(z) \to \frac{\partial}{\partial n}\omega(z)$ by $\omega'(z) \Rightarrow \omega(z)$, where $\omega(z) = \omega(B \frown G_2, z, G_1)$. Then by Fatou's lemma

$$\int_{C_L} \frac{\partial}{\partial n}\omega(z)\, ds \leqq \lim_n \int_{C_L \frown R_n} \frac{\partial}{\partial n}\omega'_n(z)\, ds$$

$$= \frac{1}{L} \lim_n D_{\Omega^L \frown R_n}(\omega'_n(z)) = \frac{1}{L} D_{\Omega^L}(\omega(z)) = D_{G_1}(\omega(z)).$$

Thus

$$A_L = \int_{C_L} \frac{\partial}{\partial n}\omega(z)\, ds \leqq D_{G_1}(\omega(z)) \quad \text{for every niveau curve } C_L. \tag{13}$$

Now we can take $p + iq = \omega(z) + i\tilde{\omega}(z)$ as the local parameter of every point z in G_1 except at most enumerably infinite number of branch points of $p + iq$, where $\tilde{\omega}(z)$ is the conjugate function of $\omega(z)$. Then $\frac{\partial}{\partial p}\omega(z) = 1$, $\frac{\partial}{\partial q}\omega(z) = 0$ and

$$D_{G_1}(\omega(z))=\iint_{G_1}\left\{\left(\frac{\partial}{\partial p}\omega(z)\right)^2+\left(\frac{\partial}{\partial q}\omega(z)\right)^2\right\}dp\,dq=\int_0^1\left[\int_{C_p}dq\right]dp=\int_0^1 A_p\,dp, \quad (14)$$

because $dq=d\tilde{\omega}=\frac{\partial}{\partial s}\tilde{\omega}\,ds=\frac{\partial}{\partial n}\omega\,ds$, where ds is the line element along C_p. Hence by (14) and (13)

$$A_L=D_{G_1}(\omega(B\frown G_2, z, G_1)) \quad \text{for almost } L.$$

Thus we have P.C.6.

Theorem 3. *a). Let $C_j(j=1,2)$ be regular niveau curve of* C.P. $\omega(z)$ $(=\omega(B\frown G_2,z,G_1)$ *such that* $\omega(z)=L_j$: $0<L_1<L_2<1$ *and* $\int_{C_{L_j}}\frac{\partial}{\partial n}\omega(z)ds=D(\omega(z))$. *Since $\omega(z)$ has* M.D.I. *over* $\Omega=E[z\in G_1: L_1<\omega(z)<L_2]$, $\omega_n'(z)\Rightarrow\omega(z)$ *as* $n\to\infty$, *where $\omega_n'(z)$ is a harmonic function in $\Omega\frown R_n$ such that $\omega_n'(z)=\omega(z)$ on $\partial\Omega\frown R_n$ and $\frac{\partial}{\partial n}\omega_n'(z)=0$ on $\Omega\frown\partial R_n$. Let $A_n(z)$ be a continuons function on C_{L_j} such that $A_n(z)\to A(z)$* as $n\to\infty$ *and $M\geqq A_n(z)\geqq 0$ for every n. Then*

$$\int_{C_{L_j}} A(z)\frac{\partial}{\partial n}\omega(z)\,ds=\lim_n\int_{C_{L_j\frown R_n}} A_n(z)\frac{\partial}{\partial n}\omega_n'(z)\,ds.$$

b). Let $U_n(z)$ be a harmonic function in $\Omega\frown R_n$ such that $U_n(z)=0$ on C_{L_1}, $0\leqq U_n(z)\leqq N$ on C_{L_2} and $\frac{\partial}{\partial n}U_n(z)=0$ on $\Omega\frown\partial R_n$. *If $U_n(z)\to U(z)$, then*

$$\int_{C_{L_1}} A(z)\frac{\partial}{\partial n}U(z)\,ds=\lim_n\int_{C_{L_1\frown R_n}} A_n(z)\frac{\partial}{\partial n}U_n(z)\,ds.$$

Proof of a). Assume that there exist a positive number δ and infinitely many numbers n and i such that

$$\int_{C_{L_j\frown(R_{n+i}-R_n)}}\frac{\partial}{\partial n}\omega_n'(z)\,ds>\delta>0.$$

Then by $(L_2-L_1)\int_{C_{L_j\frown R_{n+i}}}\frac{\partial}{\partial n}\omega_{n+i}'(z)\,ds=D_{\Omega\frown R_{n+i}}(\omega_{n+i}'(z))$,

$$\begin{aligned}\int_{C_{L_j\frown R_n}}\frac{\partial}{\partial n}\omega_{n+i}'(z)\,ds&=\int_{C_{L_j\frown R_{n+i}}}\frac{\partial}{\partial n}\omega_{n+i}'(z)\,ds-\int_{C_{L_j\frown(R_{n+i}-R_n)}}\frac{\partial}{\partial n}\omega_{n+i}'(z)\,ds\\&\leqq D_{\Omega\frown R_{n+i}}(\omega_{n+i}'(z))\big/(L_2-L_1)-\delta.\end{aligned} \quad (15)$$

On the other hand, $\frac{\partial}{\partial n}\omega_{n+i}'(z)\to\frac{\partial}{\partial n}\omega(z)$ on C_{L_j} and $D_{\Omega\frown R_{n+i}}(\omega_{n+i}'(z))\uparrow D_\Omega(\omega(z))$ $=D_{G_1}(\omega(z))\,(L_2-L_1)$.

Then by Fatou's lemma and by (15)

$$\int_{C_{L_j}\frown R_n} \frac{\partial}{\partial n}\omega(z)\,ds=\int_{C_{L_j}\frown R_n}\Big(\lim_i \frac{\partial}{\partial n}\omega'_{n+i}(z)\Big)ds\leqq \varliminf_i \int_{C_{L_j}\frown R_{n+i}} \frac{\partial}{\partial n}\omega'_{n+i}(z)ds-\delta=D_{G_1}(\omega(z))-\delta.$$

Let $n\to\infty$. Then $\int_{C_{L_j}} \frac{\partial\omega}{\partial n}(z)\,ds\leqq D_{G_1}(\omega(z))-\delta$.

This contradicts the regularity of C_{L_j}. Hence for any given positive number ε, there exists a number n_0 such that

$$0\leqq \int_{C_{L_j}\frown(R_{n+i}-R_n)} \frac{\partial}{\partial n}\omega'_{n+i}(z)\,ds<\frac{\varepsilon}{M}\quad \text{for } i\geqq 0 \text{ and } n\geqq n_0. \tag{16}$$

At present fix $m(\geqq n_0)$. Then by $\frac{\partial}{\partial n}\omega'_{n+i}(z)\to\frac{\partial}{\partial n}\omega(z)$ and $A_n(z)\to A(z)$ on $C_{L_j}\frown R_n$, there exists a number i_0 such that

$$\int_{C_{L_j}\frown R_n} A_{m+i}(z)\frac{\partial}{\partial n}\omega'_{m+i}(z)\,ds-\varepsilon\leqq\int_{C_{L_j}\frown R_m} A(z)\frac{\partial}{\partial n}\omega(z)\,ds\quad \text{for } i\geqq i_0. \tag{17}$$

By (16) and (17)

$$\int_{C_{L_j}} A(z)\frac{\partial}{\partial n}\omega(z)\,ds\geqq\int_{C_{L_j}\frown R_m} A(z)\frac{\partial}{\partial n}\omega(z)\,ds\geqq\int_{C_{L_j}\frown R_m} A_{m+i}(z)\frac{\partial}{\partial n}\omega'_{n+i}(z)\,ds-\varepsilon$$

$$=\int_{C_{L_j}\frown R_{m+i}} A_{m+i}(z)\frac{\partial}{\partial n}\omega'_{m+i}(z)\,ds-\int_{C_{L_j}\frown(R_{m+i}-R_m)} A_{m+i}(z)\frac{\partial}{\partial n}\omega'_{m+i}(z)\,ds-\varepsilon$$

$$\geqq\int_{C_{L_j}\frown R_{m+i}} A_{m+i}(z)\frac{\partial}{\partial n}\omega'_{m+i}(z)\,ds-2\varepsilon.$$

Let $\varepsilon\to 0$. Then

$$\int_{C_{L_j}} A(z)\frac{\partial}{\partial n}\omega(z)\,ds\geqq\varlimsup_n\int_{C_{L_j}\frown R_n} A_n(z)\frac{\partial}{\partial n}\omega'_n(z)\,ds.$$

On the other hand, by Fatou's Lemma

$$\int_{C_{L_j}} A(z)\frac{\partial}{\partial n}\omega(z)\,ds=\int_{C_{L_j}}\Big(\lim_n A_n(z)\frac{\partial}{\partial n}\omega'_n(z)\Big)\,ds\leqq\varliminf_n\int_{C_{L_j}} A_n(z)\frac{\partial}{\partial n}\omega'_n(z)\,ds.$$

Hence

$$\int_{C_{L_j}} A(z)\frac{\partial}{\partial n}\omega(z)\,ds=\lim_n\int_{C_{L_j}\frown R_n} A_n(z)\frac{\partial}{\partial n}\omega'_n(z)\,ds.$$

Proof of b). By the maximum principle $U_n(z)\leqq N\frac{(\omega'_n(z)-L_1)}{L_2-L_1}$. Hence $0\leqq\frac{\partial}{\partial n}U_n(z)\leqq\frac{N}{L_2-L_1}\frac{\partial}{\partial n}\omega'_n(z)$ on C_{L_1}, whence there exists a number n_0 for any given positive number ε such that

$$0 \leqq \int_{C_{L_1} \frown (R_{n+i}-R_n)} \frac{\partial}{\partial n} U_{n+i}(z)\, ds \leqq \frac{L_2-L_1}{MN}\varepsilon \quad \text{for } n \geqq n_0 \text{ and } i \geqq 0. \tag{18}$$

Thus similarly as a) we have b).

4. Harmonic measures and capacity potentials of a closed set F and of a decreasing sequence of compact or non compact domains.

Let F be a closed set in $\overline{R}-R_0$ with respect to δ-metric. Put $F_m = E\left[z \in \overline{R}: \delta(z, F) \leqq \frac{1}{m}\right]$. Let $w_{m,n}(z)(\omega_{m,n}(z))$ be a function in $(R_n \frown G_1)$ such that $w_{m,n}(z)(\omega_{m,n}(z))$ is harmonic in $(R_n \frown G_1)-(F_m \frown G_2)$, $w_{m,n}(z)=\omega_{m,n}(z)=1$ on $F_m \frown G_2$, $w_{m,n}(z)=\omega_{m,n}(z)=0$ on $\partial G_1 \frown R_n$ and $w_{m,n}(z)=\frac{\partial}{\partial n}\omega_{m,n}=0$ on $\partial R_n -(F_m \frown G_2)$. Then $w_{m,n}(z) \uparrow w_m(z)$ as $n \to \infty$ and $w_m(z) \downarrow w(z)$ as $m \to \infty$. If $D(\omega_{m,n}(z)) < M$ for a certain number m_0 and for every $n \geqq 0$, $\omega_{m,n}(z) \Rightarrow \omega_m(z)$ as $n \to \infty$ and $\omega_m(z) \Rightarrow \omega(z)$ as $m \to \infty$. We denote $w(z)$ and $\omega(z)$ by $w(F \frown G_2, z, G_1)$ and $\omega(F \frown G_2, z, G_1)$ respectively. Let $\{V_m\}$ $(m=1,2,\cdots)$ be a decreasing sequence of compact or non compact domains. We define H.M. $w(\{V_m\} \frown G_2, z, G_1)$ and C.P. $\omega(\{V_m\} \frown G_2, z, G_1)$ of $\{V_m\} \frown G_2$ as H.M. and C.P. of $F \frown G_2$ by replacing $V_m \frown G_2$ instead of $F_m \frown G_2$. We proved the properties of H.M.(C.P.) only by the fact that $w_{n,n+i}(z) \uparrow w_n(z)$ and $w_n(z) \downarrow w(z)$ $(\omega_{n,n+i}(z) \Rightarrow \omega_n(z)$ and $\omega_n(z) \Rightarrow \omega(z))$. Hence these H.M.'s and C.P.'s have all the properties stated before. In this paper we denote by P.H.N.(C.P.N.) $(N=1,2,\cdots 7)$ the properties of the above H.M.'s and C.P.'s respectively

If $G_1 = R-R_0$ and $G_2 \frown F_{m_0} \frown R_1 = 0$ or $G_2 \frown V_{m_0} \frown R_1 = 0$, by the Dirichlet principle

$$D(\omega_{m,n}(z)) < D(\hat{\omega}(z)) \quad \text{for } m \geqq m_0 \text{ and } n \geqq 0,$$

where $\hat{\omega}(z)$ is a harmonic function in $R_1 - R_0$ such that $\hat{\omega}(z)=0$ on ∂R_0 and $\omega(z)=1$ on ∂R_1. In this case, we omit $R-R_0$ and denote it by $w(G_2 \frown F, z)$ $(\omega(G_2 \frown F, z))$ simply.

5. Superharmonic function in $\overline{R}-R_0$.

Let G be a compact or non compact domain in $R-R_0$. If $U(z)$ is continuous in G except a closed set of capacity zero and $U(z)$ has partial derivatives almost everywhere in G, we call $U(z)$ a *C_1-class function.*

Let $U(z)$ be a positive function of C_1-class in G and continuous on ∂G except a set of capacity zero such that $U(z)=0$ on $\partial R_0 \frown G$ (may be void) and $D_G(\min(M, U(z))) < \infty$ for $0 < M < \infty$. Let ${}_{CG}U^M(z)$ be a harmonic function in G such that ${}_{CG}U^M(z) = \min(M, U(z))$ on ∂G and ${}_{CG}U^M(z)$ has M.D.I. $\leqq D(\min(M, U(z)))$ over G. Then ${}_{CG}U^M(z)$ is uniquely determined by Lemma 1. a). ${}_{CG}U^M(z) \uparrow$ as $M \uparrow \infty$. Put ${}_{CG}U(z) = \lim_{M=\infty} {}_{CG}U^M(z)$. If ${}_{CG}U(z) = U(z)$, we call $U(z)$ a $\overline{harmonic}$ *function* in G with boundary value $U(z)$ on ∂G. Let $U(z)$ be a positive function of C_1-class in $R - R_0$ such that $U(z) = 0$ on ∂R_0 and $D(\min(M, U(z)) < \infty$ for $M < \infty$. Let D be a compact or non compact domain in $R - R_0$. Let ${}_DU^M(z)$ be a function such that ${}_DU^M(z) = \min(M, U(z))$ on $\partial D + D$ and ${}_DU^M(z)$ is $\overline{\text{harmonic}}$ in $R - R_0 - D$ with ${}_DU^M(z) = 0$ on ∂R_0 (may be void). Put ${}_DU(z) = \lim_{M=\infty} {}_DU^M(z)$. If ${}_DU(z) \leqq U(z)$ for every *domain D such that ∂D is compact*, we say that $U(z)$ *is $\overline{superharmonic}$ in $\overline{R} - R_0$*. From this definition, if $U(z)$ is $\overline{\text{superharmonic}}$ in $\overline{R} - R_0$ and if $U(z)$ is continuous in an open compact set G in $R - R_0$, $U(z)$ is superharmonic in G (in ordinary sense).

Lemma 2. a). $\overset{*}{M}$*aximum principle. Let $U_i(z)$ $(i = 1, 2)$ be a $\overline{harmonic}$ function in a compact or non compact domain G such that $U_1(z) \geqq U_2(z)$ on ∂G. Then*

$$U_1(z) \geqq U_2(z).$$

b). Let $U(z)$ be a $\overline{harmonic}$ function in G such that $M \geqq U(z)$ on ∂G. Then

$$U(z) \leqq M \text{ in } G.$$

c). Let D be a compact or non compact domain in $R - R_0$. Let $U(z)$ be a positive function of C_1-class in $R - R_0$ such that $U(z) = 0$ on ∂R_0 and $D(\min(M, U(z))) < \infty$ for $M < \infty$ and ${}_DU(z) \leqq U(z)$. Then

$$D(min(M, {}_DU(z))) \leqq D(min(M, U(z))).$$

Proof of a). Let ${}_iU_n^M(z)$ be a harmonic function in $G \frown R_n$ such that ${}_iU_n^M(z) = \min(M, U_i(z))$ on $\partial G \frown R_n$ and $\frac{\partial}{\partial n} {}_iU_n^M(z) = 0$ on $\partial R_n \frown G$. Then ${}_1U_n^M(z) \geqq {}_2U_n^M(z)$. Let $n \to \infty$ and then $M \to \infty$. Then

$$\lim_M \lim_n {}_1U_n^M(z) = U_1(z) \geqq U_2(z) = \lim_M \lim_n {}_2U_n^M(z).$$

Similarly we have b).

Proof of c).

$$E[z \in R : {}_DU^L(z) < M] = {}^L\Omega^M \downarrow {}_D\Omega^M$$
$$= E[z \in R : {}_DU(z) < M] \supset \Omega^M = E[z \in R : U(z) < M] \text{ as } L \to \infty.$$

Suppose $L \geqq M$. ${}_DU^L(z)$ has M.D.I. over $R - R_0 - D$ with value $\min(U(z), L)$

on $\partial R_0+\partial D$. This implies by Lemma 1. *a*) that ${}_DU^L(z)$ has also M.D.I. over ${}^L\Omega^M-D$ with value ${}_DU^L(z)=M=\min(M, U(z))$ on $\partial^L\Omega^M-D$ and with value ${}_DU^L(z)=\min(M, {}_DU(z))=U(z)$ on $\partial D\frown{}^L\Omega^M$, by ${}_DU^L(z)\leqq U(z)$, i.e. ${}_DU^L(z)=\min(M, U(z))$ on $\partial(R-R_0-{}^L\Omega^M-D)$.

Hence $\qquad D_{{}^L\Omega^M-D}(\min(M, {}_DU^L(z)))\leqq D_{{}^L\Omega^M-D}(\min(M, U(z)))$.

On the other hand, ${}_DU^L(z)=\min(L, U(z))$ in D and by $M\leqq L$

$$D_D(\min(M, {}_DU^L(z)))=D_D(\min(M, L, U(z)))=D_D(\min(M, U(z)))$$

and $M\leqq {}_DU^L(z)\leqq U(z)$ in $R-R_0-D-{}^L\Omega^M$, whence $\min(M, {}_DU^L(z))=\min(M, U(z))=M$ in $R-R_0-{}^L\Omega^M$ and

$$D_{R-R_0-D-{}^L\Omega^M}(\min(M, {}_DU^L(z)))=D_{R-R_0-D-{}^L\Omega^M}(\min(M, U(z)))=0.$$

Thus $\quad D_{{}^L\Omega^M}(\min(M, {}_DU^L(z)))=D_{R-R_0}(\min(M, {}_DU^L(z)))\leqq D_{R-R_0}(\min(M, U(z)))=D_{\Omega^M}(\min(M, U(z)))$.

Let $U(z)$ be a positive function of C_1-class and $U(z)=0$ on ∂R_0. If ${}_DU(z)$ exists (${}_DU(z)$ has M.D.I. $<\infty$ over $R-R_0-D$ and ${}_DU(z)=U(z)$ on $\overline{D}$) for any compact or non compact domain such that ∂D is compact and $\sup_{z\in\partial D} U(z)<\infty$ and if ${}_DU(z)\leqq U(z)$, we say that $U(z)$ is $\overline{\overline{su}}$*perharmonic in* $\overline{R}-R_0$ *in the weak sense.*

Theorem 4. *a*). *Let $U(z)$ be a positive function of C_1-class with $U(z)=0$ on ∂R_0 and $D(\min(M, U(z)))<\infty$. Let D be a domain (compact or non compact) and G be a domain with compact ∂G. If ${}_DU(z)\leqq U(z)$ and ${}_GU(z)\leqq U(z)$, then*

$${}_G({}_DU(z))\leqq{}_DU(z).$$

Therefore if $U(z)$ is $\overline{\overline{su}}$perharmonic (${}_GU(z)\leqq U(z)$ for any domain G with compact ∂G) and ${}_DU(z)\leqq U(z)$, ${}_G({}_DU(z))\leqq{}_DU(z)$, i.e. ${}_DU(z)$ is $\overline{\overline{su}}$per-harmonic in $\overline{R}-R_0$.

If $U(z)$ is $\overline{\overline{su}}$perharmonic in $\overline{R}-R_0$ (${}_DU(z)\leqq U(z)$ and ${}_GU(z)\leqq U(z)$ for any domains D and G with compact ∂D and ∂G, ${}_DU(z)$ is also $\overline{\overline{su}}$per-harmonic in $\overline{R}_0-R$.

a'). *If $U(z)$ is $\overline{\overline{su}}$perharmonic in $\overline{R}-R_0$ in the weak sense and if ${}_DU(z)$ and ${}_GU(z)$ are defined ($\sup_{z\in(\partial D+\partial G)} U(z)<\infty$), then ${}_G({}_DU(z))$ is defined and*

$${}_G({}_DU(z))\leqq{}_DU(z),$$

where ∂D and ∂G are compact.

b). *Let $U(z)$ be $\overline{\overline{su}}$perharmonic in $\overline{R}-R_0$, then for any domains D_1 and D_2 with compact relative boundaries*

$${}_{D_2}({}_{D_1}U(z)) = {}_{D_1}U(z) \leqq {}_{D_2}U(z) : D_1 \subset D_2.$$

b′). Let $U(z)$ be $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$ in the weak sense. Then for any domains D_1 and D_2 with compact relative boundary such that $\sup_{z \in \partial D_i} U(z) < \infty$ $(i=1,2)$, we have

$${}_{D_2}({}_{D_1}U(z)) = {}_{D_1}U(z) \leqq {}_{D_2}U(z) : D_1 \subset D_2.$$

c). Let $U(z)$ be $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$ and put $D_n = D \frown R_n$. Then $\lim_n {}_{D_n}U(z) = {}_DU(z)$.

c′). Let $U(z)$ be $\overline{\text{superharmonic}}$ $\overline{R}-R_0$ in the weak sense. Suppose $D_n \supset D'_n \supset D_{n+1}$, $D_n = D \frown R_n$ $(n=1,2,\cdots)$ and $\sup_{z \in \partial D'_n} U(z) < \infty$. Then ${}_{D'_n}U(z) \uparrow U^*(z) \leqq U(z)$. If ${}_DU^M(z)$ exists for every $M < \infty$, then $U^*(z) = \lim_{M=\infty} {}_DU^M(z)$.

d). Let $U(z)$ be $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$. Then ${}_DU(z) \leqq U(z)$ for compact or non compact domain D.

e). Let $U(z)$ be $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$. Then ${}_DU(z)$ is $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$ for a compact or non compact domain D.

f). Let $U(z)$ be $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$. Then for compact or non compact domains D_1 and $D_2 : D_2 \supset D_1$

$${}_{D_2}({}_{D_1}U(z)) = {}_{D_1}U(z) \leqq {}_{D_2}U(z).$$

g). Let $U_n(z)$ $(n=1,2,\cdots)$ be a $\overline{\text{superharmonic}}$ function in $\overline{R}-R_0$ and $U_n(z) \to U(z)$ in every compact domain in $R-R_0$. If $D(\min(M, U(z)))$ $< \infty$ for $M < \infty$, then $U(z)$ is $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$.

g′). Let $U_n(z)$ $(n=1,2,\cdots)$ be $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$ in the weak sense such that $\sup_{z \in \partial D} U_n(z) < \infty$. If $U_n(z) \to U(z)$ in $R-R_0$ and if ${}_DU(z)$ exists, $U(z)$ is $\overline{\text{superharmonic}}$ in the weak sense.

h). Let $U_n(z)$ $(n=1,2,\cdots)$ be $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$ such that $U_n(z)$ is continuous in $R-R_0$ and $U_n(z) \uparrow U(z)$. If $U(z)$ is finite in $R-R_0$ and *$D(\min(M, U(z))) < \infty$ for $M < \infty$, then for compact or non compact domain D*

$${}_D(\lim_n U_n(z)) = {}_DU(z) = \lim_n {}_DU_n(z).$$

i). Let $U(z)$ and $V(z)$ be $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$. Then for a compact or non compact domain D

$${}_DU(z) + {}_DV(z) = {}_D(U(z)+V(z)),$$
$$V(z) \geqq U(z) \text{ implies } {}_DV(z) \geqq {}_DU(z),$$
$$C({}_DU(z)) = {}_D(CU(z)) \text{ for } C \geqq 0.$$

j). Let D_1 and D_2 be two compact or non compact domains. Then

$${}_{D_1+D_2}U(z) \leqq {}_{D_1}U(z) + {}_{D_2}U(z).$$

k). $\min(M, U(z))$ *and* $\min(U(z), V(z))$ *are* $\overline{\text{superharmonic}}$ *in* $\bar{R}-R_0$.

Proof of a) Since $\lim_{L=\infty} {}^{L}_{D}U(z)={}_{D}U(z)$ on ∂G, for any given numbers M and ε there exists a number L_0 such that $L_0>M$ and $\varepsilon+{}^{L_0}_{D}U(z)\geqq\min(M, {}_{D}U(z))$ on ∂G.

In fact, assume that there exists a sequence $\{z_i\}$ on ∂G such that

$$ {}^{L_i}_{D}U(z_i)\leqq\min(M, {}_{D}U(z_i))-\delta_0:\quad \delta_0>0 $$

for infinitely many numbers L_i such that $\lim_i L_i=\infty$.

Since ∂G is compact, there exists a point z^* such that $z'_i\to z^*$, where $\{z'_i\}$ is a subsequence of $\{z^i\}$. Then two cases occur.

Case 1. $z\in\partial D$. In this case ∂D is composed of analytic curves and every point of ∂D is regular for Dirichlet problem. Now ${}_{D}U^{M}(z)=\min(M, U(z))$ on ∂D. Hence

$$ \lim_{i=\infty} U(z'_i)\geqq\lim_{i=\infty}(\min(L_0, U(z'_i)))\geqq\lim_i {}_{D}U^{M}(z'_i)=\min(M, U(z'_i))\quad\text{for } L_0\geqq M. $$

Case 2. $z\in\partial G-D$. In this case, there exists a neighbourhood $\upsilon(z^*)$ of z^* such that $\upsilon(z^*)\frown\bar{D}=0$. ${}_{D}U^{L_i}(z): i=1,2,\cdots$ are $\overline{\text{harmonic}}$ in $\upsilon(z^*)$ and ${}^{L_i}_{D}U(z)\uparrow{}_{D}U(z)$ uniformly in $\upsilon(z^*)$, whence $\lim_i {}_{D}U^{L_i}(z'_i)\geqq\min(M, {}_{D}U(z^*))$. Cases 1 and 2 are contradinctions. Hence

$$ \varepsilon+{}^{L_0}_{D}U(z)\geqq\min(M, {}_{D}U(z))\quad\text{on}\quad\partial G. $$

Let ${}_{G}V^{M}(z)$ be a $\overline{\text{harmonic}}$ function in $R-R_0-G$ such that ${}_{G}V^{M}(z)=\min(M, {}_{D}U(z)))$ on $\bar{G}+\partial R_0$. This can be defined by $D(\min(M, U_D(z)))\leqq D(\min(M, U(z)))<\infty$ by Lemma 2. c). By the assumption: $U(z)\geqq{}_{G}U(z)$ and $U(z)\geqq U_D(z)$, which imply

$$ {}^{M}_{G}U(z)=\min(M, U(z))\geqq\min(M, {}_{D}U(z))={}_{G}V^{M}(z)\quad\text{on}\quad\partial G. $$

Both ${}_{G}U^{M}(z)$ and ${}_{G}V^{M}(z)$ have M.D.I. over $R-R_0-G$. Hence by the $\overset{*}{\text{maximum}}$ principle

$$ U(z)\geqq{}_{G}U^{M}(z)\geqq{}_{G}V^{M}(z)\quad\text{in}\quad R-R_0-G. $$

Whence

$$ {}_{D}U^{L}(z)=\min(L, {}_{D}U(z))=\min(L, U(z)) $$
$$ \geqq\min(M, {}_{G}U^{M}(z))\geqq{}^{M}_{G}V(z)\geqq{}_{G}V^{M}(z)-\varepsilon\text{ on }\bar{D}-G. $$

On the other hand, $\varepsilon+{}^{L}_{D}U(z)\geqq\min(M, {}_{D}U(z)-\varepsilon)$ on ∂G for $L\geqq L_0$, and both ${}_{G}V^{M}(z)$ and ${}_{D}U^{L}(z)$ have M.D.I. over $R-R_0-D-G$.

Hence by the $\overset{*}{\text{maximum}}$ principle

$$ {}_{D}U^{L}(z)\geqq{}_{G}V^{M}(z)-\varepsilon\quad\text{in}\quad R-R_0-G-D\quad\text{for}\quad L\geqq L_0. $$

Let $L\to\infty$ and then $M\to\infty$ and then $\varepsilon\to 0$. Then

$${}_DU(z)=\lim_M {}_GV^M(z)=\lim_M {}^M_G({}_DU(z))={}_G({}_DU(z)) \quad \text{in } R-R_0-D-G.$$

${}_GU^M(z)\geqq {}_GV^M(z)$ by $U(z)\geqq {}_DU(z)$ in $R-R_0-G$, whence by the $\overset{*}{\text{maximum}}$ principle

$${}_DU^L(z)=\min(U(z), L)\geqq {}_GU^M(z)\geqq {}_GV^M(z) \quad \text{on} \quad \overline{D}-G.$$

Next ${}_DU(z)\geqq \min(M, {}_DU(z))={}_GV^M(z)$ on G.

Let $L\to\infty$ and then $M\to\infty$. Then ${}_DU(z)\geqq {}_G({}_DU(z))$ on $\overline{D}+\overline{G}$.

Thus $${}_DU(z)\geqq {}_G({}_DU(z)).$$

The latter part of a) is proved at once by the definition of $\overline{\text{su}}$perharmonicity in $\overline{R}-R_0$.

Proof of a'). By the assumption ${}_DU(z)$ and ${}_GU(z)$ exist and

$$D_{R-R_0-D}({}_DU(z))<\infty \text{ and } D_{R-R_0-G}({}_GU(z))<\infty.$$

Put $T(z)=\min({}_DU(z), {}_GU(z))$. Then

$$\left|\frac{\partial T(z)}{\partial x}\right|\leqq \max\left(\left|\frac{\partial_D U(z)}{\partial x}\right|, \left|\frac{\partial_G U(z)}{\partial x}\right|\right), \quad \left|\frac{\partial T(z)}{\partial y}\right|\leqq \max\left(\left|\frac{\partial_D U(z)}{\partial y}\right|, \left|\frac{\partial_G U(z)}{\partial y}\right|\right)$$

whence $D_{R-R_0-G-D}(T(z))\leqq D_{R-R_0-G}({}_GU(z))+D_{R-R_0-D}({}_DU(z))<\infty$.

By $T(z)={}_GU(z)$ in $D-G$, because ${}_DU(z)=U(z)\geqq {}_GU(z)$ in D,

$$D_{D-G}(T(z))=D({}_GU(z))<\infty.$$

Hence $$D_{R-R_0-G}(T(z))<\infty.$$

On the other hand,

$$T(z)={}_GU(z)=U(z)={}_DU(z) \text{ on } D\frown\partial G \text{ by } {}_GU(z)\leqq U(z)={}_DU(z) \text{ in } \overline{D}\frown\overline{G},$$

$$T(z)={}_DU(z) \text{ on } \partial G-D \text{ by } {}_DU(z)\leqq U(z)={}_GU(z) \text{ on } \partial G,$$

hence $$T(z)={}_DU(z) \text{ on } \partial R_0+\partial G \text{ and } D_{R-R_0-G}(T(z))<\infty.$$

Hence there exists a harmonic function $H(z)$ in $R-R_0-G$ such that $H(z)$ has M.D.I. ($\leqq D_{R-R_0-G}(T(z))<\infty$) over $R-R_0-G$ and $H(z)={}_DU(z)$ on $\partial G+\partial R_0$. Thus ${}_G({}_DU(z))$ ($=H(z)$ in $R-R_0-G$ and $={}_DU(z)$ in G) is defined. Now as above

$${}_G({}_DU(z))\leqq {}_DU(z) \text{ on } \partial R_0+\partial D+\partial G$$

and by the $\overset{*}{\text{maximum}}$ principle

$${}_G({}_DU(z))\leqq {}_DU(z) \quad \text{in } R-R_0-G-D.$$

$${}_G({}_DU(z))={}_DU(z) \text{ in } G \text{ and } {}_G({}_DU(z))\leqq {}_GU(z)\leqq U(z)={}_DU(z) \text{ in } D.$$

Thus

$${}_G({}_DU(z))\leqq {}_DU(z).$$

Proof of b). ${}_{D_1}U^L(z)$ has M.D.I. over $R-R_0-D_1$, whence by Lemma 1. a) ${}_{D_1}U^L(z)$ has M.D.I. over $R-R_0-D_2$ with value ${}_{D_1}U^L(z)$ in $\overline{D}_2+\partial R_0$, i.e. ${}_{D_1}U^L(z)$ is $\overline{\text{h}}$armonic in $R-R_0-D_2$.

Let $V^L(z)$ be a $\overline{\text{h}}$armonic function in $R-R_0-D_2$ such that $V^L(z)={}_{D_1}U^L(z)$ on $\overline{D}_2+\partial R_0$. Then by the $\overset{*}{\text{m}}$aximum principle $V^L(z)={}_{D_1}U^L(z)$.

Hence $$\lim_{L=\infty} V^L(z)=\lim_{L=\infty} {}_{D_1}U^L(z)={}_{D_1}U(z).$$

Let $W^L(z)$ be a $\overline{\text{h}}$armonic function in $R-R_0-D_2$ with $W^L(z)=\min(L, {}_{D_1}U(z))$ on $\overline{D}_2+\partial R_0$. Then by $\min(L, {}_{D_1}U(z))\geqq {}_{D_1}U^L(z)$ on $\partial R_0+\overline{D}_2$ and by the $\overset{*}{\text{m}}$aximum principle $W^L(z)\geqq V^L(z)$. Let $L\to\infty$.

Then $${}_{D_2}({}_{D_1}U(z))=\lim_{L=\infty} W^L(z)\geqq \lim_{L=\infty} V^L(z)={}_{D_1}U(z).$$

On the other hand, by a) ${}_{D_1}U(z)$ is $\overline{\text{s}}$uperharmonic, whence ${}_{D_2}({}_{D_1}U(z))\leqq {}_{D_1}U(z)$. Thus
$${}_{D_2}({}_{D_1}U(z))={}_{D_1}U(z).$$

Next from $U(z)\geqq {}_{D_1}U(z)$,
$${}_{D_2}U(z)\geqq {}_{D_2}({}_{D_1}U(z))={}_{D_1}U(z).$$

Proof of b′). $D_{R-R_0-D_2}({}_{D_1}U(z))\leqq D_{R-R_0-D_2}({}_{D_1}U(z))<\infty$, whence as above we have
$${}_{D_2}U(z)\geqq {}_{D_2}({}_{D_1}U(z))={}_{D_1}U(z).$$

Proof of c). $D_n=D\frown R_n$ is compact, ${}_{D_n}U(z)$ increases to a function $U^*(z)(\leqq U(z))$ as $n\to\infty$ by b). By lemma 2. c) $D(\min(M, {}_{D_n}U(z)))\leqq D(\min(M,U(z)))<\infty$. $\min(M, {}_{D_n}U(z))$ is harmonic or a constant M in $R-R_0-D$ and $=\min(M, U(z))$ in D_n. Hence by Fatou's lemma
$$D_{R-R_0-D}(\min(M, U^*(z))\leqq \underline{\lim}_{n=\infty} D(\min(M, {}_{D_n}U(z)))\leqq D(\min(M, U(z)))<\infty.$$
By the $\overline{\text{s}}$uperharmonicity of $U(z)$ ${}_{D_n}U^M\leqq U(z)$ on ∂D and has M.D.I. over $R-R_0-D$ by $(R-R_0-D)\subset(R-R_0-D_n)$, whence by the $\overset{*}{\text{m}}$aximum principle
$${}_{D_n}U^M(z)\leqq \sup_{z\in\partial D_n} {}_{D_n}U^M(z)\leqq M \quad \text{in} \quad R-R_0-D_n.$$
By Lemma 1. a) ${}_{D_n}U^M(z)$ has M.D.I. over $R-R_0-D$ with value $\leqq\min(M, U(z))$ on $\overline{D}+\partial R_0$. On the other hand, ${}_DU^M(z)$ has M.D.I. over $R-R_0-D$ with value $\min(M, U(z))$ $(\geqq {}_{D_n}U^M(z))$ on $\overline{D}$. Hence by the $\overset{*}{\text{m}}$aximum principle
$${}_DU^M(z)\geqq {}_{D_n}U^M(z) \quad \text{in} \quad R-R_0-D.$$
Clearly $\min(M, U(z))={}_DU^M(z)\geqq {}_{D_n}U^M(z)$ in D. Let $M\to\infty$ and then $n\to\infty$. Then
$${}_DU(z)\geqq U^*(z)=\lim_n {}_{D_n}U(z). \tag{19}$$

Since $D_{R-R_0}({}_DU^M(z))<\infty$, for any given positive number $\varepsilon>0$, there exists a number n_0 such that $D_{D-D_n}({}_DU^M(z))<\varepsilon$ for $n\geqq n_0$. Now ${}_{D_n}U^M(z)$

$=\min(M, U(z))={}_DU^M(z)$ on $\overline{D}_n+\partial R_0$ and ${}_{D_n}U^M(z)$ has M.D.I. over $R-R_0-D_n$. Hence

$$D_{R-R_0-D_n}({}_{D_n}U^M(z))\leqq D_{R-R_0-D_n}({}_DU^M(z))$$
$$=D_{R-R_0-D_n}({}_DU^M(z))+D_{D-D_n}({}_DU^M(z))\leqq D_{R-R_0-D}({}_DU^M(z))+\varepsilon.$$

${}_{D_n}U^M(z)\uparrow$ as $n\to\infty$. Put $V^M(z)=\lim_{n=\infty}{}_{D_n}U^M(z)$ $(\leqq\lim_n{}_{D_n}U(z)=U^*(z))$. (20)

Then $V^M(z)={}_DU^M(z)$ on $\overline{D}$ and ${}_{D_n}U^M(z)$ is harmonic in $R-R_0-D_n$ and further derivatives of ${}_{D_n}U^M(z)\to$ those of $V^M(z)$. Hence

$$D_{R-R_0-D}(V^M(z))\leqq\underline{\lim}_n D_{R-R_0-D}({}_{D_n}U^M(z))\leqq D_{R-R_0-D}({}_DU^M(z))+\varepsilon.$$

Let $\varepsilon\to 0$. Then $V^M(z)$ has M.D.I. over $R-R_0-D$, because ${}_DU^M(z)(=V^M(z)$ on $\partial D)$ has M.D.I. Hence by Lemma 1. *b*)

$$V^M(z)={}_DU^M(z) \quad\text{in}\quad R-R_0. \tag{21}$$

By ${}_{D_n}U^M(z)\leqq{}_{D_n}U(z)$ and by (20), (19) and (21)

$${}_DU(z)=\lim_{M=\infty}{}_DU^M(z)=\lim_{M=\infty}V^M(z)=\lim_M(\lim_n{}_{D_n}U^M(z))\leqq\lim_n{}_{D_n}U(z)=U^*(z)\leqq{}_DU(z).$$

Thus we have *c*). *c'*) is proved similarly.

Proof of d). If ∂D is compact, it is clear by definition. If ∂D is not compact, put $D_n=D\frown R_n$. Then by *c*)

$${}_DU(z)=\lim_{n=\infty}{}_{D_n}U(z)\leqq U(z).$$

Proof of e). If ∂D is compact, this case reduces to the case of Theorem 4. *a*). Suppose that ∂D is non compact. Let G be a domain such that ∂G is compact. Then by Lemma 2. *b*) $D(\min(M, {}_DU(z)))\leqq D(\min(M, U(z)))<\infty$ and by *d*) ${}_DU(z)\leqq U(z)$. Hence by *a*)

$${}_G({}_DU(z))\leqq{}_DU(z).$$

Thus ${}_DU(z)$ is $\overline{\text{superharmonic}}$ in $\overline{R}-R_0$.

Proof of f). ${}_{D_1}U^M(z)$ has M.D.I. over $R-R_0-D_1(\supset(R-R_0-D_2))$, whence

$${}_{D_2}({}_{D_1}U(z))\geqq{}_{D_2}({}_{D_1}U^M(z))={}_{D_1}U^M(z).$$

Let $M\to\infty$. Then ${}_{D_2}({}_{D_1}U(z))\geqq{}_{D_1}U(z)$. On the other hand, ${}_{D_1}U(z)$ is $\overline{\text{super}}$-harmonic by *e*), whence ${}_{D_2}({}_{D_1}U(z))\leqq{}_{D_1}U(z)$ by *d*). Hence ${}_{D_2}({}_{D_1}U(z))={}_{D_1}U(z)$ and by ${}_{D_1}U(z)\leqq U(z)$, we have also

$${}_{D_2}U(z)\geqq{}_{D_2}({}_{D_1}U(z))={}_{D_1}U(z).$$

Thus we have *f*).

Proof of g). It is sufficient to show ${}_DU(z)\leqq U(z)$ for domain whose relative boundary ∂D is compact. Suppose that ∂D is compact. Since ${}_DU(z)$ $=\lim_M{}_DU^M(z)$, for any given positive number ε, there exists a number M_0 such that (see *a*))

$${}_DU(z)\leqq{}_DU^M(z)+\varepsilon \quad\text{for}\quad M\geqq M_0.$$

Now by $U_n^M(z)\to U^M(z)$ on ∂D, there exists a number $n_0(M)$ by the $\overset{*}{\text{maximum}}$ principle such that

$$|{}_DU_n^M(z)-{}_DU^M(z)|<\varepsilon \quad \text{in} \quad R-R_0-D;\, n\geqq n_0(M).$$

Hence ${}_DU(z)\leqq{}_DU^M(z)+\varepsilon\leqq{}_DU_n^M(z)+2\varepsilon\leqq\min(M,\ U_n(z))+2\varepsilon$ in $R-R_0-D$. Let $n\to\infty$ and then $\varepsilon\to 0$. Then ${}_DU(z)\leqq U(z)$ in $R-R_0-D$ and ${}_DU(z)=U(z)$ in $\overline{D}$. Thus $U(z)$ is $\overline{\text{super}}$harmonic in $\overline{R}-R_0$. g') is proved similarly.

Proof of h). Put $D_m=D\frown R_m$. Then ∂D_m is compact. By $U(z)\geqq U_n(z)$,

$${}_{D_m}U(z)\geqq{}_{D_m}U_n(z).$$

Let $m\to\infty$. Then ${}_DU(z)\geqq{}_DU_n(z)$ and ${}_DU(z)\geqq\lim_{n=\infty}{}_DU_n(z)$.

Conversely, since $U_D(z)=\lim{}_{D_m}U(z)$, there exists a number m such that $U_D(z)\leqq{}_{D_m}U(z)+\varepsilon$. Next by c) for any given positive number ε, since ∂D_m is compact and $U_n(z)\to U(z)$ on ∂D_m, there exists a number $n(m)$ such that

$$\begin{aligned}{}_DU(z)\leqq{}_{D_m}U(z)+\varepsilon\leqq{}_{D_m}U_n(z)+2\varepsilon\leqq{}_DU_n(z)+2\varepsilon\\ \text{in } R-R_0-D_m(\supset(R-R_0-D)) \quad \text{for } n\geqq n(m).\end{aligned}$$

In D_m, ${}_DU(z)=U(z)=\lim_n U_n(z)=\lim_n{}_{D_m}U_n(z)\leqq\lim_n{}_DU_n(z)$.

Let $\varepsilon\to 0$. Then $\quad {}_DU(z)\leqq\lim_n{}_DU_n(z).$

Thus we have h).

Proof of i) and j) are clear by the definition and by the $\overset{*}{\text{maximum}}$ principle.

Proof of k). Put $T(z)=\min(U(z),\ V(z))$. Then

$$\left|\frac{\partial T(z)}{\partial x}\right|\leqq\max\left(\left|\frac{\partial U(z)}{\partial x}\right|,\left|\frac{\partial V(z)}{\partial x}\right|\right),\quad \left|\frac{\partial T(z)}{\partial y}\right|\leqq\max\left(\left|\frac{\partial U(z)}{\partial y}\right|,\left|\frac{\partial V(z)}{\partial y}\right|\right).$$

$$E[z\in R\colon T(z)<M]=E[z\in R\colon U(z)<M]+E[z\in R\colon V(z)<M].$$

Hence $D(\min(M,T(z)))\leqq D(\min(M,U(z)))+D(\min(M,V(z)))<\infty$ for every $M<\infty$.

Let G be a compact or non compact domain in $R-R_0$. Then $T(z)\leqq U(z)$ and $\leqq V(z)$ on ∂G. Hence by the $\overset{*}{\text{maximum}}$ principle

$${}_GT(z)\leqq\min({}_GU(z),\ {}_GV(z))\leqq\min(U(z),\ V(z)=T(z).$$

Thus $T(z)$ is $\overline{\text{super}}$harmonic in $\overline{R}-R_0$. The latter part is proved similarly.

6. Integral representation of $\overline{\text{super}}$harmonic functions.

Theorem 5. *a*). C.P.'s $\omega(B\frown G,z)$, $\omega(F,z)$, $\omega(\{V\},z)$ *and* $N(z,p)$: $p\in\overline{R}-R_0$ *and* $\int N(z,p)\,d\mu(p):d\mu(p)\geqq 0$ *are* $\overline{\text{super}}$*harmonic in* $\overline{R}-R_0$.

b). Let $U(z)$ be a positive $\overline{\text{su}}$perharmonic function in $\overline{R}-R_0$ with $U(z)=0$ on ∂R_0. Let F be a closed set in $\overline{R}-R_0$. Put $F_m=E\left[z\in\overline{R}: \delta(z,F)\leq\frac{1}{m}\right]$. Then F_m is a compact or non compact closed domain. By Theorem 4. f), ${}_{F_m}U(z)\downarrow$. Let ${}_FU(z)=\lim_m {}_{F_m}U(z)$. Then ${}_FU(z)$ is represented by a non negative mass distribution on F such that ${}_FU(z)=\int N(z,p)\,d\mu(p)$ for $z\in R-R_0$.

b′). Let $U(z)$ be a positive $\overline{\text{su}}$perharmonic function in $\overline{R}-R_0$ in the weak sense with $U(z)=0$ on ∂R_0. Let $G_{m,n}$ be a domain such that $K_{m,n}\subset G_{m,n}\subset K_{m-1,n-1}$ and $\sup_{z\ni\partial G_{m,n}} U(z)<\infty$, where K is a closed set in $\overline{R}-R_0$, $K_m=E\left[z\in\overline{R}: \delta(z,K)\leq\frac{1}{m}\right]$ and $K_{m,n}=K_m\frown R_n(n=1,2,\cdots)$. Such $G_{m,n}$ can be chosen by that $U(z)$ is continuous except a set of capacity zero. Then $\lim_m\lim_n G_{m,n}=K$ *and by a′) of Theorem 4* $\lim_m\lim_n {}_{G_{m,n}}U(z)$ *exists. Put ${}_KU(z)$* $=\lim_m\lim_n {}_{G_{m,n}}U(z)$. *Then ${}_KU(z)$ is represented by a mass distribution on K. Hence $U(z)$ is $\overline{\text{su}}$perharmonic in $\overline{R}-R_0$ by a). If $U(z)$ is $\overline{\text{su}}$perharmonic in the weak sense and harmonic in $R-R_0-F$, then $U(z)={}_KU(z)$* $=\int_K N(z,p)\,d\mu(p)$ *by putting $K=F+B$. Therefore $U(z)$ is $\overline{\text{su}}$perharmonic by a) and in this case there is no distinction between $\overline{\text{su}}$perharmonicity and the $\overline{\text{su}}$perharmonicity in the weak sense.*

Proof of a). Put $F_{m,n}=F_m\frown R_n$. Let $\omega_{m,n}(z)=\omega(F_{m,n},z)$ i.e. C.P. of $F_{m,n}(\omega_{m,n}(z)=1$ on $F_{m,n}$, $\omega_{m,n}(z)=0$ on ∂R_0 and $\omega_{m,n}(z)$ has M.D.I. over $R-R_0-F_{m,n})$. Let R_1' be a subset of R_1 such that $R_0\subset R_1'$, $R_1'\frown F_{m,n}=0$ for $m\geqq m_0$ and $\partial R_1'$ is compact relative boundary. Let $\tilde{\omega}(z)$ be a harmonic function in $R_1'-R_0$ such that $\tilde{\omega}(z)=0$ on ∂R_0 and $=1$ on $\partial R_1'$. Then by the Dirichlet principle

$$D(\omega_{m,n}(z))\leqq D(\tilde{\omega}(z)) \quad \text{for } m\geqq m_0 \text{ and } n\geqq 0,$$

whence

$$D(\min(M,\omega_{m,n}(z)))<\infty \quad \text{for every } M. \tag{22}$$

Let D be a domain in $R-R_0$ with compact relative boundary. Then ${}_D(\omega_{m,n}(z))=\omega_{m,n}(z)$ on $(\partial D\frown CF_{m,n})+\partial R_0$ and ${}_D\omega_{m,n}(z)<1=\omega_{m,n}(z)$ on $F_{m,n}\frown CD$. Now both ${}_D(\omega_{m,n}(z))$ and $\omega_{m,n}(z)$ have M.D.I. over $R-R-F_{m,n}-D$. Hence by the $\overset{*}{\text{m}}$aximum principle

$$ {}_D\omega_{m,n}(z)\leqq\omega_{m,n}(z). \tag{23}$$

Hence by (22) and (23) $\omega_{m,n}(z)$ is $\overline{\text{su}}$perharmonic in $\overline{R}-R_0$. Now $\omega_{m,n}(z)$ $\Rightarrow\omega_m(z)$ and $\omega_m(z)\Rightarrow\omega(z)$, whence $\omega(F,z)=\omega(z)$ is $\overline{\text{su}}$perharmonic in $\overline{R}-R_0$ by Theorem 4. g). For other C.P.'s we can prove similarly. We show that $N(z,p)$ is $\overline{\text{su}}$perharmonic in $\overline{R}-R_0$. $D(\min(M,N(z,p)))\leqq 2\pi M$ by Theorem 1. b). Next let D be a domain with compact relative boundary ∂D. Put $V_M(p)=E\left[z\in R: N(z,p)>M\right]$.

Case 1. $p\in D$. In this case $V_M(p)\subset D$ for sufficiently large M by Theorem 1. a). Since $N(z,p)$ has M.D.I. over $R-R_0-V_M(p)\supset(R-R_0-D)$, ${}_DN(z,p)=N(z,p)$.

Case 2. $p\notin\overline{D}$: $N(z,p)={}_{D+V_M(p)}N(z,p)$ by case 1. Let $M>\sup_{z\in\partial D}N(z,p)$. Then ${}_DN(z,p)$ has M.D.I. over $R-R_0-D$, whence by the $\overset{*}{\text{m}}$aximum principle ${}_DN(z,p)<M$ in $R-R_0-D-V_M(p)$. ${}_DN(z,p)$ and ${}_{D+V_M(p)}N(z,p)$ have M.D.I. over $R-R-V_M(p)-D$. Hence by the $\overset{*}{\text{m}}$aximum principle

$${}_DN(z,p)\leqq{}_{D+V_M(p)}N(z,p)=N(z,p) \quad\text{in}\quad R-R_0-V_M(p)-D,$$

because ${}_DN(z,p)\leqq{}_{D+V_M(p)}N(z,p)$ on $\partial D+\partial V_M(p)+\partial R_0$.
And $N(z,p)\geqq M\geqq{}_DN(z,p)$ in $V_M(p)$ and ${}_DN(z,p)=N(z,p)$ in D. Hence $N(z,p)$ $\geqq{}_DN(z,p)$.

Case 3. $p\in\partial D$. In this case ${}_DN^M(z,p)\leqq M$ on $V_M(p)\frown CD$.[5] Hence as in case 2

$${}_DN(z,p)\leqq{}_{D+V_M(p)}N(z,p)=N(z,p) \quad\text{in}\quad R-R_0-D-V_M(p).$$

Let $M\to\infty$. Then $V_M(p)\to p\in\partial D$ and ${}_DN(z,p)=\lim_M{}_DN^M(z,p)\leqq N(z,p)$ in $R-R_0-D$. Now ${}_DN(z,p)=N(z,p)$ in $\overline{D}$. Hence ${}_DN(z,p)\leqq N(z,p)$. Thus by case 1, 2 and 3 $N(z,p)$ is $\overline{\text{su}}$pperharmonic in $\overline{R}-R_0$ for $p\in R-R_0$.

Next suppose $N(z,p)=\lim_i N(z,p_i)$: $p\in B$ and $p_i\in R-R_0$, where $\{p_i\}$ is a fundamental sequence. Then by the $\overline{\text{su}}$perharmonicity of $N(z,p_i)$, $N(z,p)$ is $\overline{\text{su}}$perharmonic in $\overline{R}-R_0$ by Theorem 4. g) and by $D(\min(M,N(z,p)))$ $\leqq 2\pi M$.

Let $V(z)=\int N(z,p)\,d\mu(p)$. Since $N(z,p)$ is continuous (for fixed z) with respect to p, the approximation to $V(z)$ is done in every compact domain in $R-R_0$ by $V_n(z)=\sum_{i=1}^{n}c_iN(z,p):c_i>0$, $\sum c_i=\int d\mu(p)$, $p_i\in R-R_0$ $(n=1,2,\cdots)$.

Since $N(z,p_i)=\infty$ at p_i, there exists a neighbourhood υ of $\sum p_i$ such

5) CD means the complementary set of D.

that $\sum c_i N(z, p_i) > N$ in υ for any given large number N. Now $N(z, p_i)$ has M.D.I. over $R-R_0-\upsilon$. Hence $V_n(z)$ has M.D.I. over Ω_M ($\subset R-R_0-\upsilon$) for $M<N$: $\Omega_M=E[z\in R: V_n(z)>M]$, whence $V_n^m(z)\Rightarrow V_n(z)$ as $m\to\infty$, where $V_n^m(z)$ is a harmonic function in $(R_m-R_0)\frown\Omega_M$ such that $V_n^m(z)=0$ on ∂R_0, $V_n^m(z)=M$ on $\partial\Omega_N\frown R_m$ and $\frac{\partial}{\partial n}V_n^m(z)=0$ on ∂R_m. Hence

$$D_{\Omega_M}(V_n(z))=\lim_m D_{\Omega_M}(V_n^m(z))=\lim_m M\int_{\partial\Omega_M\frown R_m}\frac{\partial}{\partial n}V_n^m(z)\,ds=M\lim_m\int_{\partial R_0}\frac{\partial}{\partial n}V_n^m(z)\,ds$$
$$=M\int_{\partial R_0}\frac{\partial}{\partial n}\sum^n c_i N(z, p_i)\,ds=2\pi M\sum^n c_i.$$

Now $V_n(z)\to V(z)$, whence

$$D(\min(M, V(z)))\leqq\lim_n D(\min(V_n(z), M))=2\pi M\int d\mu(p). \tag{24}$$

Clearly $V_n(z)\geqq_D(V_n(z))$ for any domain D with compact ∂D in $R-R_0$. Hence $V_n(z)$ is $\overline{\text{s}}$uperharmonic in $\overline{R}-R_0$. Now $V_n(z)\to V(z)$. Hence by (24) and by the $\overline{\text{s}}$uperharmonicity of $V_n(z)$, $V(z)$ is $\overline{\text{s}}$uperharmonic in $\overline{R}-R_0$ by Theorem 4. g).

Proof of b). Let F'_m be a closed set such that every point of $\partial F'_m$ is regular for Dirichlet problem, $U(z)$ is continuous on $\partial F'_m$ and $F_m\subset F'_m\subset F_{m-1}$ ($m=1,2,\cdots$). Put $F'_{m,n}=F'_m\frown R_n$. Now $U(z)$ is superharmonic (in ordinary sense) at every point of F'_m. Hence it can be proved by the method of F. Riesz-Frostmann that the functional

$$J(\mu)=\frac{1}{2}\frac{1}{4\pi^2}\iint N(z, p)\,d\mu(p)\,d\mu(z)-\frac{1}{2\pi}\int U(z)\,d\mu(z)$$

is minimized by a unique mass distribution $\mu_{m,n}$ on $F'_{m,n}$ among all non negative mass distributions. The function $V(z)$ given by $\frac{1}{2\pi}\int N(z,p)\,d\mu_{m,n}(p)$ is equal to $U(z)$ on $F'_{m,n}$ and $U(z)=V(z)$ on $\partial F'_{m,n}$ by the regularity of $\partial F'_{m,n}$. $V(z)$ is continuous ($=U(z)$) on $\partial F'_{m,n}$. Since $F'_{m,n}$ is compact, the continuity principle of the potential in euclidean space is valid, whence $V(z)$ is continuous in $R-R_0-F'_{m,n}+\partial F'_{m,n}$. Put $K=F'_{m,n}$ and $K_l=E\left[z\in R: \delta(z, K)\leqq\frac{1}{l}\right]$. Since K is closed and compact, $U(z)-V(z)$ and $U(z)-{}_KU(z)$ are uniformly continuous in $R-R_0-K$. Hence for any given positive number ε, there exists a number l_0 such that

$$|U(z)-V(z)|<\varepsilon \text{ and } |U(z)-{}_KU(z)|<\varepsilon \text{ on } \partial K_{l_0},$$

by $U(z)-V(z)=0=U(z)-{}_KU(z)$ on ∂K.

We can find a sequence $V_m(z)=\sum_{i=1}^{m} c_i N(z,p_i)$ $(m=1,2,\cdots)$ such that the total mass of $V_m(z)=\sum^{m} c_i$ and $V_m(z)\to V(z)$ in $R-R_0-K_l$ and every pole p_i of $V_m(z)$ is contained in $K_l(l>2l_0)$. Since $V_m(z)\to V(z)$, there exists a number m_0 such that

$$|V_m(z)-V(z)|<\varepsilon \text{ on } \partial K_{l_0} \text{ for } m\geqq m_0.$$

Hence

$$|{}_K U(z)-V_m(z)|<|{}_K U(z)-U(z)|+|U(z)-V(z)|+|V(z)-V_m(z)|<3\varepsilon \text{ on } \partial K_{l_0}.$$

${}_K U(z)$ $(={}_{K_{l_0}}U(z))$ and $V_m(z)$ have M.D.I. over $R-R_0-K_l$, whence by the *maximum principle

$$|{}_K U(z)-{}_m V(z)|<3\varepsilon \text{ in } R-R_0-R_{l_0}.$$

Let $m\to\infty$ and then $\varepsilon\to 0$. Then ${}_K U(z)=V(z)$ in $R-R_0-K_{l_0}$, whence ${}_K U(z)=V(z)$ in $R-R_0-F'_{m,n}$, where the total mass of ${}_{F'm,n}U(z)$ is given by $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}{}_{F'm,n}U(z)\,ds\leqq\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds$ for every n and m. Since $N(z,p)$ is a continuous function of $p\in\overline{R}-R_0$ for fixed z and the total mass of $\mu_{m,n}$ is less than $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds$ by ${}_{F'm,n}U(z)\leqq{}_{F'm}U(z)\leqq U(z)$, $\{\mu_{m,n}\}$ has an weak limit μ_m on F'_m as $n\to\infty$. Hence ${}_{F'm}U(z)=\frac{1}{2\pi}\int N(z,p)\,d\mu_m(p)$ and by letting $m\to\infty$, ${}_F U(z)=\frac{1}{2\pi}\int_F N(z,p)\,d\mu(p)$.

Proof of b'). The former part is proved similarly, and the latter part is easily proved by taking account of the fact that $U(z)={}_{K_m}U(z)$, because $R-R_0-K_m$ is compact, where $K_m=E\left[z\in\overline{R}:\ \delta(z,(F+B))\leqq\frac{1}{m}\right]$.

Theorem 6. *a). Let A be a closed set of capacity zero. If $U(z)$ is positively superharmonic in $\overline{R}-R_0$ and harmonic in $R-R_0-A$ with $U(z)=0$ on ∂R_0. Then $U(z)-{}_A U(z)$ is superharmonic and*

$${}_A U(z)={}_A({}_A U(z)).$$

b). Let $\{A_m\}$ $(m=1,2,\cdots)$ be a sequence of decreasing domain such that $\omega(\{A_m\},z)=0$. If $U(z)$ is positively superharmonic in $\overline{R}-R_0$ and harmonic in $R-R_0-A_m$ with $U(z)=0$ on ∂R_0. Then $U(z)-\lim_m {}_{A_m}U(z)$ is superharmonic in $\overline{R}-R_0$ and $\lim_m {}_{A_m}U(z)=\lim_{m=\infty}{}_{A_m}(\lim_{m=\infty}{}_{A_m}U(z))$.

Proof. Let G be a domain in $R-R_0$ such that ∂G is compact and $\sup_{z\in\partial G}U(z)<N<\infty$. Put

$$U(z)={}_G U(z)+V(z).$$

Then $V(z)=0$ on $\partial R_0+\partial G$ and $V(z)$ *is* $\overline{\textit{superharmonic}}$ *in* $\overline{R}-R_0-G$.

In fact, $V(z)=0$ on $\partial R_0+\partial G$. $V(z)\leqq M$ implies $U(z)\leqq M+N$ in $R-R_0-G$, because ${}_GU(z)\leqq N$ in $R-R_0-G$ by the $\overset{*}{\mathrm{m}}$aximum principle. Hence

$$E[z\in R-R_0-G: V(z)<M]=\Omega_V^M\subset\Omega_U^{M+N}=E[z\in R-R_0-G: U(z)<M+N].$$

Hence $D(\min(M, V(z)))=D_{\Omega_V^M}(V(z))\leqq D_{\Omega_U^{M+N}}(U(z)-{}_GU(z))\leqq D_{\Omega_U^{M+N}}(U(z))$

$$+D_{R-R_0-G}({}_GU(z))+2\sqrt{D_{\Omega_U^{M+N}}(U(z),{}_GU(z))}\leqq L(M)<\infty. \tag{25}$$

Let Ω be a domain in $R-R_0$ (not necesarily $\Omega\frown G=0$). *Let* ${}_\Omega V^M(z,G)$ be *a function in* $R-R_0-G-\Omega$ *such that* ${}_\Omega V^M(z,G)=0=V(z)$ *on* $\partial R_0+\partial G$, ${}_\Omega V^M(z,G)=\min(M, V(z))$ *on* $\partial\Omega$, ${}_\Omega V^M(z,G)$ *is harmonic in* $R-R_0-G-\Omega$ *and* ${}_\Omega V^M(z,G)$ *has* M.D.I. *over* $R-R_0-G-\Omega$, which can be defined by (25).

Put $\Omega'=E[z\in\Omega: U(z)>M+N]$. Then $U(z)\geqq M+N$ on $\partial\Omega'$ and $U(z)\leqq M+N$ on $\partial\Omega-\Omega'$. Now ${}_GU(z)\leqq N$ on ∂G, whence by the $\overset{*}{\mathrm{m}}$aximum principle

$$ {}_GU(z)\leqq N \quad \text{in } R-R_0-G. \tag{26}$$

${}_{G+\Omega}U^{M+N}(z)=\min(M+N, U(z))=M+N$ and ${}^{M+N}_{G+\Omega}({}_GU(z))=$ $\min(M+N, {}_GU(z))={}_GU(z)\leqq N$ by (26) on $(\Omega\frown\partial\Omega')-G$, whence

$${}_{G+\Omega}U^{M+N}(z)-{}^{M+N}_{G+\Omega}({}_GU(z))=M+N-{}_GU(z)\geqq M\geqq{}_\Omega V^M(z,G) \quad \text{on } (\partial\Omega'\frown\Omega)-G.$$

${}_{G+\Omega}U^{M+N}(z)=\min(M+N, U(z))=U(z)$ and ${}^{M+N}_{G+\Omega}({}_GU(z))=\min(M+N, {}_GU(z))$ $={}_GU(z)$ on $\partial\Omega-\Omega'-G$, whence

$${}_{G+\Omega}U^{M+N}(z)-{}^{M+N}_{G+\Omega}({}_GU(z))=U(z)-{}_GU(z)\geqq{}_\Omega V^M(z,G) \quad \text{on } \partial\Omega-\Omega'-G.$$

${}_{G+\Omega}U^{M+N}(z)=U(z)={}^{M+N}_{G+\Omega}({}_GU(z))={}_GU(z)$ on ∂G and ${}_{G+\Omega}U^{M+N}(z)-{}^{M+N}_{G+\Omega}({}_GU(z))$ $=0={}_\Omega V^M(z,G)$ on ∂G, whence

$${}_{G+\Omega}U^{M+N}(z)-{}^{M+N}_{G+\Omega}({}_GU(z))\geqq{}_\Omega V^M(z,G) \quad \text{on } \partial R_0+\partial G+\partial\Omega.$$

Now ${}_{G+\Omega}U^{M+N}(z)$, ${}^{M+N}_{G+\Omega}({}_GU(z))$ and ${}_\Omega V^M(z,G)$ are $\overline{\text{harmonic}}$ in $R-R_0-G-\Omega$. Hence by the $\overset{*}{\mathrm{m}}$aximum principle

$${}_{G+\Omega}U^{M+N}(z)-{}^{M+N}_{G+\Omega}({}_GU(z))\geqq{}_\Omega V^M(z,G) \quad \text{in } R-R_0-G-\Omega.$$

On the other hand, since ${}_GU(z)\leqq N$ in $R-R_0-G$,

$${}^{M+N}_{G+\Omega}({}_GU(z))={}_{G+\Omega}({}_GU(z))={}_GU(z) \quad \text{in } R-R_0-G \text{ by } G+\Omega\supset G.$$

By $U(z)\geqq{}_{G+\Omega}U^{M+N}(z)$

$$V(z)=U(z)-{}_GU(z)\geqq{}_{G+\Omega}U^{M+N}(z)-{}^{M+N}_{G+\Omega}({}_GU(z))\geqq{}_\Omega V^M(z,G) \quad \text{in } R-R_0-G-\Omega.$$

Let $M\to\infty$. Then $V(z)\geqq{}_\Omega V(z,G)$ in $R-R_0-G-\Omega$. Put ${}_\Omega V(z,G)=V(z)$ in Ω. Then $V(z)\geqq{}_\Omega V(z,G)$ in $R-R_0-G$. *Thus by* (25) $V(z)$ *is* $\overline{\textit{superharmonic}}$ *in* $\overline{R}-R_0-G$.

Let D be a domain with compact ∂D in $R-R_0$ such that $\sup_{z\in\partial D} U(z) < L < \infty$. Then ${}_DU(z)$ has M.D.I. over $R-R_0-D$ $(<D(\min(L, U(z))))$, whence ${}_DU(z)$ has also M.D.I. over $R-R_0-G-D$. Put $V(z,G)=U(z)-{}_GU(z)$ and ${}_\Omega V(z,G)=\lim_{M=\infty} {}_\Omega V^M(z,G)$.

Since ${}_DU(z)-{}_G({}_DU(z))=0={}_DV(z,G)$ on $\partial G-D+\partial R_0$,

$$ {}_DU(z)-{}_G({}_DU(z))-{}_DV(z,G)=0 \text{ on } \partial G-D+\partial R_0. \tag{27}$$

Since ${}_DV(z,G)=U(z)-{}_GU(z)=V(z)$ on $\partial D-G$ and since $U(z)={}_DU(z)$ on $\partial D-G$,

$$ {}_DU(z)-{}_G({}_DU(z))-{}_DV(z,G)=U(z)-{}_G({}_DU(z))-{}_DV(z)=U(z) \\ -{}_G({}_DU(z))-(U(z)-{}_GU(z))={}_GU(z)-{}_G({}_DU(z)) \text{ on } \partial D-G. \tag{28}$$

Since ${}_DU(z)-{}_G(U(z))-{}_DV(z,G)$ is harmonic in $R-R_0-G-D$ and has M.D.I. $<\infty$ over $R-R_0-G-D$ (because ${}_DU(z)$, ${}_G({}_DU(z))$ and ${}_DV(z,G)$ have M.D.I. $<\infty$ over $R-R_0-G-D$), we have by (27) and (28)

$$ T(z,D,G)={}_DU(z)-{}_G({}_DU(z))-{}_DV(z,G) \text{ in } R-R_0-G-D, \tag{29}$$

where $T(z,D,G)$ is a harmonic function in $R-R_0-G-D$ such that $T(z,D,G)=0$ on $\partial R_0+\partial G-D$ and $T(z,D,G)={}_DU(z)-{}_G({}_DU(z))-{}_DV(z,G)={}_GU(z)-{}_G({}_DU(z))\geqq 0$ on $\partial D-G$. Let $\omega(D,z)$ be C.P. of D. Then since $N\omega(D,z)\geqq T(z,D,G)=0$ on $\partial R_0+\partial G$ and by (28) $T(z,D,G)={}_GU(z)-{}_G({}_DU(z))\leqq {}_GU(z)\leqq N=N\omega(D,z)$ on $\partial D-G$, where $N\geqq \sup_{z\in\partial G} U(z)$. Now $T(z,D,G)$ and $\omega(D,z)$ have M.D.I. over. $R-R_0-G-D$, whence by the $\overset{*}{\text{m}}$aximum principle

$$ T(z,D,G)\leqq N\omega(D,z). $$

Put $D=A'_{m,n}=A'_m\frown R_n$, where A'_m is a domain such that $A_{m+1}\subset A'_m\subset A_m$, $A_m=E\left[z\in\overline{R}: \delta(z,A)\leqq\frac{1}{m}\right]$ and $\sup_{z\in\partial A'_{m,n}} U(z)<\infty$ *for every m and n.*

Then ${}_{A'_{m,n}}U(z)\uparrow {}_{A'_m}U(z)$ and ${}_{A'_m}U(z)\downarrow {}_AU(z)$. Since $V(z)$ is superharmonic in $\overline{R}-R_0-G$, ${}_{A'_{m,n}}V(z,G)\uparrow {}_{A'_m}V(z,G)$ and ${}_{A'_m}V(z,G)\downarrow {}_AV(z,G)$. Whence $T(z,A'_{m,n},G)\to T(z,A'_m,G)$ and $T(z,A'_m,G)\to T(z,A,G)$. Now by the assumption $0\leqq T(z,A,G)\leqq N\omega(A,z)=0$.

Thus
$$ {}_AU(z)={}_G({}_AU(z))+{}_AV(z,G) \text{ in } R-R_0-G-A. \tag{30}$$

By ${}_AV(z,G)\leqq V(z)$ (since $V(z)$ is superharmonic in $\overline{R}-R_0-G$) and ${}_G({}_AU(z))\leqq {}_GU(z)$ (because ${}_AU(z)\leqq U(z)$) and by $U(z)={}_GU(z)+V(z)$, we have by (30)

$$ U(z)-{}_AU(z)\geqq({}_GU(z)-{}_G({}_AU(z)))+(V(z)-{}_AV(z,G))\geqq {}_GU(z)-{}_G({}_AU(z)). \tag{31}$$

Now ${}_GU(z)-{}_G({}_AU(z))$, ${}_GU(z)$ and ${}_G({}_AU(z))$ have M.D.I. $(\leqq 4D(\min(N, U(z))))$: $N>\sup_{z\in\partial G} U(z))$ over $R-R_0-G$. Hence by the $\overset{*}{\text{m}}$aximum principle

$${}_{G}U(z)-{}_{G}({}_{A}U(z))={}_{G}(U(z)-{}_{A}U(z)).$$

By (31) $\qquad U(z)-{}_{A}U(z)\geqq{}_{G}(U(z)-{}_{A}U(z))$ in $R-R_0-G-A$. $\qquad$ (32)

Put $U(z)-{}_{A}U(z)={}_{G}U(z)-{}_{G}({}_{A}U(z))$ in G. Then

$$U(z)-{}_{A}U(z)\geqq{}_{G}(U(z)-{}_{A}U(z)) \quad \text{in } R-R_0-A. \tag{33}$$

But G is any compact domain such that $\sup_{z\in\partial G} U(z)<\infty$. *Hence* $K(z)=U(z)$ $-{}_{A}U(z)$ *is* $\overline{\text{superharmonic}}$ *in the weak sense by* (33). *Whence* $K(z)$ *is* $\overline{\text{superharmonic}}$ *in* $\overline{R}-R_0$ *by Theorem* 5. *b*).

By (30) and (29) and by putting $D=A'_{m,n}$ and by letting $n\to\infty$,

$${}_{A'_m}U(z)-{}_{A}U(z)={}_{G}({}_{A'_m}U(z))-{}_{G}({}_{A}U(z))+({}_{A'_m}V(z,G)-{}_{A}V(z,G))+T(z,A'_m,G)$$
$$\geqq{}_{G}({}_{A'_m}U(z))-{}_{G}({}_{A}U(z))={}_{G}({}_{A'_m}U(z)-{}_{A}U(z)) \text{ (similarly as (31))},$$

because ${}_{A'_m}V(z,G)\geqq{}_{A}V(z,G)$ and $T(z,A'_m,G)\geqq 0$.

Put $G=A'_{m',n}=A'_{m'}\frown R_n$ such that $A'_{m'}\supset A'_m$ (i.e. $m'<m$) and $\sup_{z\in\partial A'_{m,n}} U(z)$ $<\infty$. Then

$${}_{A'_m}U(z)-{}_{A}U(z)\geqq{}_{A'_{m',n}}({}_{A'_m}U(z)-{}_{A}U(z))={}_{A'_{m',n}}({}_{A'_m}U(z))-{}_{A'_{m',n}}({}_{A}U(z)). \tag{33'}$$

By the $\overline{\text{superharmonicity}}$ of ${}_{A'_m}U(z)$ and ${}_{A}U(z)$, because ${}_{A}U(z)$ is limit of ${}_{A'_m}U(z)$,

$${}_{A'_{m',n}}({}_{A'_m}U(z))\uparrow{}_{A'_{m'}}({}_{A'_m}U(z)) \text{ and } {}_{A'_{m',n}}({}_{A}U(z))\uparrow{}_{A'_{m'}}({}_{A}U(z)) \text{ as } n\to\infty.$$

On the other hand, since $A'_{m'}$ and A'_m can be considered as domains. Then by Theorem 4. *f*) ${}_{A'_{m'}}({}_{A'_m}U(z))={}_{A'_m}U(z)$. Hence by (33′) we have by letting $n\to\infty$

$${}_{A}U(z)\leqq{}_{A'_{m'}}({}_{A}U(z)).$$

Let $m'\to\infty$. Then ${}_{A}({}_{A}U(z))\geqq{}_{A}U(z)$. On the other hand, ${}_{A}U(z)$ is $\overline{\text{super}}$-harmonic (because ${}_{A}U(z)$ is the limit of $\overline{\text{superharmonic}}$ functions ${}_{A'_m}U(z)$), whence ${}_{A}({}_{A}U(z))\leqq{}_{A}U(z)$. Thus

$${}_{A}({}_{A}U(z))={}_{A}U(z).$$

b) is proved similarly.

Theorem 7. *a*). *Let A be a closed set in $\overline{R}-R_0$. Then* $\omega(A,z)={}_{A}(\omega(A,z))=\int_A N(z,p)\,d\mu(p)$ *for* $\omega(A,z)\geqq 0$.

b). $\omega(p,z)=0$ *for* $p\in R-R_0$. *If p is an ideal boundary point such that* $\omega(p,z)>0$, *then*

$$\omega(p,z)=KN(z,p),\ K>0.$$

We call such a point a singular boundary point and denote by B_S *the* *set of singular boundary points.*

c). $\upsilon_n(p)=E\left[z\in\overline{R}:\ \delta(z,p)<\frac{1}{n}\right]$. *Then* ${}_{\upsilon_n(p)}N(z,p)\downarrow$ *and has limit* $(={}_pN(z,p))$ *as* $n\to\infty$. *Put* $\phi(\upsilon_n(p))=\int_{\partial R_0}\frac{\partial}{\partial n}\,{}_{\upsilon_n(p)}N(z,p)\,ds$ *and* $\phi(p)=\lim_{n=\infty}\phi(\upsilon_n(p))$. *Then* $\phi(p)=\int_{\partial R_0}\frac{\partial}{\partial n}\,{}_pN(z,p)\,ds$ *and* $\phi(p)=1$ *for* $p\in R-R_0+B_S$ *dan further* $\phi(p)=1$ *or* 0 *for* $p\in\overline{R}-R_0$.

$\phi(\upsilon_n(p))$ *is lower semicontinuous with respect to* δ*-metric. Denote by* B_0 *and* B_1 *the set of points* p *of* B *for which* $\phi(p)=0$ *and* $\phi(p)=1$ *respectively. Then by b)* $B_S\subset B$, $B=B_0+B_1$ *and* B_0 *is an* F_σ *set or void.*

d). B_0 *is an* F_σ *set of capacity zero, whence* $B_S\subset B_1$.

e). If $U(z)$ *is given by* $\int_{B_0}N(z,p)\,d\mu(p)\ (\mu(p)\geqq 0)$, *then* ${}_{B_0}U(z)=0$.

f). If $U(z)$ *is positively harmonic in* $R-R_0-F$ *with* $U(z)=0$ *on* ∂R_0 *and superharmonic in* $\overline{R}-R_0$,

$$U(z)=\int_{B+F}N(z,p)\,d\mu(p),$$

where F is a closed in $\overline{R}-R_0$.

Proof of a). Put $A_n=E\left[z\in\overline{R}:\ \delta(z,A)\leqq\frac{1}{n}\right]$. Then by Theorem 2, P.C.1 $\omega(A,z)=\lim_n {}_{A_n}\omega(A,z)$, whence $\omega(A,z)={}_A\omega(A,z)$. Next by Theorem 5.*b*) ${}_A\omega(A,z)=\int_A N(z,p)\,d\mu(p)$.

Proof of b). By Theorem 1, a) $\lim_{M=\infty}V_M(p)=p$ and $N(z,p)$ has M.D.I. over $R-R_0-V_M(p)$ for $p\in R-R_0$. Hence by the $\overset{*}{\text{m}}$aximum principle

$$\omega(p,z)\leqq\omega(V_M(p),z)\leqq\frac{N(z,p)}{M}.$$

Let $M\to\infty$. Then $\omega(p,z)=0$.

Put $p=A$ in *a*). Then $d\mu(p)$ is a point mass, whence we have at once b).

Proof of c). ${}_{\upsilon_n(p)}N(z,p)$ is superharmonic in $\overline{R}-R_0$, whence ${}_pN(z,p)=\lim_n {}_{\upsilon_n(p)}N(z,p)$ and since ∂R_0 is compact $\phi(p)=\int_{\partial R_0}\lim_{n=\infty}\frac{\partial}{\partial n}\,{}_{\upsilon_n(p)}N(z,p)\,ds=\lim_n\int_{\partial R_0}\frac{\partial}{\partial n}\,{}_{\upsilon_n(p)}N(z,p)\,ds=\lim_n\phi(\upsilon_n(p))$. $N(z,p)$: $p\in R-R_0$ has M.D.I. over $R-R_0-\upsilon_n(p)$, because N-Martin's topology is homeomorphic in $R-R_0$ to the original topology, $v_n(p)\ni p$ and $\sup_{z\in\partial v_n(p)} N(z, p)<\infty$, whence ${}_{v_n(p)}N(z, p)$ $=N(z, p)$. Hence ${}_pN(z, p)=N(z, p)$ and $\phi(p)=1$. For $p\in B_S$, $N(z, p)=K\omega(p, z)$ $K_p\omega(p, z)={}_pN(z, p)$: $K>0$ by *b*). Hence $\phi(p)=1$. We consider the case: $\omega(p, z)=0$ and $p\in B$. In this case p is closed and of capacity zero. Hence by Theorem 6, ${}_pN(z, p)={}_p({}_pN(z, p))$. But ${}_pN(z, p)$ has a point mass at p, i.e. $N(z, p)=\phi(p)N(z, p)$. ${}_p({}_pN(z, p))=\phi^2(p){}_pN(z, p)=\phi(p)N(z, p)$. Hence $\phi(p)$ $=0$ or $=1$. The set Γ_m is defined as the set (possible void) of all points of $\overline{R}-R_0$ such that $\phi(v_m(p))=\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}{}_{v_m(p)}N(z, p)\,ds\leqq\frac{1}{2}$(this means $\phi(p)$ $=0$). Then $B_0=\bigcup_{m>1}\Gamma_m$. By definition ${}_{v_m(p)}N(z, p)=\lim_{n=\infty}{}_{v_m(p)\frown R_n}N(z, p)$. Hence for any given positive number ε, there exists a number n_0 such that

$$\phi(v_m(p)\frown R_n))=\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}{}_{v_m(p)\frown R_n}N(z, p)\,ds$$

$$\geqq\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}{}_{v_m(p)}N(z, p)\,ds-\varepsilon=\phi(v_m(p))-\varepsilon \quad \text{for } n\geqq n_0.$$

Suppose $p_i\to p$. Then $N(z, p_i)\to N(z, p)$ uniformly on compact $\partial(v_m(p)\frown R_n)$ and $v_m(p_i)\frown R_n\to v_m(p)\frown R_n$. Now ${}_{v_m(p_i)\frown R_n}N(z, p_i)$ and ${}_{v_m(p)\frown R_n}N(z, p)$ are determined by the values of $N(z, p_i)$ and of $N(z, p)$ on $\partial(v_m(p_i)\frown R_n)$ and on $\partial(v_m(p)\frown R_n)$ respectively. Hence

$$\varliminf_{i=\infty}{}_{v_m(p_i)}N(z, p_i)\geqq\varliminf_{i=\infty}{}_{v_m(p)\frown R_n}N(z, p_i)={}_{v_m(p)\frown R_n}N(z, p).$$

Thus $\varliminf_{i=\infty}\phi(v_m(p_i))\geqq\varliminf_{i}\phi(v_m(p_i)\frown R_m)\geqq\phi(v_m(p))-\varepsilon$, whence by letting $\varepsilon\to 0$, $\varliminf_{i=\infty}\phi(v_m(p_i))\geqq\phi(v_m(p))$. Therefore $\phi(v_m(p_i))$ is lower semicontinuous with respect to p and by $\phi(v_m(p))\downarrow\phi(p)$ $\phi(p)$ is also lower semicontinuous, whence Γ_m is closed and B_0 is an F_σ set.

Proof of d). The set Γ_m, being closed and compact, may be covered by a finite number of its closed subsets whose diameteres are less than $\frac{1}{m}$. It is sufficient by P.C.5 to prove *d*) for any closed subset A of Γ_m whose diameter is less than $\frac{1}{m}$. Assume Cap$(A)>0$. Then $\omega(A, z)={}_A\omega(A, z)$ $=\int_A N(z, p)\,d\mu(p)$. On the other hand, since ${}_A\omega(A, z)=\lim_m\lim_n\omega_{A_{2m,n}}(A, z)$ $(A_{2m,n}=A_{2m}\frown R_n$ and $A_{2m}=E\left[z\in\overline{R}: \delta(z, A)\leqq\frac{1}{2m}\right]$, for any given positive number ε, there exist numbers m and n such that

$$\text{Cap}(A)=\int_{\partial R_0}\frac{\partial}{\partial n}\omega(A, z)\,ds\leqq\int_{\partial R_0}\frac{\partial}{\partial n}{}_{A_{2m,n}}\omega(A, z)\,ds+\varepsilon.$$

Now $\omega(A,z)$ can be approximated on $A_{2m,n}$ by a sequence of functions $V_l(z)=\sum_i^l c_iN(z,p_i): p_i\in A(l=1,2\cdots)$. Then by Fatou's lemma

$$\begin{aligned}\mathrm{Cap}\,(A)-\varepsilon&\leqq\int_{\partial R_0}\frac{\partial}{\partial n}{}_{A_{2m,n}}\omega(A,z)\,ds\leqq\varliminf_{l=\infty}\int_{\partial R_0}\frac{\partial}{\partial n}{}_{A_{2m,n}}V_l(z)\,ds\\&=\varliminf_{l=\infty}\int_{\partial R_0}\sum_i^l c_i\frac{\partial}{\partial n}{}_{A_{2m,n}}N(z,p_i)\,ds\leqq\varliminf_{l=\infty}\int_{\partial R_0}\frac{1}{2}\sum_i^l c_iN(z,p_i)\,ds\\&=\int_{\partial R_0}\frac{1}{2}\frac{\partial}{\partial n}\omega(A,z)\,ds=\mathrm{Cap}\,(A),\end{aligned}$$

because $A_{2m}\subset v_m(p_i)=E\left[z\in\overline{R}:\ \varepsilon(z,p_i)\leqq\frac{1}{m}\right]$ for every $p_i\in A$ implies

$$\int_{\partial R_0}\frac{\partial}{\partial n}{}_{A_{2m}}N(z,p_i)ds\leqq\int_{\partial R_0}\frac{\partial}{\partial n}{}_{v_m(p_i)}N(z,p_i)\,ds\leqq\frac{1}{2}\int_{\partial R_0}\frac{\partial}{\partial n}N(z,p_i)\,ds.$$

Let $\varepsilon\to 0$. Then $\mathrm{Cap}\,(A)\leqq\frac{1}{2}\,\mathrm{Cap}\,(A)$. Hence $\mathrm{Cap}\,(A)=0$, $\mathrm{Cap}\,(\Gamma_m)=0$ and $\omega(B_0,z)=0$ by P.C.5. Since $\omega(p,z)>0$ for $p\in B_S$. Hence $B_S\subset B_1$.

Proof of e). Let A be a closed subset of Γ_m whose diameter $\leqq\frac{1}{m}$. By Theorem 5, *b)* ${}_AU(z)=\int_A N(z,p)\,d\mu(p)$. Hence ${}_AU(z)$ can be expressed by a limit of linear forms $V_l(z)=\sum^l c_iN(z,p_i): p_i\in A$ $(l=1,2,\cdots)$. Hence as above

$$\int_{\partial R_0}\frac{\partial}{\partial n}{}_A({}_AU(z))\,ds\leqq\int_{\partial R_0}\frac{\partial}{\partial n}{}_{A_m}({}_AU(z))\,ds\leqq\varliminf_{l=\infty}\int_{\partial R_0}\frac{\partial}{\partial n}{}_{A_m}V_l(z)\,ds\leqq\frac{1}{2}\int_{\partial R_0}\frac{\partial}{\partial n}{}_AU(z)ds. \tag{34}$$

On the other hand, by Theorem 6. *a)* ${}_AU(z)={}_A({}_AU(z))$. Hence by (34) ${}_AU(z)=0$, ${}_{\Gamma_m}U(z)=0$ and ${}_{B_0}U(z)=0$.

Proof of f). Since $U(z)$ is harmonic in $R-R_0-F'_m$, where $F'_m=\left[z\in\overline{R}:\ \delta(z,F+B)\leqq\frac{1}{m}\right]$, $U(z)={}_{F'_m}U(z)$. Hence by Theorem 5. *b)* $U(z)=\int_{F+B}N(z,p)\,d\mu(p)$.

7. Canonical mass distributions. Let $U(z)$ the superharmonic function in $\overline{R}-R_0$. Let $U_n^*(z)$ be a function in $\overline{R}-R_0$ such that $U_n^*(z)=U(z)$ in R_n-R_0 and $U_n^*(z)$ is harmonic in $R-R_n$. Then $U(z)=\lim_n U_n^*(z)$ in $R-R_0$. Clearly $U_n^*(z)={}_{R_n}U(z)$. Hence $U_n^*(z)$ is representable by a uniquely determined mass distribution $\mu_n(p)$ on $\overline{R}_n-R_0$, because $\overline{R}_n-R_0$ is compact.

Operation ${}_D[U(z)]^*$. *Let D be a compact or non compact domain in $R-R_0$. Let ${}_{D_n}[U(z)]^*$ be a function in $\overline{R}-R_0$ such that $U_n^*(z)-{}_{D_n}[U(z)]^*$*

is harmonic in $D_n = D \frown R_n$ *and superphemonic in* $\overline{R} - R_0$ *and further* ${}_{D_n}[U(z)]^*$ *is harmonic in* $R - R_0 - D_n$, ${}_{D_n}[U(z)]^* = 0$ *on* ∂R_0 *and superharmonic in* $\overline{R} - R_0$. Such ${}_{D_n}[U(z)]^*$ is uniquely determined. In fact, let ${}_1\mu_n(p)$ be the restriction of $\mu_n(p)$ on $\overline{D} \frown \overline{R}_n$. Then

$$ {}_{D_n}[U(z)]^* = \int N(z, p)\, d_1\mu_n(p). $$

Now ${}_2\mu_n(p) = \mu_n(p) - {}_1\mu_n(p)$ is also a positive mass distribution, which implies that $U_n^*(z) - {}_{D_n}[U(z)]^*$ is superharmonic in $\overline{R} - R_0$. Let $\{n'\}$ be a subsequence of $\{n\}$ such that ${}_{D_{n'}}[U(z)]^*$ converges uniformly in $R - R_0$. Put ${}_D[U(z)]^* = \lim\limits_{n'} {}_{D_{n'}}[U(z)]^*$. ${}_D[U(z)]^*$ *depends on* D *and the subsequence* $\{n'\}$.

Theorem 7. *Let* D_1 *and* D_2 *be two domains and* $\{n'\}$ *be a subsequence such that both* ${}_{D_1, n'}[U(z)]^*$ *and* ${}_{D_2, n'}[U(z)]^*$ *converge uniformly in* $R - R_0$. *Then*

a). ${}_{D_1 + D_2}[U(z)]^* \leqq {}_{D_1}[U(z)]^* + {}_{D_2}[U(z)]$.

b). ${}_D[CU(z)]^* = C_D[U(z)]^*$ *for any constant* $C \geqq 0$.

c). ${}_D[U(z)]^* \leqq {}_D U(z) \leqq U(z)$.

d). *Both* ${}_D[U(z)]^*$ *and* $U(z) - {}_D[U(z)]^*$ *are superharmonic in* $\overline{R} - R_0$.

e). ${}_{D_1}[U(z)]^* \leqq {}_{D_2}[U(z)]^*$ *for* $D_1 \subset D_2$.

f). ${}_D[U(z)]^*$ *is representable by a mass distribution on* $\overline{D}$, *where* $\overline{D}$ *is the closure of* D.

g). *Let* $p \in R - R_0$. *Then* $N(z, p) = \lim\limits_m {}_{v_m(p)}[N(z, p)]^*$ *for every* $v_m(p)$. *Let* B_0^* *be the set of points of* $\overline{R} - R_0$ *such that* $\lim\limits_{n_i = \infty} {}_{v_{n_i}(p)}[N(z, p)]^* = 0$ *for every sequence*: $n_1 < n_2 < n_3 \cdots$. *Then by the above fact* $B_0^* \frown (R - R_0) = 0$ *and by c*) ${}_{v_m(p)}N(z, p) \geqq {}_{v_m(p)}[N(z, p)]^*$, *whence* $B_0^* \supset B_0$.[6]

h). ${}_{B_0}[(z)]^* = 0$ *for* $U(z) = \int_{B_0} N(z, p(d\mu(p)$.

Proof of a), *b*), *d*) and *e*) is clear by the definition.

Proof of c). ${}_D U(z) = \lim\limits_n {}_{D_n} U(z)$: $D_n = D \frown R_n$. Now $U(z) = {}_{D_n} U(z) \geqq {}_{D_n}[U(z)]^*$ on D_n and both ${}_{D_n}U(z)$ and ${}_{D_n}[U(z)]^*$ are harmonic in $R - R_0 - D$, whence by the maximum principle ${}_{D_n}[U(z)]^* \leqq {}_{D_n}U(z)$. Hence

$$ {}_D[U(z)]^* = \lim_n {}_{D_n}[U(z)]^* \leqq \lim_n {}_{D_n}U(z) = {}_D U(z). $$

Proof of f). ${}_{D_n}[U(z)]^*$ and $U_n^*(z) - {}_{D_n}[U(z)]^*$ are representable by

6) $B_0 = B_0^*$ will be proved in Theorem 9.

positive mass distributions ${}_1\mu_n$ and ${}_2\mu_n=\mu_n-{}_1\mu_n$ on $R_n\frown\overline{D}$ and $R_n\frown\overline{CD}$ respectively. But the total masses of ${}_1\mu_n$ and ${}_2\mu_n$ are bounded $\leqq \int_{\partial R_0}\frac{\partial}{\partial n}U(z)ds$. We can find a subsequence $\{n'\}$ of $\{n\}$ such that both $\{{}_1\mu_n\}$ and $\{{}_2\mu_n\}$ have weak limits ${}_1\mu$ on $\overline{D}\frown\overline{R}$ and ${}_2\mu$ on $\overline{CD}\frown\overline{R}$ respectively. Clearly by $\{n'\}\subset\{n\}$, $U(z)=\int N(z,p)\,d\mu(p):\mu={}_1\mu+{}_2\mu$, ${}_D[U(z)]^*=\int N(z,p)\,d\mu(p)$ and $U(z)-{}_D[U(z)]^*=\int N(z,p)\,d_2\mu(p)$. Hence ${}_D[U(z)]^*$ and $U(z)-{}_D[U(z)]^*$ are superharmonic in $\overline{R}-R_0$.

Proof of g). Since $p\in R-R_0$, there exists a number n_0 for any given neighbourhood $v_m(p)$ of p such that $v_m(p)\subset R_n-R_0$ for $n\geqq n_0$ Then $N(z,p)$ has M.D.I. over $R-R_n$, whence $N(z,p)$ is harmonic in $\overline{R}-R_n$ and $N_n^*(z,p)=N(z,p)$ in $R-R_n$. In this case $N_n^*(z,p)=\int_p N(z,p)\,d\mu(p)$ and ${}_{v_m(p)}[N(z,p)]^*=N_n^*(z,p)=N(z,p)$ for $n\geqq n_0$. Hence ${}_p[N(z,p)]^*=N(z,p)$ and $B_0^*\frown(R-R_0)=0$.

Proof of h). ${}_{B_0}U(z)=0$ implies $h)$ by $c)$.

Theorem 8. *Every positive superharmonic function in $\overline{R}-R_0$ such that $U(z)=0$ on ∂R_0 is representable by a positive mass distribution μ on $\overline{R}-R_0+B_1$ such that*

$$U(z)=\int N(z,p)\,d\mu(p)\quad\text{for } z\in R-R_0.$$

We call such a canonical mass distribution.

Remark. It seems that Theorem 8 can be improved to the following: $U(z)$ is representable by a mass distribution on $R-R_0+B_1^*:B_1^*=B-B_0^*\subset B-B_0=B_1$. But in Theorem 9 it is proved that $B_0^*=B_0$. Hence the above two are equal.

Proof. Suppose $V(z)=\int_{B_0}N(z,p)\,d\mu(p)$. Then by Theorem 7. $e)$ ${}_{B_0}V(z)=0$ and by $c)$ of Theorem 7 ${}_{\Gamma_m}V(z)=0$. This implies $\lim_n {}_{\Gamma_{m,n}}[V(z)]^*=0$, where $\Gamma_{m,n}=E\left[z\in\overline{R}:\,\delta(z,\Gamma_m)\leqq\frac{1}{n}\right]$ Let z_0 be a point in $R-R_0$. Then for any given positive number ε, there exists a number $n_0(m)$ such that

$${}_{\Gamma_{m,n}}[V(z_0)]^*\leqq{}_{\Gamma_{m,n}}V(z_0)\leqq\frac{\varepsilon}{2^m}\quad\text{for } n\geqq n_0(m).$$

For each m select $\Gamma'_m(=\Gamma_{m,n})$ in this fashion. Put $C_m=\sum_{i=1}^{m}\Gamma'_i$. Then C_m

is closed and increases as $m\rightarrow\infty$. Denote by $\tilde{A}_m$ and A_m the closure of the of the complement of C_m in B and $\bar{R}-R_0$ respectively. Then the distance between $\tilde{A}_m$ and Γ_m (Γ_m is contained in B_0 by the definition of Γ_m) is at least $\frac{1}{n(m)}$. Thus $\{\tilde{A}_n\}$, which forms a decreasing sequence, has an intersection $\tilde{A}$ which is closed and, having no points in common with any Γ_m, is a subset of B_1. Now

$${}_{C_m}[V(z)]^*\leqq{}_{C_m}V(z)\leqq\sum_i^m{}_{\Gamma'_i}V(z)\leqq\sum_{i=1}^m 2^{-i}\varepsilon<\varepsilon\quad\text{for } z=z_0.$$

Observing $\tilde{A}_m+\tilde{C}_m=B$, we obtain for a subsequence $\{n'\}$ of $\{n\}$ such that

$${}_{A_m\frown CR_{n'}}[V(z)]^*\rightarrow{}_{\tilde{A}_m}[V(z)]^*\text{ as } n'\rightarrow\infty,[7]$$

where $\tilde{C}_m=C_m\frown B$ and $A_m\frown B=\tilde{A}_m$ and A_m is a closed domain in $\bar{R}-R_0$.

$${}_{\tilde{A}_m}[V(z)]^*\leqq{}_B[V(z)]^*=V(z)\leqq{}_{\tilde{A}_m}[V(z)]^*+{}_{C_m}[V(z)]^*,$$

whence $V(z)\geqq{}_{\tilde{A}_m}[V(z)]^*\geqq V(z)-\varepsilon$ for $z=z_0$.

Now $V(z)-{}_{\tilde{A}_m}[V(z)]^*$ and ${}_{\tilde{A}_m}[V(z)]^*$ are harmonic in $R-R_0$, superharmonic in $\bar{R}-R_0$ and are representable by positive mass distributions μ'_m and μ''_m over $(C_m\frown B)$ and $\tilde{A}_m$ respectively. Let $\{n''\}$ be a subsequence of $[n']$ such that ${}_{A_{m+1}\frown CR_{n''}}[V(z)]^*\rightarrow{}_{\tilde{A}_{m+1}}[V(z)]^*$. Then ${}_{\tilde{A}_{m+1}}[V(z)]^*$ is representable by μ''_{m+1} over $\tilde{A}_{m+1}$ and ${}_{C_m}[V(z_0)]^*<\varepsilon$. Proceeding in this way, by *e*) of Theorem 7 ${}_{\tilde{A}_m}[V(z)]^*\downarrow{}_{\tilde{A}}[V(z)]^*$ and ${}_{C_m}[V(z)]^*\uparrow{}_{\tilde{C}}[V(z)]^*$, where ${}_{\tilde{C}}[V(z_0)]^*<\varepsilon$ by ${}_{C_m}[V(z_0)]<\sum_{i=1}^m\frac{\varepsilon}{2^i}$. $\{\mu'_m\}$ and $\{\mu''_m\}$ $(m=1,2,\cdots)$ have weak limits μ' and μ'' over $B\frown\bar{C}(=\overline{\sum C_m})$ and $\tilde{A}=\frown\tilde{A}_m\subset B_1$ respectively. Hence

$$V(z_0)\leqq{}_{\tilde{A}}[V(z_0)]^*+\varepsilon,$$

where ${}_{\tilde{A}}[V(z)]^*$ and $V(z)-{}_{\tilde{A}}[V(z)]^*$ are superharmonic in $\bar{R}-R_0$ and representable by $\mu'_1(=\mu')$ and ${}_1\mu'''(=\mu'')$ respectively.

Let ${}_1\mu'''$ be the restriction of ${}_1\mu''$ on B_1 and put

$$V_1(z)=\int_{B_0}N(z,p)\,d({}_1\mu''-{}_1\mu''')(p).$$

Then $0\leqq V_1(z)\leqq V(z)-{}_{\tilde{A}}[V(z)]^*<\varepsilon$ for $z=z_0$ and $V(z)-{}_1V(z)=\int_{B_1}N(z,p)d(\mu'_1+{}_1\mu''')(p)$. Put ${}_1\mu^*={}_1\mu'+{}_1\mu'''$ and ${}_1\mu^{**}={}_1\mu''-{}_1\mu'''$.

Repeat the proçess (used for $V(z)$) for $V_1(z)$, writting $V_1(z)=V_2(z)+(V_1(z)-V_2(z))$, where $V_2(z)$ and $V_1(z)-V_2(z)$ are representable by positive

7) $CR_{n'}$ means the complementary set of $R_{n'}$.

mass distributions ${}_1\mu^{**}$ and ${}_1\mu^{*}$ on B_0 and B_1 respectively such that $V_2(z_0)<\frac{\varepsilon}{2}$.

Proceeding in this way,

$$V_n(z)=V_{n+1}(z)-(V_n(z)-V_{n+1}(z)),$$

where $V_{n+1}(z)$ and $V_n(z)-V_{n+1}(z)$ are representable by positive mass distributions ${}_n\mu^{**}$ and ${}_n\mu^{*}$ over B_0 and B_1 respectively such that $V_{n+1}(z_0)<\frac{\varepsilon}{2^n}$. Then

$$V(z)=V(z)-V_1(z)+\sum_{n=1}^{\infty}(V_n(z)-V_{n+1}(z))$$

and $V(z)$ is represented by a positive mass distribution $\mu=\sum_{n=1}^{\infty}{}_n\mu^*$ over B_1.

Let $U(z)=\int\limits_{R-R_0+B} N(z,p)\,d\mu(p)$. Let μ' be the restriction of μ over B_0. Then μ' can be replaced by another distribution over $\left(\text{in putting } V(z)=\int\limits_{B_0} N(z,p)\,d\mu(p)\right)$ B_1 without any change of $U(z)$. Hence we have the theorem.

8. N-minimal functions and N-minimal points. Let $U(z)$ be a positively superharmonic function in $\overline{R}-R_0$ with $U(z)=0$ on ∂R_0. If $U(z)\geqq V(z)\geqq 0$ implies $V(z)=KU(z)$ $(0\leqq K\leqq 1)$ for every function $V(z)$ such that both $U(z)-V(z)$ and $V(z)$ are positively superharmonic in $\overline{R}-R_0$, $U(z)$ is called *N-minimal function.*

Theorem 9. a). *Let $U(z)$ be a N-minimal function such that $U(z)=\int\limits_A N(z,p)\,d\mu(p)$. Then $U(z)$ is a multiple of some $N(z,p): p\in(R-R_0+B_1)\frown A$.*

b). *$N(z,p)$ is N-minimal or not according as $\phi(p)=1$ or $=0$, i.e. $p\in R-R_0+B_1$ or $p\in B_0$.*

c). *Let $V_M(p)=E[z\in R\colon N(z,p)>M]$ and $v_n(p)=E\left[z\in \overline{R}\colon \delta(z,p)<\frac{1}{n}\right]$ andsuppose $p\in R-R_0+B_1$. Then*

$$N(z,p)={}_{V_M(p)\frown v_n(p)}N(z,p)={}_{v_n(p)}N(z,p)\colon \textit{for } M<\sup_{z\in R-R_0} N(z,p), \textit{ i.e.}$$

$$N(z,p)=M\omega(V_M(p),z) \textit{ in } R-R_0-V_M(p).$$

d). *For any given number $M<\sup\limits_{z\in R-R_0} N(z,p)$, there exists a number n such that*

$$(R\frown v_n(p))\subset V_M(p) \quad \text{for } p\in R-R_0+B_1.$$

e). $B_0^*=B_0$.

Proof of a). Suppose, $U(z)$ is N-minimal and $U(z)=\int_A N(z,p)\,d\mu(p)$. Assume, μ is not a point mass. Then for any positive mass distribution μ' such that $0<\mu'<\mu$, $\int N(z,p)\,d\mu(p)$ and $\int N(z,p)\,d(\mu-\mu')(p)$ are multiples of $U(z)$ by the N-minimality of $U(z)$, because these are superharmonic in $\overline{R}-R_0$. Since μ is not a point mass, we can find two closed sets A_1 and A_2 such that $A_i\subset A$ $(i=1,2)$, $\text{dist}(A_1,A_2)>0$ and the restriction of μ on A_i is positive $(i=1,2)$. Let $\{A_{i,n}\}$ be a decreasing sequence of closed subsets of A_i such that $A_{i,n}\to p_i$ as $n\to\infty$, $\mu_{i,n}$ (restriction of μ on $A_{i,n})>0$ and that the potential of $\mu_{i,n}$ is a multiple of $U(z)$. Put $\tilde{\mu}_{i,n}=\frac{\mu_{i,n}}{\int d\mu_{i,n}}$. Then from $\{\tilde{\mu}_{i,n}\}$ we can find weak limits $\tilde{\mu}_i$ $(i=1,2)$ of unity at $p_i\in A$ and $N(z,p_1)=\int N(z,p)\,d\tilde{\mu}_i(p)=KU(z)=\int_A N(z,p)\,d\mu(p)=N(z,p_2)$: $K=\frac{2\pi}{\int_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds}$. This contradicts $\delta(p_1,p_2)>\text{dist}(A_1,A_2)>0$. Hence μ is a point mass at $p\in A$ and $U(z)=\frac{N(z,p)}{K}$. Next we show $p\in(R-R_0+B_1)$. Assume $U(z)=K'N(z,p)$: $K'>0$ and $p\in B_0$. Every positive superharmonic function in $\overline{R}-R_0$ is representable by a canonical mass distribution μ on $R-R_0+B_1$ by Theorem 8 such that $U(z)=K'N(z,p)=\int_{B_1} N(z,p)\,d\mu(p)$. By the minimality of $U(z)$ μ is also a point mass at $q\in B_1+R-R_0$. Now we have also $N(z,p)=N(z,q)$: $q\in B_1+R-R_0$, $p\in B_0$. This is a contradiction. Hence $U(z)=K'N(z,p)$: $p\in(R-R_0+B_1)\frown A$.

Proof of b). We show that $N(z,p)$: $p\in R-R_0+B_1$ is N-minimal. Suppose that there exists a function $U(z)$ such that $U(z)>0$ and that both $U(z)$ and $N(z,p)-U(z)$ are superharmonic in $\overline{R}-R_0$. Then

$$N(z,p)={}_pN(z,p)={}_pU(z)+{}_pV(z)\leqq U(z)+V(z)=N(z,p),$$

where $V(z)=N(z,p)-U(z)$.

By ${}_pU(z)\leqq U(z)$ and $V_p(z)\leqq V(z)$, we have $U(z)={}_pU(z)$ and $V(z)={}_pV(z)$.

But by *b*) of Theorem 5, ${}_pU(z)=K_1N(z,p)$ and ${}_pV(z)=K_2N(z,p)$. Hence $N(z,p)$ is N-minimal. Thus by *a*) $N(z,p)$ is N-minimal if and only if $p\in R-R+B_1$. Hence we have *b*).

Proof of c). For $p\in R-R_0+B_1$, ${}_pN(z,p)=N(z,p)$. Hence $N(z,p)=$

${}_{p}N(z, p) \leqq {}_{\upsilon_n(p)}N(z, p) \leqq N(z, p)$. We show ${}_{\upsilon_n(p)\frown V_m(p)}N(z, p) = N(z, p)$.

Case 1. $p \in R - R_0 + B_1 - B_S$. In this case we remark $\sup_{z \in R - R_0} N(z, p) = \infty$. In fact, assume $N(z,p) \leqq M$ and $p \in R - R_0 + B_1 - B_S$. Then $N(z,p) \leqq M\omega(\upsilon_m(p), z)$. Let $m \to \infty$. Then $N(z, p) \leqq M\omega(p, z) = 0$. This contradicts $p \in R - R_0 + B_1 - B_S$. Hence $\sup_{z \in R} N(z, p) = \infty$.

Put $\lim_{n=\infty} {}_{CV_m(p)\frown \upsilon_n(p)}N(z, p) = N'(z, p)$. Then by $\upsilon_n(p) \supset (\upsilon_n(p) \frown CV_M(p))$ $N'(z, p)$ has no mass except at p. Hence $N'(z, p) = KN(z, p)$ $(0 \leqq K \leqq 1)$. But $\sup_{z \in R} N(z, p) = \infty$ and $\sup_{z \in R} N'(z, p) \leqq M$ imply $K = 0$ and $N'(z, p) = 0$. Hence

$$\begin{aligned} {}_{p}N(z, p) &= {}_{p}N(z, p) + N'(z, p) = \lim_{n} \left({}_{V_M(p)\frown \upsilon_n(p)}N(z, p) + {}_{CV_M(p)\frown \upsilon_n(p)}N(z, p)\right) \\ &= \lim_{n} {}_{V_M(p)\frown \upsilon_n(p)}N(z, p) \geqq \lim_{n} {}_{\upsilon_n(p)}N(z, p) = {}_{p}N(z, p) = N(z, p). \end{aligned}$$

Therefore $N(z, p) = {}_{p}N(z, p) = \lim_{M=\infty} {}_{V_M(p)\frown p}N(z, p) = \lim_{M=\infty} {}_{V_M(p)}N(z, p) \leqq {}_{V_M(p)}N(z, p) \leqq N(z, p)$. Now ${}_{V_M(p)}N(z, p)$ has M.D.I. $\leqq 2\pi M$ over $R - R_0 - V_M(z)$ and $N(z, p) = M$ on $\partial V_M(p)$. Hence $N(z, p) = M\omega(V_M(p), z)$ in $R - R_0 - V_M(p)$.

Case 2. $p \in B_S$. In this case by Theorem 7, *b*) $N(z, p) = K\omega(p, z)$. Hence ${}_{\upsilon_n(p)}\omega(p, z) = \omega(p, z)$ by P.C.1 and ${}_{V_M(p)}\omega(p, z) = \omega(p, z)$: $M < 1$ by P.C.4. On the other hand, ${}_{V_M(p)\frown \upsilon_n(p)}\omega(p, z) + {}_{CV_M(p)\frown \upsilon_n(p)}\omega(p, z) \geqq {}_{p}\omega(p, z) = \omega(p, z)$.

Let $\upsilon_n(p) \to p$. Then by P.C.3 ${}_{CV_M(p)\frown p}\omega(p, z) = 0$. Hence $\omega(p, z) \geqq {}_{V_M(p)}\omega(p, z) \geqq {}_{V_M(p)\frown \upsilon_n(p)}\omega(p, z) = \omega(p, z)$. Thus we have c).

Proof of d). Let $q_i \in R - R_0 - V_M(p)$ and let M'' be a number such that $M < M'' < \sup_{z \in R} N(z, p)$. Now by *c*) $N(z, p) = M''\omega(V_{M''}(p), z)$. Hence by P.C.6 there exists a regular niveau curve $C_{M'} = E[z \in R: N(z, p) = M']$ such that $M < M' < M''$ and

$$\int_{C_{M'}} \frac{\partial}{\partial n} N(z, p)\, ds = \int_{C_{M'}} M''\omega(V_{M''}(p), z)\, ds = \int_{\partial R_0} \frac{\partial}{\partial n} N(z, p)\, ds = 2\pi.$$

Let $N_n(z, q_i)$ be a harmonic function in $R_n - R_0 - q_i$ such that $N_n(z, q_i) = 0$ on ∂R_0, $\frac{\partial}{\partial n} N(z, q_i) = 0$ on ∂R_n and has a logarithmic singularity at $q_i: q_i \in R - R_0 - V_M(p)$. Then by the definition of $N(z, q_i)$, $N_n(z, q_i) \Rightarrow N(z, q_i)$ as $n \to \infty$. Let $N'_n(z, p)$ be a harmonic function in $R_n - R_0 - V_{M'}(p)$ such that $N'_n(z, p) = 0$ on ∂R_0, $N'_n(z, p) = M'$ on $\partial V_{M'}(p) = C_{M'}$ and $\frac{\partial}{\partial n} N'_n(z, p) = 0$ on ∂R_n. Then since $N(z, p) = M'\omega(V_{M'}(p), z)$ has M.D.I. over $R - R_0 - V_M(p)$, $N'_i(z, p) \Rightarrow N(z, p)$ as $n \to \infty$. By the Green's formula

$$\int_{C_{M'}} N(z, q_i) \frac{\partial}{\partial n} N'_n(z, p)\, ds = 2\pi N'_n(q_i, p) \quad \text{by } q_i \notin V_M(p).$$

Since $C_{M'}$ is regular and $N_n(z, q_i)$ is uniformly bounded on $C_{M'}$ by $q_i \in R-R_0-V_M(p): V_{M'}(p) \subset V_M(p)$, (by Theorem 6) we have by letting $n \to \infty$,

$$M > \frac{1}{2\pi} \int_{C_{M'}} N(z, q_i) \frac{\partial}{\partial n} N(z, p)\, ds = N(q_i, p) \quad \text{by } q_i \notin V_M(p). \tag{34}$$

Assume that *d*) is false. Then there exists a sequence of points $\{q_i\}$ such that $q_i \in CV_M(p) \frown (R-R_0)$ and $\delta(p, q_i) \to 0$. Let $M < M^* < M'$ and put $\varepsilon_0 = 2\pi\left(1-\frac{M}{M^*}\right) > 0$. By the regularity of $C_{M'}$ there exists a number n_0 such that

$$\int_{C_{M'} \frown R_n} \frac{\partial}{\partial n} N(z, p)\, ds \geqq 2\pi - \varepsilon_0 \quad \text{for } n \geqq n_0.$$

If $N(z, q_i) > M^*$ on $R_{n_0} \frown C_{M'}$,

$$\int_{C_{M'} \frown R_{n_0}} N(z, q_i) \frac{\partial}{\partial n} N(z, p)\, ds \geqq M^*(2\pi - \varepsilon_0) = M. \tag{35}$$

But $N(q_i, p) < M$ for $q_i \notin V_M(p)$. Hence (34) contradicts (35). Hence $N(z, q_i) \not\geqq M^*$ on $C_{M'} \frown R_{n_0}$ and there exists at least one point z_i on $C_{M'} \frown R_{n_0}$ such that $N(z_i, q_i) \leqq M^* < M'$. Since $R_{n_0} \frown C_{M'}$ is compact, there exists a point $\tilde{z}$ which is one of limiting points $\{z_i\}$. Now $N(\tilde{z}, q) \leqq \overline{\lim_{i=\infty}} N(\tilde{z}, q_i) \leqq M^*$, where $q = \lim_i q_i$. On the other hand, $N(\tilde{z}, q) = M' = N(\tilde{z}, p)$ by $\lim_{i=\infty} \delta(p, q_i) = 0$ ($\delta(p, q) = 0$ is equivalent to $N(z, q) = N(z, p)$) and by $\tilde{z} \in C_{M'}$. This is a contradiction. Hence we have *d*).

Proof of e). By *g*) of Theorem 7, $B_0 \subset B_0^*$. We show $B - B_0 = B_1 \subset B - B_0^*$. Let $p \in B_1$. Let $N_n^*(z, p)$ be a function in $\overline{R} - R_0$ such that $N_n^*(z, p) = N(z, p)$ in $R_n - R_0$ and $N_n^*(z, p)$ has M.D.I. over $R - R_n$. Clearly $N_n^*(z, p) = {}_{R-R_n}N(z, p)$ and $N_n^*(z, p)$ is superharmonic in $\overline{R} - R_0$. Then since $p \in B_1$, $N_n^*(z, p) = N(z, p)$ is harmonic in $R_n - R_0$ and $N_n^*(z, p)$ is represented by a mass distribution on ∂R_n.

Hence ${}_{B_m}[N_n(z, p)]^* = N_n^*(z, p)$, where $B_m = R - R_m$ and $n > m$.

Now ${}_{B_m \frown v_l(p)}[N_n^*(z, p)]^* + {}_{B_m \frown Cv_l(p)}[N_n^*(z, p)]^* = {}_{B_m}N_n^*(z, p) = N_n^*(z, p)$.[8]

Let $\{n'\}$ be a subsequence of $\{n\}$ such that ${}_{B_m \frown v_l(p)}[N_n^*(z, p)]$ converges uniformly. Then by letting $n' \to \infty$,

$${}_{B_m \frown v_l(p)}[N(z, p)]^* + {}_{B_m \frown Cv_l(p)}[N(z, p)]^* = N(z, p) \geqq {}_{B_m \frown v_l(p)}[N(z, p)]^*.$$

${}_{B_m \frown v_l(p)}[N_n(z, p)]^*$ and ${}_{B_m \frown Cv_l(p)}[N_,(z, p)]^*$ have masses ${}_1\mu_n$ and ${}_2\mu_n$ on

8) $Cv_l(p)$ means the complementary set of $v_l(p)$.

$v_l(p)\frown\partial R_n$ and $Cv_l(p)\frown\partial R_n$ respectively and $\{{}_1\mu_{n'}\}$ and $\{{}_2\mu_{n'}]$ have weak limits ${}_1\mu$ and ${}_2\mu$ on $B\frown\bar{v}_l(p)$ and $B\frown Cv_l(p)$ respectively and ${}_{B_m\frown v_l(p)}[N(z,p)]^*$ $=\int N(z,p)\,d_1\mu(p)$ and ${}_{B_m\frown Cv_l(p)}[N(z,p)]^*=\int N(z,p)d_2\mu(p)$. Since by the assumption $N(z,p)$ is N-minimal, whence ${}_{B_m\frown v_l(p)}[N(z,p]^*$ and ${}_{B_m\frown Cv_l(p)}[N(z,p)]^*$ are N-minimal and $=K_iN(z,p)$ $(i=1,2)$, because $N(z,p)-{}_{B_m\frown v_l(p)}[N(z,p)]^*$, ${}_{B_m\frown v_l(p)}[N(z,p)]$ are superharmonic by Theorem 7. d). Assume ${}_{B_m\frown Cv_l(p)}[N(z,p)]^*$ $=K_2N(z,p)>0$. Then by a) of Theorem 9 $p\in(B\frown Cv_l(p))$. On the other hand $p\in v_l(p)$. This is a contradiction. Hence $K_2=0$ and ${}_{B_m\frown v_l(p)}[N(z,p)]^*$ $=N(z,p)$ for every $v_l(p)$. Now $\{n'\}$ is any subsequence of $\{n\}$ such that ${}_{B_m\frown v_l(p)}[N_n(z,p)]^*$ converges, hence ${}_{v_l(p)}[N_n^*(z,p)]^*\to N(z,p)$ as $n\to\infty$ for every $v_l(p)$. Whence $p\in B-B_0^*$. Hence $B_0\supset B_0^*$ and $B_0=B_0^*$.

Theorem 10. *Put* $V_M(p)=E[z\in R:\ N(z,p)>M]$ *for* $p\in R-R_0+B_1$. *Then* $V_M(p)$ *may consist of at most an enumerably infinite number of domains* D_l $(l=1,2,\cdots)$.

a). $D_{R-R_0-V_M(p)}(N(z,p))=2\pi M$ *and* $\min(M,N(z,p))=M\omega(V_M(p),z)$.

b). *Let* D_l *be a component of* $V_M(p)$. *Then* D_l *contains a subset* D *of* $V_{M'}(p)$ *for* $M<M'<\sup_{z\in R}N(z,p)$.

c). *Let* $U(z)$ *be a positive superharmonic function with* $U(z)=0$ *on* ∂R_0 *and let* $C_{M_i}=\partial V_{M_i}(p)$ $(i=1,2,\cdots)$ *be a regular niveau curve of* $N(z,p)$: $p\in R-R_0+B_1$ *such that* $\int_{C_i}\frac{\partial}{\partial n}N(z,p)\,ds=2\pi$. *Then for* $M_i<M_{i+1}$

$$\text{mean}\,(U(z)\text{ on }C_{M_i})=\frac{1}{2\pi}\int_{C_{M_i}}U(z)\frac{\partial}{\partial n}N(z,p)\,ds$$

$$\leqq\frac{1}{2\pi}\int_{C_{M_i+1}}U(z)\frac{\partial}{\partial n}N(z,p)\,ds=\text{mean}\,(U(z)\text{ on }C_{M_{i+1}}).$$

d). *Let* C_{M_i} $(i=1,2,\cdots)$ *and* C_M *be a regular curve of* $N(z,p)$: $p\in R-R_0+B_1$ *such that* $M_i\uparrow M$. *Then*

$$\lim_i(\text{mean}\,(U(z)\text{ on }C_{M_i}))=\text{mean}\,(U(z)\text{ on }C_M).$$

If C_M *is not regular, we define* mean $(U(z)$ on $C_M)$ *by* $\lim_{i=\infty}$(mean $U(z)$ on C_{M_i}) *where* $M_i\uparrow M$ *and* $\{C_{M_i}\}$ *are regular. Then* mean $(U(z)$ on $C_M)$ *is defined for every* $M<\sup_{z\in R}N(z,p)$ *and c) holds for every* M.

Proof of a). By Theorem 9. c) we have $M\omega(V_M(p),z)=\min(M,N(z,p))$ and $N(z,p)=\lim_n N'_n(z,p)$, where $N'_n(z,p)$ is a harmonic function in R_n-R_0 $-V_M(p)$ such that $N'_n(z,p)=0$ on ∂R_0, $N'_n(z,p)=M$ on $\partial V_M(p)$ and

$\frac{\partial}{\partial n}N'(z,p)=0$ on $\partial R_n-V_M(p)$. Clearly $D_{R-R_0-V_{M}(p)}(N(z,p))\geqq \lim_n D_{R-R_0-V_M(p)}(N'_n(z,p))=\lim_{n=\infty} M\int_{\partial V_M(p)}\frac{\partial}{\partial n}N'_n(z,p)\,ds=M\lim_n\int_{\partial R_0}\frac{\partial}{\partial n}N'_n(z,p)\,ds= M\int_{\partial R_0}\frac{\partial}{\partial n}N(z,p)\,ds=2\pi M$. On the other hand, by Fatou's lemma $D(\min(M,N(z,p))\leqq \lim_n D(N'_n(z,p))\leqq 2\pi M$. Thus $D(\min(M,N(z,p)))=2\pi M$.

Proof of b). Assume that $D_l\frown V_{M'}(p)=0$. Consider $D(\min(M,N(z,p)))$. Put $N'(z,p)=N(z,p)$ in $R-R_0-D_l$, $N'(z,p)=M$ in D_l. Then since $N(z,p)$ is non constant in $R-R_0$, $D_{R-R_0-V_{M'}(p)}(N'(z,p))<D_{R-R_0-V_{M'}(p)}(N(z,p))$ and $N'(z,p)=N(z,p)=M'$ on $\partial V_{M'}(p)$. This contradicts that $N(z,p)$ has M.D.I. over $R-R_0-V_{M'}(p)$ among all functions with value M' on $\partial V_{M'}(p)$ and 0 on ∂R_0. Hence we have *b*).

Proof of c). By *a*) $N(z,p)=M\omega(V_M(p),z)$ in $R-R_0-V_M(p)$ for every $M<\sup N(z,p)$, whence C_M is regular for almost all constants $M'<\sup_{z\in R} N(z,p)$. i.e. $\int_{\partial V_{M'}(p)}\frac{\partial}{\partial n}N(z,p)ds=2\pi$. Let $C_{M_{i+1}}$ be a regular niveau curve and let ${}_{CV_{M_{i+1}}(p)}U^L(z)$ be a superharmonic function such that ${}_{CV_{M_{i+1}}(p)}U^L(z)= \min(L,U(z))$ on $R-R_0-V_{M_{i+1}(p)}(p)$ and ${}_{CV_{M_{i+1}}(p)}U^L(z)$ is harmonic in $V_{M_{i+1}}(p)$. Then ${}_{CV_{M_{i+1}}(p)}U^L(z)\uparrow U(z)$ on $C_{M_{i+1}}$ as $L\uparrow\infty$ and $U^L_m(z)\Rightarrow{}_{CV_{M_{i+1}}(p)}U^L(p)$ as $n\to\infty$ in $V_{M_{i+1}}(p)$, where $U^L_n(z)$ is a harmonic function in $V_{M_{i+1}}(p)\frown R_n$ such that $U^L_n(z)=\min(L,U(z))$ on $\partial V_{M_{i+1}}(p)\frown R_n$ and $\frac{\partial}{\partial n}U^L_n(z)=0$ on $\partial R_n\frown V_{M_{i+1}}(p)$.

Let $N'_n(z,p)$ be a harmonic function in $R_n\frown(V_{M_i}(p)-V_{M_{i+1}}(p))$ such that $N'_n(z,p)=M_i$ on $\partial V_{M_i}(p)$, $N'_n(z,p)=M_{i+1}$ on $\partial V_{M_{i+1}}(p)$ and $\frac{\partial}{\partial n}N'_n(z,p)=0$ on $\partial R_n\frown(V_{M_i}(p)-V_{M_{i+1}}(p))$. Then $N'_n(z,p)\Rightarrow N(z,p)$.

Then $$\int_{\partial V_{M_i}\frown R_n} N'_n(z,p)\frac{\partial}{\partial n}U^L_n(z)\,ds=\int_{\partial V_{M_{i+1}}(p)\frown R_n} N'_n(z,p)\frac{\partial}{\partial n}U^L_n(z)\,ds=0.$$

Hence by the Green's formula

$$\int_{\partial V_{M_{i+1}}(p)\frown R_n} U^L_n(z)\frac{\partial}{\partial n}N'_n(z,p)\,ds=\int_{\partial V_{M_{i+1}}(p)\frown R_n} U^L(z)\frac{\partial}{\partial n}N'_n(z,p)\,ds.$$

Then by Theorem 3. *a*) by letting $n\to\infty$

$$\int_{\partial V_{M_i}(p)} {}_{CV_{M_i}(p)}U^L(z)\frac{\partial}{\partial n}N(z,p)\,ds=\int_{\partial V_{M_{i+1}}(p)} {}_{CV_{M_i}(p)}U^L(z)\frac{\partial}{\partial n}N(z,p)\,ds.$$

By the superharmonicity of $U(z)$ ${}_{CV_{M_i}(p)}U^L(z)\leqq U(z)$ in $V_{M_i}(p)$ and

$\lim_{L=\infty} {}_{CV_{M_i}(p)}U^L(z)=U(z)$ on $\partial V_{M_i}(p)$ and $\lim_{L=\infty} {}_{CV_{M_i}(p)}U^L(z)\leqq U(z)$ on $\partial V_{M_{i+1}}(p)$, whence by letting $L\to\infty$

$$\int_{\partial V_{M_i}(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds=\lim_{L=\infty}\int_{\partial V_{M_i}(p)} {}_{CV_{M_i}(p)}U^L(z)\frac{\partial}{\partial n}N(z,p)\,ds$$

$$=\lim_{L=\infty}\int_{\partial V_{M_{i+1}}(p)} {}_{CV_{M_i}(p)}U^L(z)\frac{\partial}{\partial n}N(z,p)\,ds\leqq\int_{\partial V_{M_{i+1}}(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds.$$

Thus we have c).

Proof of d). By c) $\int_{\partial V_M(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds\geqq\lim_{i=\infty}\int_{\partial V_{M_i}(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds$ is clear. Since $\int_{\partial V_M(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds=\lim_{L=\infty}\lim_{n=\infty}\int_{\partial V_M(p)\frown R_n}\min(L,U(z))\frac{\partial}{\partial n}N(z,p)\,ds$, for any given positive number ε, there exist L_0 and n_0 such that

$$\int_{\partial V_M(p)\frown R_n}\min(L,U(z))\frac{\partial}{\partial n}N(z,p)\,ds\geqq\int_{\partial V_M(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds-\varepsilon \text{ for } n\geqq n_0 \text{ and } L\geqq L_0.$$

Suppose $z_i\in\partial V_{M_i}(p)$, $z\in\partial V_M(p)$ and $z_i\to z$. Then $\frac{\partial}{\partial n}N(z_i,p)\to\frac{\partial}{\partial n}N(z,p)$ and since $\min(L,U(z))$ is continuous in R_n-R_0, $U(z_i)\to U(z)$ in R_n-R_0. Hence

$$\lim_i\int_{\partial V_{M_i}(p)} U(z)N(z,p)\,ds\geqq\lim_i\int_{\partial V_{M_i}(p)\frown R_n} U(z)\frac{\partial}{\partial n}N(z,p)\,ds$$

$$\geqq\int_{\partial V_M(p)\frown R_n}\min(L,U(z))\frac{\partial}{\partial n}N(z,p)\,ds\geqq\int_{\partial V_M(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds-\varepsilon.$$

Hence by letting $\varepsilon\to 0$, $\lim_i\int_{\partial V_{M_i}(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds=\int_{\partial V_M(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds.$

9. The value of a $\overline{\text{superharmonic}}$ function on B. Till now the value of a $\overline{\text{superharmonic}}$ function is defined in $R-R_0$ only. We shall consider it on the ideal boundary.

Let $U(z)$ be a positive $\overline{\text{superharmonic}}$ function in $\overline{R}-R_0$ with $U(z)=0$ on ∂R_0. Then mean ($U(z)$ on $\partial V_M(p)$) (if $\partial V_M(p)$ is not regular, we use d) of Theorem 9)) $\uparrow$ as $M\uparrow\sup_{z\in R}N(z,p)$ for $p\in R-R_0+B_1$. We define the value $U(z)$ at $p\in R-R_0+B_1$ by

$$\lim_M(\text{mean}\,(U(z)\text{ on }\partial V_M(p))\text{ as }M\uparrow\sup_{z\in R}N(z,p).$$

It is clear, if $U(z)$ is continuous or ∞ at a point $z\in R-R_0$, this coincides with $U(z)$. Next at $p\in B_0$ we shall define the value of $U(z)$.

For $$p\in B_0,\ N(z,p)=\int_{B_1} N(z,p_\alpha)\,d\mu(p_\alpha),$$

where $\mu(p_\alpha)$ is a canonical distribution and not necessarily uniquely determined. In this case *we define $U(p)$ by*

$$\int_{B_1} U(p_\alpha)d\mu(p_\alpha).$$

This definition reduces to the former definition, if $p\in B_1$, because the canonical mass distributiou of $N(z,p): p\in R-R_0+B_1$ must be a point mass at p. Hence our definition is natural. If $U(p)\geqq \frac{1}{2\pi}\int_{\partial V_M(p)} U(z)\frac{\partial}{\partial n}N(z,p)ds$: $p\in R-R_0+B_1$, we say that $U(z)$ is *superharmonic locally at a point p.*

Theorem 11. *a). $N(p,q)=N(q,p)$ for p and $q\in R-R_0$.*

b). $N(p,p)=\sup_{z\in R} N(z,p): p\in R-R+B_1$.

c). Let $U(z)$ be a positive superharmonic function in $\overline{R}-R_0$ with $U(z)=0$ on ∂R_0 (of course $N(z,p)$ is a superharmonic function by Theorem 5. a). Then $U(z)$ is lower semicontinuous in $\overline{R}-R_0$ and $U(z)$ is superharmonic locally at every point of $R-R_0+B_1$. There exists at least one canonical distribution μ by Theorem 8 such that

$$U(z)=\int_{R-R_0+B_1} N(z,p)\,d\mu(p)\ \text{for}\ z\in R-R_0,$$

where the uniqueness of μ is not proved.

By the definition of the value of $U(z)$ on B, $U(z)$ is well defined at any point $p\in R-\overline{R}_0$ and the value of $U(z)$ at a point of B does not depend on a particular distribution and

$$U(z)=\int_{R-R_0+B_1} N(z,p)\,d\mu(p)$$

is valid not only in $R-R_0$ but also on B.

Proof of a). Case 1. *p and q are contained in $R-R_0$.* In this case, by the Green's formula

$$N(p,q)=N(q,p).$$

Case 2. *One of p and q is contained in $R-R_0$.*

Case. 2. a). $p\in R-R_0$ *and* $q\in B_1$.

Then $N(z,q)$ is harmonic in $R-R_0$ and by the maximum principle $V_M(q)$

clusters at B as $M\uparrow\sup_{z\in R} N(z,q)$. Hence we can find a number M such that $p\notin V_M(q)$ and $\partial V_M(q)$ is regular. Then by (34)

$$N(p,q)=\frac{1}{2\pi}\int_{\partial V_M(q)} N(z,p)\frac{\partial}{\partial n}N(z,q)\,ds$$

$$=\frac{1}{2\pi}\lim_{M\to M^*}\int_{\partial V_M(q)} N(z,p)\frac{\partial}{\partial n}N(z,q)\,ds=N(q,p), \tag{36}$$

where $M^*=\sup_{z\in R} N(z,p)$.

Case 2. b). *$p\in R-R_0$, $q\in B_0$. Then* $N(z,q)=\int_{B_1} N(z,q_\beta)\,d\mu(q_\beta): z\in R-R_0$, where $\mu(q_\beta): q_\beta\in B_1$ is a canonical distribution of $N(z,q)$. Then by case 2, a) $N(q_\beta,p)=N(p,q_\beta)$ and by the definition of the value of $N(z,q)$ at $p\in B_0$, we have

$$N(q,p)=\int_{B_1} N(q_\beta,p)\,d\mu(q_\beta)=\int_{B_1} N(p,q_\beta)\,d\mu(q_\beta)=N(q,p) \text{ by } p\in R-R_0.$$

Now $N(p,q): p\in R-R_0$ and $q\in B$ is well defined and $N(q,p)=N(p,q)$, hence $N(q,p)$ does not depend on a particular distribution $\mu(q_\beta)$.

Case 3. *$p\in B$ and $q\in B$.*

Case 3. a) *$p\in B_1$ and $q\in B_1$.*

Let ξ and $\eta\in R-R_0$. Then by (36)

$$N(p,\eta)=N(\eta,p)=\frac{1}{2\pi}\int_{\partial V_M(p)} N(z,\eta)\frac{\partial}{\partial n}N(z,p)\,ds \quad \text{for } \eta\notin V_M(p). \tag{37}$$

$$N(p,\eta)=N(\eta,p)\geqq\frac{1}{2\pi}\int_{\partial V_M(p)} N(z,\eta)\frac{\partial}{\partial n}N(z,p)\,ds \quad \text{for } \eta\in V_M(p), \tag{38}$$

where $\partial V_M(p)$ is regular.

Since mean $(N(z,q)$ on $\partial V_M(p))=\frac{1}{2\pi}\int_{\partial V_M(p)} N(\xi,q)\frac{\partial}{\partial n}N(\xi,p)\,ds$ and since $V_M(q)$ clusters at B as $M\uparrow\sup_{z\in R} N(z,p)$, there exists a number M' for any given positive number ε such that

$$\text{mean }(N(z,q) \text{ on } \partial V_M(p))-\varepsilon\leqq\frac{1}{2\pi}\int_{\underline{\partial V_M(p)}} N(\xi,q)\frac{\partial}{\partial n}N(\xi,p)\,ds,$$

where $\underline{\partial V_M}(p)$ is the part of $\partial V_M(p)$ outside of $V_{M'}(q)$ and $\partial V_{M'}(q)$ is regular.

Suppose $\xi\in\underline{\partial V_M}(p)$, then $\xi\notin V_{M'}(q)$, whence

$$N(\xi, q) = N(q, \xi) = \frac{1}{2\pi} \int_{\partial V_{M'(q)}} N(\eta, \xi) \frac{\partial}{\partial n} N(\eta, q)\, ds.$$

Accordingly we have

$$\text{mean}\,(N(z, q) \text{ on } \partial V_M(p)) - \varepsilon \leqq \frac{1}{4\pi^2} \int_{\underline{\partial V_{M}(p)}} \left(\int_{\partial V_{M'}(p)} N(\eta, \xi) \frac{\partial}{\partial n} N(\eta, q)\, ds \right) \frac{\partial}{\partial n} N(\xi, p)\, ds$$

$$= \frac{1}{4\pi^2} \int_{\partial V_{M'}(q)} \left(\int_{\underline{\partial V_{M}(p)}} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, p)\, ds \right) \frac{\partial}{\partial n} N(\eta, q)\, ds. \tag{39}$$

By (37) and (38)

$$\frac{1}{2\pi} \int_{\underline{\partial V_{M}(p)}} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, p)\, ds \leqq \frac{1}{2\pi} \int_{\partial V_{M}(p)} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, p)\, ds$$

$$= N(\eta, p) = N(p, \eta) \quad \text{for } \eta \notin V_M(p)$$

$$\frac{1}{2\pi} \int_{\underline{\partial V_{M}(p)}} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, p)\, ds \leqq \frac{1}{2\pi} \int_{\partial V_{M}(p)} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, p)\, ds$$

$$= N(\eta, p) = N(p, \eta) \quad \text{for } \eta \in V_M(p).$$

On the other hand,

$$\text{mean}\,(N(z, p) \text{ on } \partial V_{M'}(q)) = \frac{1}{2\pi} \int_{\partial V_{M'}(q)} N(p, \eta) \frac{\partial}{\partial n} N(\eta, q)\, ds.$$

Hence by (37), (38) and (39)

$$\text{mean}\,(N(z, p) \text{ on } \partial V_M(p)) - \varepsilon \leqq \frac{1}{4\pi^2} \int_{\partial V_{M'}(q)} \left(\int_{\partial V_{M}(p)} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, p)\, ds \right) \frac{\partial}{\partial n} N(\eta, q)\, ds$$

$$\leqq \frac{1}{2\pi} \int_{\partial V_{M'}(q)} N(\eta, p) \frac{\partial}{\partial n} N(\eta, q)\, ds = \text{mean}\,(N(z, p) \text{ on } \partial V_{M'}(q)).$$

Thus by letting $\varepsilon \to 0$

$$\text{mean}\,(N(z, q) \text{ on } \partial V_M(p)) \leqq \text{mean}\,(N(z, p) \text{ on } \partial V_{M'}(q)).$$

Since the inverse inequality holds for the other pair of $V_{M''}(p)$ and $V_{M'''}(q)$ and since mean $(N(z, q)$ on $\partial V_M(p)) \uparrow N(p, q)$ and mean $(N(z, p)$ on $\partial V_M(q)) \uparrow N(q, p)$, we have

$$N(p, q) = N(p, q).$$

Case 3, b). *$p \in B_1$ and $q \in B_0$ or $p \in B_0$ and $q \in B_1$.* Without loss of generality we can suppose $p \in B_1$ and $q \in B_0$. In this case $N(z, q) = \int_{B_1} N(z, q_\beta)\, d\mu(q_\beta)$ and similarly as in case 2, b) we have $N(p, q) = N(q, p)$.

Case 4. $p\in B_0$: $N(z,p)=\int_{B_1} N(z,p_\alpha)\,d\mu(p_\alpha)$ *and* $q\in B_0$: $N(z,q)=\int_{B_1} N(z,q_\beta)\,d\mu(q_\beta)$, $\mu(p_\alpha)$ and $\mu(q_\beta)$ are canonical distributions. By Case 3. a) and b)

$$N(p,q)=\int_{B_1} N(p_\alpha,\ q)\,d\mu(p_\alpha)=\int_{B_1}\Big(\int_{B_1} N(p_\alpha,q_\beta)\,d\mu(p_\alpha)\Big)d\mu(q_\beta)$$
$$=\int_{B_1}\Big(\int_{B_1} N(q_\beta,p_\alpha)\,d\mu(p_\alpha)\Big)d\mu(q_\beta)=\int_{B_1} N(q_\beta,p)\,d\mu(q_\beta)=N(q,p).$$

By the second and 5-th terms we see that $N(p,q)$ does not depend on particular distributions $\mu(p_\alpha)$ and $\mu(q_\beta)$, whence $N(q,p)$: $p\in \bar{R}-R_0$ and $q\in\bar{R}-R_0$ is well defined and $N(p,q)=N(q,p)$ by Cases 1, 2, 3 and 4.

Proof of b). By the definition of the value of $N(z,p)$ at $p\in R-R_0+B_1$

$$N(p,p)=\lim_{M\to M^*}\frac{1}{2\pi}\int_{\partial V_M(p)} N(z,p)\frac{\partial}{\partial n}N(z,p)\,ds=\lim_{M\to M^*}M=\sup_{z\in R}N(z,p),$$

where $M^*=\sup N(z,p)$.

Proof of c). At first we show that $U(p): p\in R-R_0+B_1$ is well defined and the representation $U(z)=\int_{R-R_0+B_1} N(z,p)\,d\mu(p)$ is valid not only in $R-R_0$ but also in $\bar{R}-R_0$.

Case 1. $p\in R-R_0+B_1$ and $U(z)$ is given by

$$\int_{R-R_0+B_1} N(z,p_\alpha)\,d\mu(p_\alpha)\ \text{ in } R-R_0 \tag{39}$$

(μ_α is not uniquely determined).

Since $\int_{\partial V_M(q)} N(z,p)\frac{\partial}{\partial n}N(z,q)\,ds\uparrow$ as $M\uparrow\sup_{z\in R}N(z,q)$, the order of the integration can be changed. Hence

$$U(p)=\lim_{M\to M^*}\frac{1}{2\pi}\int_{\partial V_M(p)} U(z)\frac{\partial}{\partial n}N(z,p)\,ds$$
$$=\lim_{M\to M^*}\frac{1}{2\pi}\int_{\partial V_M(p)}\Big(\int_{R-R_0+B_1} N(z,p_\alpha)\,d\mu(p_\alpha)\Big)\frac{\partial}{\partial n}N(z,p)\,ds$$
$$=\frac{1}{2\pi}\int_{R-R_0+B_1}\Big(\lim_{M\to M^*}\int_{\partial V_M(p)} N(z,p_\alpha)\frac{\partial}{\partial n}N(z,p)\,ds\Big)d\mu(p_\alpha)=\int_{R-R+B_1} N(p,p_\alpha)d\,\mu(p_\alpha). \tag{40}$$

By the second term we see that $U(p)$: $p\in B_1$, depends on the behaviour of

$U(z)$ in $R-R_0$ and does not depend on a particular distribution $\mu(p_\alpha)$. $U(p)$ is uniquely determined and by (40) the representation (39) is also valid on B_1.

Case 2. $p\in B_0$: $N(z,p)=\int\limits_{B_1} N(z,p_\beta)\,d\mu(p_\beta)$. In this case by (40) $U(p) =\int\limits_{R-R_0+B_1} N(p,p_\alpha)\,d\mu(p_\alpha)$ for $p\in B_1+R-R_0$ and by the definition of $U(p)$ at $p\in B_1$,

$$U(p)=\int\limits_{R-R_0+B_1} U(p_\beta)\,d\mu(p_\beta)=\int\limits_{B_1}\Big(\int\limits_{R-R_0+B_1} N(p_\beta,p_\alpha)\,d\mu(p_\alpha)\Big)d\mu(p_\beta)$$
$$=\int\limits_{R-R_0+B_1}\Big(\int\limits_{B_1} N(p_\beta,p_\alpha)\,d\mu(p_\beta)\Big)d\mu(p_\alpha)=\int\limits_{R-R_0+B_1} N(p,p_\alpha)\,d\mu(p_\alpha).$$

We see that by the second term $U(p)$ does not depend on μ_α and by the last term it does not depend on μ_β. Hence $U(p)$ is uniquely determined. Hence (39) is valid also on B_0. Thus the representation is valid not only in $R-R_0$ but also on $\overline{R}-R_0$.

Next we show that $U(z)$ is lower semicontinuous in $\overline{R}-R_0$.

1). *$N(z,p)$ is lower semicontinuous in $\overline{R}-R_0$ for $p\in R-R+B_1$.*

Let $\{z_i\}$ be a sequence in $\overline{R}-R_0$ such that $\delta(z_i,z_0)\to 0$. Then $\varliminf_i N(z_i,p) \geqq N(z_0,p)$.

Proof of 1). By *b)* $N(z_i,p)=N(p,z_i)$ and $N(z_0,p)=N(p,z_0)$. Hence it is sufficient to show $\varliminf_i N(p,z_i)\geqq N(p,z_0)$. Since $N(p,z_0)=$ $\frac{1}{2\pi}\lim_{M\to M^*}\int\limits_{\partial V_M(p)} N(\zeta,z_0)\frac{\partial}{\partial n}N(\zeta,p)\,ds$, for any given positive number ε, there exist numbers M_0 and m_0 such that

$$N(p,z_0)-\varepsilon\leqq\frac{1}{2\pi}\int\limits_{\partial V_M(p)\frown R_m} N(\zeta,z_0)\frac{\partial}{\partial n}N(\zeta,p)\,ds \text{ for } M^*>M\geqq M_0 \text{ and } m\geqq m_0,$$

where $M^*=\sup\limits_{z\in R} N(z,p)$ and $\partial V_M(p)$ is regular.

$\delta(z_i,z_0)\to 0$ implies that $N(\zeta,z_i)\to N(\zeta,z_0)$ in $R-R_0$ and $N(\zeta,z_i)$ converges uniformly to $N(\zeta,z_0)$ on R_m. Hence

$$N(p,z_0)-\varepsilon\leqq\frac{1}{2\pi}\int\limits_{\partial V_M(p)\frown R_m} N(\zeta,z_0)\frac{\partial}{\partial n}N(\zeta,p)\,ds=\frac{1}{2\pi}\lim_i\int\limits_{\partial V_M(p)\frown R_m} N(\zeta,z_i)\frac{\partial}{\partial n}N(\zeta,p)\,ds$$
$$\leqq\frac{1}{2\pi}\varliminf_i\Big(\lim_{M\to M^*}\int\limits_{\partial V_M(p)} N(\zeta,z_i)\frac{\partial}{\partial n}N(\zeta,p)\,ds\Big)=\varliminf_i N(p,z_i).$$

Let $\varepsilon\to 0$. Then $N(p,z_0)\leqq\varliminf_i N(p,z_i)$. Hence $N(z_0,p)\leqq\varliminf_i N(z_i,p)$ and $N(z,p)$ is lower semicontinuous in $\bar{R}-R_0$ for $p\in R-R_0+B_1$.

2). *$U(z)$ is lower semicontinuous in $\bar{R}-R_0$* (of course $N(z,p):p\in B_0$ is a superharmonic function, hence $N(z,p)$ is lower semicontinuous).

Proof of 2). By (39), (40) and others the representation $U(z)=\int_{R-R_0+B_1}N(z,p_0)\,d\mu(p_\alpha):\ p_\alpha\in R-R_0+B_1$ is valid in $\bar{R}-R_0$.

Let $\{z_i\}$ be a sequence such that $\delta(z_i,z_0)\to 0:\ z_i,z_0\in\bar{R}-R_0$. Then by Fatou's lemma

$$\varliminf_{i=\infty}\int_{R-R_0+B_1}N(z_i,p_\alpha)\,d\mu(p_\alpha)\geqq\int_{R-R_0+B_1}\varliminf N(z_i,p_\alpha)\,d\mu(p_\alpha).$$

$N(z,p_\alpha)$ is lower semicontinuous. Hence

$$\varliminf_i U(z_i)=\varliminf_i\int_{R-R_0+B_1}N(z_i,p_\alpha)\,d\mu(p_\alpha)$$

$$\geqq\int_{R-R_0+B_1}\varliminf_i N(z_i,p_\alpha)\,d\mu(p_\alpha)\geqq\int_{R-R_0+R_1}N(z_0,p_\alpha)\,d\mu(p_\alpha)=U(z_0).$$

Thus $U(z)$ is lower semicontinuous in $\bar{R}-R_0$.

It is clear that $U(z)$ is superharmonic locally at $p\in R-R+B_1$ by the definition of the value $U(z)$ at $R-R_0+B_1$.

We have discussed the capacitary potentials of $(G\frown B)$, of F and of that determined by a sequence of decreasing domain and obtained some properties. Now the method to define the value, on B, of superharmonic functions is established. We consider the behaviour of C.P.'s and we shall prove some classical theorems which hold in euclidean space.

10. Capacitary potentials of closed sets, F_σ sets and of $F_{\sigma\delta}$ sets.

Theorem 12. *a). Let $p\in R-R_0+B_1-B_S$, then $\omega(p,z)=0$ and $\sup_{z\in R}N(z,p)=\infty$. Then*

$$\lim_{M\to\infty}{}_{V_{M(p)}\frown C_{v_n(p)}}N(z,p)=0\quad \textit{for every } v_n(p).$$

Let C_M be a regular niveau curve of $N(z,p)$. Then

$$\lim_{M\to\infty}\int_{C_M\frown v_n(p)}\frac{\partial}{\partial n}N(z,p)\,ds=2\pi.$$

b). Let $\omega(F,z)$ be C.P. *of a closed set F in $\bar{R}-R_0$ of positive capacity. Then*

$$\sup_{z\in F}\omega(F,z)=1.$$

c). (P.C.7). *Let* $\omega(F, z)$ *be* C.P. *of a closed set* F *of positive capacity. Then* $\omega(F, z)=1$ *except at most an* F_σ *set of capacity zero.*

d). *Let* $G(z, p)$ $p\in R_0$ *be the Green's function of a Riemann surface* $R=\bigcup_n R_n$ *with positive boundary. Then* $G(z, p)=0$ *on* B *except at most an* F_σ *set of capacity zero.*

e). (P.C.8). *Let* $F: F\frown\partial R_0=0$ *be a closed set of positive capacity:* $\omega(F, z)>0$. *Let* Ω *be the component of* $R-R_0-F$ *containing* ∂R_0 *as its boundary. Then* $E[z\in\Omega\frown CG_n: \omega(F, z)=1]$ *does not contain a closed set of positive capacity:* $G_n=E\left[z\in\overline{R}: \delta(z, F)<\frac{1}{n}\right]$.

Proof of a). Assume $N(z, p)\leqq M$. Then $\frac{N(z, p)}{M}\leqq\omega(V_M(p), z)$. Hence $\frac{N(z, p)}{M}\leqq\omega(p, z)=0$. This is a contradiction. Hence $\sup_{z\in R} N(z, p)=\infty$. By ${}_{V_{M(p)}}N(z, p)=N(z, p)$ in $R-R_0-V_M(p)$, $\omega(V_M(p), z)=\frac{\min(M, N(z, p))}{M}$ and $\lim_M \omega(V_M(p), z)=0$. Hence by Theorem 6. *b*) $N(z, p)-N'(z,p)$ is superharmonic, where $N'(z, p)=\lim_{M=\infty} {}_{V_{M(p)}\frown Cv_n(p)}N(z, p)$ and $N'(z, p)$ is represented by a mass distribution over $V_M(p)\frown Cv_m(p)$ by Theorem 5. *b*). If $N'(z, p)>0$, $N'(z, p)= KN(z, p)$ by the minimality of $N(z, p)$, because $N'(z, p)$ and $N(z, p)-N'(z, p)$ are superharmonic. Hence $KN(z, p)$ must be a point mass over $Cv_n(p)$ by Theorem 9. *a*), whence $N(z, p)=N(z, q): q\in Cv_n(p)$ by $\int_{\partial R_0}\frac{\partial}{\partial n}N(z, p)\,ds =\int_{\partial R_0}\frac{\partial}{\partial n}N(z, q)\,ds$. But $N(z, p)=N(z, q)$ implies $p=q$. This is a contradiction. Hence $N'(z, p)=0$.

Let $\omega_m(z)$ be a harmonic function in $R_m-R_0-(V_M(p)\frown Cv_n(p))$ such that $\omega_m(z)=0$ on ∂R_0, $\omega_m(z)=1$ on $\partial(V_M(p)\frown Cv_n(p))\frown R_m$ and $\frac{\partial}{\partial n}\omega_m(z)=0$ on $\partial R_m-(V_M(p)\frown Cv_n(p))$. Then $\omega_m(z)\Rightarrow\omega(V_M(p)\frown Cv_n(p), z)$ as $m\to\infty$. Hence by Fatou's lemma and by the compactness of ∂R_0 $\int_{\partial R_0}\frac{\partial}{\partial n}\omega(V_M(p)\frown Cv_n(p), z))\,ds$

$$=\lim_m\int_{\partial R_0}\frac{\partial}{\partial n}\omega_m(z)\,ds=\lim_m\int_{\partial(V_M(p)\frown Cv_n(p)\frown R_m}\frac{\partial}{\partial n}\omega_m(z)\,ds\geqq\int_{\partial(V_M(p)\frown Cv_n(p))}\frac{\partial}{\partial n}\omega(V_M(p)\frown Cv_n(p), z)\,ds$$

$$\geqq\int_{\partial V_M(p)\frown Cv_n(p)}\frac{\partial}{\partial n}\omega(V_M(p)\frown Cv_n(p), z)\,ds. \tag{41}$$

By $(V_M(p)\frown Cv_n(p))\subset V_M(p)$ and $N(z, p)\geqq M$ on $Cv_n(p)\frown V_M(p)$, by the

$\overset{*}{\text{m}}$aximum principle

$$M\omega(V_M(p)\frown Cv_n(p),z)\leqq {}_{V_M(p)\frown Cv_n(p)}N(z,p)\leqq N(z,p)$$
$$\text{in } R-R_0-(V_M(p)\frown Cv_n(p)). \tag{42}$$

On the other hand,

$$M\omega(V_M(p)\frown Cv_n(p),z)=M={}_{V_M(p)\frown Cv_n(p)}N(z,p)=N(z,p)$$
$$\text{on } \partial V_M(p)\frown Cv_n(p). \tag{43}$$

Hence by (41) and (43)

$$\frac{\partial}{\partial n}M\omega(V_M(p)\frown Cv_n(p),z))\geqq\frac{\partial}{\partial n}\,{}_{V_M(p)\frown Cv_n(p)}N(z,p)$$
$$\geqq\frac{\partial}{\partial n}N(z,p)\geqq 0 \text{ on } \partial V_M(p)\frown Cv_n(p). \tag{44}$$

Hence by (41) and (43)

$$\int_{\partial R_0}\frac{\partial}{\partial n}M\omega(V_M(p)\frown Cv_n(p),z)\,ds\geqq\int_{\partial V_M(p)\frown Cv_n(p)}\frac{\partial}{\partial n}\,{}_{V_M(p)\frown Cv_n(p)}N(z,p)\,ds$$
$$\geqq\int_{\partial V_M(p)\frown Cv_n(p)}\frac{\partial}{\partial n}N(z,p)\,ds. \tag{45}$$

Assume $\overline{\lim\limits_{M=\infty}}\int_{\partial V_M(p)\frown Cv_n(p)}\frac{\partial}{\partial n}N(z,p)\,ds>\delta_0>0$. Then by (45)

$$\overline{\lim_{M=\infty}}\int_{\partial R_0}\frac{\partial}{\partial n}M\omega(V_M(p)\frown Cv_n(p),z)\,ds\geqq\delta_0 \text{ and } \overline{\lim_{M=\infty}}M\omega(V_M(p)\frown Cv_n(p),z)>0,$$

whence by (42) $N'(z,p)=\lim\limits_{M=\infty}{}_{V_M(p)\frown Cv_n(p)}N(z,p)>0$. This contradicts $N'(z,p)=0$. Hence

$$\overline{\lim_{M=\infty}}\int_{\partial V_M(p)\frown Cv_n(p)}\frac{\partial}{\partial n}N(z,p)\,ds=0 \text{ and } \lim_{M=\infty}\int_{\partial V_M(p)\frown v_n(p)}\frac{\partial}{\partial n}N(z,p)\,ds=2\pi$$

by the regularity of $\partial V_M(p)$. Thus we have a).

Proof of b). Let $F_m=E\left[z\in\overline{R}:\ \delta(z,F)\leqq\frac{1}{m}\right]$. Then $F=\frown F_m$ and F_m can be considered as a non compact domain. Hence $\sup\limits_{z\in R}\omega(F,z)=1$ by P.C.2. But our assertion *is not so trivial*. If F has a closed subset F' of positive capacity in $R-R_0$, *our assertion is clear*. If F has a point $p\in B_S$, $1=\omega(p,p)\leqq\sup\limits_{z\in F}\omega(z,F)=1$ by Theorem 10, b). Hence we can suppose without loss of generality that $F\subset B$ and $F\frown B_S=0$ and Cap $(F)>0$. Since B_0 is a set of capacity zero, F has at least one point $p\in B_1-B_S$. Assume

$\omega(F,z)<K<1$ on F. By the definition $\omega(F,p)=\frac{1}{2\pi}\lim_{M=\infty}\int_{\partial V_M(p)}\omega(F,z)\frac{\partial}{\partial n}N(z,p)ds$ for $p\in B_1-B_S$. Hence by $a)$

$$\omega(F,p)=\frac{1}{2\pi}\lim_{M=\infty}\int_{\partial V_M(p)\frown v_n(p)}\omega(F,z)\frac{\partial}{\partial n}N(z,p)\,ds.$$

Let $G_{K+\delta}=E[z\in R:\omega(F,z)<K+\delta]:\delta>0$ and $1>K+\delta>K$. Then by $\omega(F,z)\leqq K$ on F, there exists a positive constant ε_0 such that

$$\varliminf_{M=\infty}\int_{\partial V_M(p)\frown G_{K+\delta}\frown v_n(p)}\frac{\partial}{\partial n}N(z,p)\,ds>2\pi\varepsilon_0 \text{ and } 0<\varepsilon_0<\frac{\delta}{K+\delta}. \tag{46}$$

In fact, if $\overline{\lim_{M=\infty}}\int_{\partial V_M(p)\frown CG_{K+\delta}\frown v_n(p)}\frac{\partial}{\partial n}N(z,p)\,ds>2\pi(1-\varepsilon_0)$,

$$\omega(F,p)\geqq\overline{\lim_{M=\infty}}\left(\frac{1}{2\pi}\int_{\partial V_M(p)\frown CG_{K+\delta}\frown v_n(p)}\omega(F,z)\frac{\partial}{\partial n}N(z,p)ds+\frac{1}{2\pi}\int_{\partial V_M(p)\frown G_{K+\delta}\frown v_n(p)}\omega(F,z)\frac{\partial}{\partial n}N(z,p)ds\right)$$

$$\geqq\frac{1}{2\pi}\overline{\lim_{M=\infty}}\left(\int_{\partial V_M(p)\frown CG_{K+\delta}\frown v_n(p)}\omega(F,z)\frac{\partial}{\partial n}N(z,p)\,ds\right)\geqq\frac{(K+\delta)}{2\pi}2\pi(1-\varepsilon_0)>K.$$

This contradicts $\omega(F,p)<K$. Hence (46) holds for every $v_n(p)$. Now by P.C.3, $\omega(G_{K+\delta}\frown F,z)=0$ for $K+\delta<1$, i.e. $\lim_{m=\infty}\omega(G_{K+\delta}\frown F_m,z)=0$, where $F_m=E\left[z\in\bar{R}:\delta(z,F)\leqq\frac{1}{m}\right]$. Choose a subsequence $m_1,m_2,\cdots$ of $1,2,\cdots$ such that $\omega(G_{K+\delta}\frown F_{m_i},z)<\frac{1}{2^i}$ for $z=z_0$ $(i=1,2,\cdots)$. Then

$$\omega^*(z)=\sum_{i=1}^{\infty}\omega(G_{K+\delta}\frown F_{m_i},z)<\infty,$$

and $\omega^*(z)$ is superharmonic by Theorem 4. $h)$ and $\omega^*(z)\geqq i_0$ for $z\in(\bigcap_{i=1}^{i_0}F_{m_i}\frown G_{K+\delta})\frown R-R_0)$, hence $\omega^*(z)\to\infty$ as $z\to p\in F$ inside of $G_{K+\delta}$.

Let $p\in R-R_0+B_1-B_S$. Then

$$\omega^*(p)=\lim_{M=\infty}\frac{1}{2\pi}\int_{\partial V_M(p)}\omega^*(z)\frac{\partial}{\partial n}N(z,p)\,ds$$

$$\geqq\frac{1}{2\pi}\lim_{M=\infty}\int_{\partial V_M(p)\frown G_{K+\delta}\frown v_n(p)}\omega^*(z)\frac{\partial}{\partial n}N(z,p)ds\geqq i_0\varepsilon_0,$$

for $v_n(p)\subset F_{m_{i_0}}$.

This holds for every $v_n(p)$. Hence let $i_0\to\infty$. Then $\omega^*(p)=\infty$. Now by the lower semicontinuity of $\omega^*(z)$, $\omega^*(z)\to\infty$ as $z\to p\in(F\frown(R-R_0+B_1-B_S))$ not only inside of $G_{K+\delta}$ but also $\omega^*(z)\to\infty$ only if $z\to p$.

B_0 is a sum of closed sets of capacity zero. We can construct as above a $\overline{\text{su}}$perharmonic function $\omega^{**}(z)$ such that $\lim_{z\to p\in B_0}\omega^{**}(z)=\infty$. Hence $\lim_{z\to p\in F}\varepsilon(\omega^*(z)+\omega^{**}(z))=\infty$ for any $\varepsilon>0$. Put $\Delta_\varepsilon=E[z\in\overline{R}:\varepsilon(\omega^*(z)+\omega^{**}(z))\leqq 2]$. Then Δ_ε is closed and $\Delta_\varepsilon\frown F=0$, which implies $\text{dist}(\Delta_\varepsilon, F)>d_\varepsilon>0$ and $C\Delta_\varepsilon\supset F$. Put $F_{d_\varepsilon}=E[z\in\overline{R}:\ \delta(z,F]\leqq d_\varepsilon]$. Let ${}_{d_\varepsilon}\omega(z)$ be C.P. of $F_{d_\varepsilon}(\supset F)$. Then $\varepsilon(\omega^*(z)+\omega^{**}(z))\geqq{}_{d_\varepsilon}\omega(z)\geqq\omega(F,z)$. Let $\varepsilon\to 0$. Then $\omega(F,z)=0$. This is a contradiction. Hence $\sup_{z\in F}\omega(F,z)=1$.

Proof of c). Let $\omega(E_k,z)$ be C.P. of $E_k=E\left[z\in F:\ \omega(F,z)\leqq 1-\frac{1}{k}\right]$ $(k=1,2,\cdots)$. Then $\omega(E_k,z)\leqq\omega(F,z)$, whence $\sup_{R-R_0}(F_k,z)\leqq 1-\frac{1}{k}<1$. Hence by *b)* E_k is of capacity zero. Then $E=\smile E_k$ is an F_σ set of capacity zero, because E_k is closed by the semicontinuity of $\omega(F,z)$.

Proof of d). Let $\omega_n(z)$ be a $\overline{\text{su}}$perharmonic function in $\overline{R}-R_0$ such that $\omega_n(z)=1$ in $R-R_n$, $\omega_n(z)=0$ on ∂R_0 and $\omega_n(z)$ is harmonic in R_n-R_0. Then $\lim_n\omega_n(z)=\omega(B,z)$. Now $G(z,p)\leqq N<\infty$ in $R-R_0$. Hence by the maximum principle $0<G(z,p)\leqq N(1-\omega(B,z))$. On the other hand, $(1-\omega(B,z))=0$ on B by *c)* except an F_σ set of capacity zero. Whence we have at once *d)*.

Proof of e). Assume $\omega(F,z)=1$ on a closed set F^* of positive capacity in $\Omega\frown CG_{n_0}$: $CG_{n_0}=E\left[z\in\overline{R}:\ \delta(z,F)\geqq\frac{1}{n_0}\right]$. Clearly $\omega(F,z)<1$ in $\Omega\frown(R-R_0)$ by the maximum principle and $F^*\subset B$.

$$\omega(F,z)\geqq{}_{F^*}\omega(F,z)=\omega(F^*,z)>0. \tag{47}$$

Let $F^*_{n_0}=E\left[z\in\overline{R}:\ \delta(z,F^*)<\frac{1}{n_0}\right]$. Then $F^*_{n_0}\subset\Omega$ and $\omega(F,z)$ is $\overline{harmonic}$ *and non constant in* $F^*_{n_0}\frown(R-R_0)$ and $\text{dist}(CF^*_{n_0},F^*)\geqq\frac{1}{n_0}$. By $\omega(F^*,z)={}_{F^*}\omega(F^*,z)=\int_{F^*\frown(R-R_0+B_1)}N(z,p)\,d\mu(p)$ by Theorem 13. *d)*[9] and by Theorem 13. *b)*

$${}_{CF^*_{n_0}}\omega(F^*,z)<\omega(F^*,z).$$

Put $V(z)=\omega(F^*,z)-{}_{CF^*_{n_0}}\omega(F^*,z)$. Then $V(z)>0$, $V(z)=0$ on $\partial F^*_{n_0}$ and $D_{F^*_{n_0}}(V(z))<\infty$. Since $\omega(F,z)$ has M.D.I. over $R-R_0-F^*_m$ and ${}_{CF^*_{n_0}}\omega(F^*,z)$ has M.D.I. over $F^*_{n_0}$. Hence $V(z)$ has M.D.I. over $F^*_{n_0}-F^*_m$: $F^*_m=E\left[z\in\overline{R}:\ \delta(z,F^*)\leqq\frac{1}{m}\right]$ and $m>2n_0$, whence

9) Theorem 13. *d)* and *a)* will be proved independently soon. See p. 60.

$$ {}_{F^*_m + CF^*_{n_0}}V(z) = V(z). $$

Put $G_\delta = E\left[z \in F^*_{n_0} : V(z) > \frac{\delta}{2}\right]$: $\delta = \sup_{z \in F^*_{n_0}} V(z)$. Let $\omega(G_\delta, z, F^*_{n_0})$ be C.P. of G_δ relative $F^*_{n_0}$, then $\omega(G_\delta, z, F^*_{n_0})$ has M.D.I. over $F^*_{n_0} - G_\delta$ among all functions $S(z)$ such that $S(z)=0$ on $\partial F^*_{n_0}$ and $S(z)=1$ on ∂G_δ, whence

$$ \omega(G_\delta \frown F^*_m, z, F^*_{n_0}) \leqq \omega(G_\delta, z, F^*_{n_0}) \text{ and} $$
$$ D_{F^*_{n_0}}(\omega(G_\delta \frown F^*_m, z, F^*_{n_0})) \leqq D_{F^*_{n_0}}(\omega(G_\delta, z, F^*_{n_0})) \leqq \frac{4}{\delta^2} D(V(z)). $$

Let $\tilde{V}(z)$ be a harmonic function in $F^*_{n_0} - F^*_m$ such that $\tilde{V}(z) = \min\left(\frac{\delta}{2}, V(z)\right)$ on $\partial F^*_{n_0} + \partial F^*_m$. Then also $D_{F^*_{n_0}}(\tilde{V}(z)) \leqq D_{F^*_{n_0}}(V(z))$. $V(z), \tilde{V}(z)$ and $\omega(G_\delta \frown F^*_m, z, F^*_{n_0})$ are harmonic in $F^*_{n_0} - (F^*_m \frown G_\delta)$ and $V(z) + \omega(G_\delta \frown F^*_m, z, F^*_{n_0}) \geqq 1 \geqq V(z)$ on $G_\delta \frown \partial F^*_m$, $\tilde{V}(z) + \omega(G_\delta \frown F^*_m, z, F^*_{n_0}) \geqq V(z)$ on $CG_\delta \frown \partial F^*_m$ and $V(z) + \omega(G_\delta \frown F^*_m, z, F^*_{n_0}) = V(z) = 0$ on $\partial F^*_{n_0}$. Hence by the maximum principle

$$ \tilde{V}(z) + \omega(G_\delta \frown F^*_m, z, F^*_{n_0}) \geqq V(z). $$

If $\lim_m \omega(G_\delta \frown F^*_m, z, F^*_{n_0}) = 0$, then $\tilde{V}(z) \geqq V(z)$ and $\sup_{z \in F^*_{n_0}} \tilde{V}(z) = \frac{\delta}{2} > \delta - \sup_{z \in F^*_{n_0}} V(z)$. This is a contradiction. Hence

$$ \lim_m \omega(G_\delta \frown F^*_m, z, F^*_{n_0}) = \omega(G_\delta \frown F^*, z, F^*_{n_0}) > 0. \tag{48} $$

Let C_M: $0 < M < 1$ be a regular niveau curve of $\omega(G_\delta \frown F^*, z, F^*_{n_0})$. Then

$$ \int_{C_M} \omega(G_\delta \frown F^*, z, F^*_{n_0}) \frac{\partial}{\partial n} \omega(G_\delta \frown F^*, z, F^*_{n_0})\, ds = D(\omega(G_\delta \frown F^*, z, F^*_{n_0})) \text{ as } M \to 1. $$

By $\omega(F, z) \geqq \omega(F^*, z) \geqq \omega(G_\delta \frown F^*, z, F^*_{n_0})$,

$$ \lim_{M=1} \int_{C_M} \omega(F, z) \frac{\partial}{\partial n} \omega(G_\delta \frown F^*, z, F^*_{n_0})\, ds \geqq D(\omega(G_\delta \frown F^*, z, F^*_{n_0})). \tag{49} $$

On the other hand, $\omega(F, z)$ is non constant and harmonic in $F^*_{n_0}$ by $F^*_{n_0} \subset \Omega$. Hence $\omega_n(z) \Rightarrow \omega(F, z)$ as $n \to \infty$, where $\omega_n(z)$ is a harmonic function in $(R_n \frown F^*_{n_0})$ such that $\omega_n(z) = \omega(F, z)$ on $\partial F^*_{n_0} \frown R_n$ and $\frac{\partial}{\partial n} \omega_n(z) = 0$ on $F^*_{n_0} \frown \partial R_n$.

Put $G^{1,2} = E[z \in R: M_1 < \omega(G_\delta \frown F^*, z, F^*_{n_0}) < M_2]$. Then $\omega(G_\delta \frown F^*, z, F^*_{n_0})$ has M.D.I. over $G^{1,2}$. Hence $\tilde{\omega}_n(z) \Rightarrow \omega(G_\delta \frown F^*, z, F^*_{n_0})$ in $G^{1,2}$, where $\tilde{\omega}_n(z)$ is a harmonic function in $R_n \frown G^{1,2}$ such that $\tilde{\omega}_n(z) = M_i$ on $C_i \frown R_n$: $C_i = E[z \in F^*_{n_0}: \omega(G_\delta \frown F^*, z, F^*_{n_0}) = M_i]$ and $\frac{\partial}{\partial n} \tilde{\omega}_n(z) = 0$ on $G^{1,2} \frown \partial R_n$. We suppose that C_1 and C_2 are regular.

Now $\int\limits_{C_{M_i}\frown R_n} \tilde{\omega}_n(z)\frac{\partial}{\partial n}\omega_n(z)\,ds = M_i \int\limits_{C_{M_i}\frown R_n} \frac{\partial}{\partial n}\omega_n(z)\,ds = 0$. Hence by the Green's formula

$$\int\limits_{C_{M_1}\frown R_n} \omega_n(z)\frac{\partial}{\partial n}\tilde{\omega}_n(z)\,ds = \int\limits_{C_{M_2}\frown R_n} \omega_n(z)\frac{\partial}{\partial n}\tilde{\omega}_n(z)\,ds.$$

By the regularity of C_{M_i} and by Theorem 3, a). By letting $n\to\infty$,

$$\int\limits_{C_{M_1}} \omega(F,z)\frac{\partial}{\partial n}\omega(G_\delta\frown F^*, z, F^*_{n_0})\,ds = \int\limits_{C_{M_2}} \omega(F,z)\frac{\partial}{\partial n}\omega(G_\delta\frown F^*, z, F^*_{n_0})\,ds.$$

On $C_{M_1}\frown R$, $\omega(F,z)<1$ *by the non constancy of* $\omega(F,z)$ in $F^*_{n_0}$, hence there exists a positive constant δ_0 such that

$$\int\limits_{C_{M_1}} \omega(F,z)\frac{\partial}{\partial n}\omega(G_\delta\frown F^*, z, F^*_{n_0})\,ds \leqq \int\limits_{C_{M_1}} \frac{\partial}{\partial n}\omega(G_\delta\frown F^*, z, F^*_{n_0}) - \delta_0$$
$$= D(\omega(G_\delta\frown F^*, z, F^*_{n_0})) - \delta_0.$$

Let $M_2\to 1$. Then

$$\lim_{M_2\to 1} \int\limits_{C_{M_2}} \omega(F,z)\frac{\partial}{\partial n}\omega(G_\delta\frown F^*, z, F^*_{n_0})\,ds < D(\omega(G_\delta\frown F^*, z, F^*_{n_0})) - \delta_0. \quad (50)$$

(49) contradicts (50). Hence $\omega(F^*, z)=0$. Thus we have e)

Let $U(z)$ be a positive superharmonic function in $\bar{R}-R_0$. Then by Theorem 8 there exists a canonical mass distribution μ of which the uniqueness is not proved. But we shall prove the following

Theorem 13. *a*). *Let* $U(z)$ *be a positive superharmonic function in* $\bar{R}-R_0$ *such that* $U(z) = \int\limits_{F\frown(\bar{R}-R_0+B_1)} N(z,p)\,d\mu(p)$. *Then* ${}_F U(z) = U(z)$.

b). *Let* $U(z)$ *be a superharmonic function in a*) *and let* F' *be a closed set such that dist* $(F, F')>0$. *Then*

$${}_{F'}U(z) < {}_F U(z) = U(z).$$

c). *Let* $U(z)$ *be a positive superharmonic function in* $\bar{R}-R_0$ *and let* F *be a closed set such that* ${}_F U(z) = U(z)$. *Then* $U(z)$ *is represented by a canonical mass distribution*[10] *on* F *such that* $U(z) = \int\limits_{F\frown(\bar{R}-R_0+B_1)} N(z,p)\,d\mu(p)$ *and any canonical distribution has no mass on* CF.

10) If $\mu=0$ on B_0, μ is called canonical.

d). Let $U(z)$ be a function in a). Let F be the kernel of a canonical mass distribution. Then the kernel of any other canonical mass distribution is also F.

e). As a corollary of c) $\omega(F, z)=\int_{F\frown(R-R_0+B_1)} N(z, p)\, d\mu(p)$.

Proof of a). By ${}_{v_n(p)}N(z, p)=N(z, p)$ for $p\in R-R_0+B_1$ and $v_n(p)\subset F_m$: $F_m=E\left[z\in \bar{R}:\ \delta(z, F)\leqq\frac{1}{n}\right]$. Whence ${}_F N(z, p)=N(z, p)$. Hence we have at once $a)$.

Proof of b). Assume ${}_{F'_n}U(z)=U(z)$: $F'_n=E\left[z\in \bar{R}:\ \delta(z, F')\leqq\frac{1}{n}\right]$ for every n. We cover F by a finite number of closed discs $\mathfrak{F}_1, \mathfrak{F}_2, \cdots \mathfrak{F}_{i_0}$ with diameter $<\frac{1}{2}$. Put $\mu=\mu_1+\mu_2+\cdots+\mu_{i_0}$, where μ_i is the restriction of μ on $\mathfrak{F}_i-\sum_{i=1}^{i-1}\mathfrak{F}_j$. Now by the superharmonicity of $\int N(z, p)\, d\mu(p)$

$$ {}_{F'_n}U(z)=\sum_{i=1}^{i_0}\left(\int_F N(z, p)\, d\mu_i(p)\right)=\sum_{i}^{i_0}\int_F N(z, p)\, d\mu_i(p)=U(z)$$

and
$$ {}_{F'_n}\left(\int N(z, p)\, d\mu_i(p)\right)\leqq\int N(z, p)\, d\mu_i(p) \text{ for every } i.$$

Hence
$$ {}_{F'_n}\left(\int N(z, p)\, d\mu_i(p)\right)=\int N(z, p)\, d\mu_i(p)\geqq 0 \text{ for every } i.$$

Hence there exists at least one μ_i and $\mathfrak{F}_i$ such that $\int N(z, p)\, d\mu_i(p)$ $={}_{F'_n}\left(\int N(z, p)\, d\mu_i(p)\right)>0$. We denote them $\mathfrak{F}^1$ and μ^1 respectively. As above we choose $\mathfrak{F}^2$ and μ^2 such that $\mathfrak{F}^2\subset\mathfrak{F}^1$, diameter of $\mathfrak{F}^2<\frac{1}{2^2}$ and ${}_{F'}\left(\int_{\mathfrak{F}^2} N(z, p)\, d\mu^2(p)\right)=\int_{\mathfrak{F}^2} N(z, p)\, d\mu^2(p)>0$, where $\mu^2(p)$ is the restriction of μ on $\mathfrak{F}^2$. In this way we can find a sequence $\mathfrak{F}^1\supset\mathfrak{F}^2\cdots$ and $\frac{\mu^1}{m_1}, \frac{\mu^2}{m_2}, \cdots$ such that $\bigcap_i^\infty \mathfrak{F}^i=p\in(R-R_0+B_1)\frown F$, where m_i is the total mass of μ^i. Because if every sequence $\mathfrak{F}^1\frown\mathfrak{F}^2\frown\cdots=p\in B_0$, μ has no mass outside of B_0. This contradicts that μ is canonical.

Now ${}_{F'}\left(\frac{1}{m_i}\int N(z, p)\, d\mu^i(p)\right)=\int\frac{1}{m_i}N(z, p)d\mu^i(p)$. We can find an weak limit μ^* of $\left\{\frac{\mu^i}{m_i}\right\}$ on $\bigcap_i^\infty\mathfrak{F}^i=p\notin B_0$ such that $\int N(z, p)\, d\mu^*(p)=\lim_{i'=\infty}\int\frac{1}{m_{i'}}N(z,p)\, d\mu^{i'}(p)$

$=N(z,p)$, where $\{i'\}$ is a subsequence such that $\int \frac{1}{m_{i'}} N(z,p)\,d\mu^{i'}(p) \to N(z,p)$. By $p \in (R-R_0+B_1) \frown F$

$$\lim_{i'=\infty} \int N(z,p)\,d\left(\frac{\mu^{i'}}{m_{i'}}\right)(p) = N(z,p) \text{ is minimal.} \tag{51}$$

On the other hand, by ${}_{F'}\left(\frac{1}{m_i}\int N(z,p)\,d\mu^{i'}(p)\right) = \frac{1}{m_i} N(z,p)\,d\mu^{i'}(p)$, $\int \frac{1}{m^{i'}} N(z,p)\,d\mu^{i'}(p)$ is represented by mass $'\mu^i$ on F' by Theorem 6. a). Hence $\lim_{i'\to\infty} \frac{1}{m_{i'}} \int N(z,p)\,d\mu^{i'}(p)$ is represented by an weak limit $'\mu^*$ of $\left\{\frac{\mu^{i'}}{m^{i'}}\right\}$ on F', i.e. $\lim_{i'\to\infty} \frac{1}{m_{i'}} \int_{F'} N(z,p)\,d'\mu^i(p) = \int_{F'} N(z,p)\,d'\mu^*(p)$. By (51) $\int_{F'} N(z,p)\,d'\mu^*(p)$ $(=N(z,p))$ is minimal, whence by Theorem 9, a) $'\mu^*$ is a point mass at $q \in F' \frown (R-R_0+B_1)$. Hence $N(z,p)=N(z,q)$ and $p \in F$ and $q \in F'$. This is a contradiction. Hence ${}_{F'}\left(\int_{F\frown(R-R_0+B_1)} N(z,p)\,d\mu(p)\right) < \int_{F\frown(R-R_0+B_1)} N(z,p)\,d\mu(p)$.

Proof of c). Since ${}_F U(z)=U(z)$, by Theorem 6, a) $U(z)=\int_F N(z,p)\,d\mu(p)$. Let μ^* be a canonical distribution of μ (μ^* may be positive over $\overline{R}-R_0-F$) and let $\mu^{*\prime}$ be the restriction of μ^* on F. Then $\mu^*-\mu^{*\prime}$ is also canonical and $\mu^*-\mu^{*\prime}=0$ on F and $\geqq 0$ on CF. Assume $\mu^*-\mu^{*\prime}>0$. Then there exists a closed set F' in CF such that the restriction $\mu^{*\prime\prime}$ of μ^* on $F'>0$ and $\text{dist}(F,F')>0$.

$${}_F U(z) = {}_F\left(\int N(z,p)\,d\mu^{*\prime\prime}(p) + \int N(z,p)\,d(\mu^*-\mu^{*\prime\prime})(p)\right) = U(z),$$

$${}_F\left(\int N(z,p)\,d\mu^{*\prime\prime}(p)\right) \leqq \int N(z,p)\,d\mu^{*\prime\prime}(p)$$

$$\text{and } {}_F\left(N(z,p)\,d(\mu^*-\mu^{*\prime\prime})(p)\right) \leqq \int N(z,p)\,d(\mu^*-\mu^{*\prime\prime})(p).$$

Whence

$${}_F\left(\int_{F'} N(z,p)\,d\mu^{*\prime\prime}(p)\right) = \int_{F'} N(z,p)\,d\mu^{*\prime\prime}(p). \tag{52}$$

(52) contradicts b) by $\text{dist}(F,F')>0$ and $\mu^{*\prime\prime}(p)$ is canonical. Hence $\mu^*-\mu^{*\prime}=0$ and any canonical distribution has no mass on CF. Hence

$$U(z) = \int_{F\frown(R-R_0+B_1)} N(z,p)\,d\mu(p).$$

Proof of d). Let μ_i $(i=1,2)$ be a canonical mass distribution of $U(z)$ whose kernel is F_i. Then by a) ${}_{F_2}U(z)=U(z)$. Hence by c) μ_1 has no mass outside of F_2, whence $F_2 \supset F_1$. Similarly $F_1 \supset F_2$. Hence $F_1 = F_2$.

Proof of e). $\omega(F, z) = {}_F\omega(F, z)$ by P.C.1, whence we have *e)* by *c)*.

Let A be an F_σ set such that $A = \cup F_n$, $F_1 \subset F_2 \cdots$ and F_n is closed. We define $\omega(A, z)$ by $\lim_n \omega(F_n, z)$. Then by Theorem 4. *h)* $\omega(A, z)$ is superharmonic in $\overline{R} - R_0$.

Theorem 14. *a)* P.C.1. $\omega(A, z) = {}_{A_m}\omega(A, z): A_m = E\left[z \in \overline{R}: \delta(z, A) \leqq \frac{1}{m}\right]$.

b). P.C.4. $\omega(A, z) = {}_{G_\delta}\omega(A, z) = (1-\delta)\omega(G_\delta, z)$ *in* $CG_\delta : G_\delta = E[z \in R : \omega(A, z) > 1-\delta]$.

c). P.C.7. *If* $\omega(A, z) > 0$, $\omega(A, z) = 1$ *on* A *except at most an* F_σ *set of capacity zero.*

Let H *be an* $F_{\sigma\delta}$ *set*: $H = \bigcap_n A_n$, $A_1 \supset A_2 \supset A_3, \cdots$ *and* A_n *is an* F_σ *set. We define* $\omega(H, z)$ *by* $\lim_n \omega(A_n, z)$. *Then* $\omega(H, z)$ *is also superharmonic in* $\overline{R} - R_0$ *by Theorem* 4, *h).*

d). $\omega(A_n, z) \Rightarrow \omega(H, z)$ *as* $n \to \infty$.

e). P.C.4. $\omega(H, z) = {}_{A_{n,m}}\omega(H, z): A_{n,m} = E\left[z \in \overline{R}: \delta(z, A_n) \leqq \frac{1}{m}\right]$.

f). P.C.4. $\omega(H, z) = {}_{G_{n,m}}\omega(H, z): G_{n,m} = E\left[z \in \overline{R}: \omega(A_n, z) > 1 - \frac{1}{m}\right]$.

g). P.C.7. *If* $\omega(H, z) > 0$, $\omega(H, z) = \left(1 - \frac{1}{m}\right)\omega(G_m, z)$ *in* $CG_m : G_m = E\left[z \in R: \omega(H, z) > 1 - \frac{1}{m}\right]$ *and* $\sup_{z \in G_m} \omega(H, z) = 1$ *and* $\sup_{z \in H} \omega(H, z) = 1$.

Proof of a). Put $F_{n,m} = E\left[z \in \overline{R}: \delta(z, F_n) \leqq \frac{1}{m}\right]$. Then $F_{n,m} \subset A_m$. Now by P.C.1. $\omega(F_n, z) \geqq {}_{A_m}\omega(F_n, z) = {}_{F_{n,m}}\omega(F_n, z) = \omega(F_n, z)$.
Hence $\omega(A, z) \geqq {}_{A_m}\omega(A, z) \geqq \lim_n {}_{A_m}\omega(F_n, z) = \lim_n \omega(F_n, z) = \omega(A, z)$.

Proof of b) Put $G_{n,m} = E\left[z \in \overline{R}: \omega(F_n, z) > 1 - \frac{1}{m}\right]$. Then $G_{n,m} \subset G_m$. Hence as above we have *b)*.

Proof of c). If $\omega(A, z) > 0$, there exists a number n_0 such that $\omega(F_n, z) > 0$ for $n \geqq n_0$. Then by P.C.7. (Theorem 12, *c)*) $\sup_{z \in A} \omega(A, z) \geqq \sup_{z \in F_n} \omega(F_n, z) = 1$. Put $L_m = E\left[z \in A: \omega(A, z) \leqq 1 - \frac{1}{m}\right]$. Then $L_m \subset \bigcup_n E\left[z \in F_n: \omega(F_n, z) \leqq 1 - \frac{1}{m}\right]$. Now $E\left[z \in F_n: \omega(F_n, z) \leqq 1 - \frac{1}{m}\right]$ is an F_σ set of capacity zero. Hence $\sum L_m$ is an F_σ set of capacity zero and we have *c)*.

Proof of d), e) and f). Since ${}_{G_{n,m}}\omega(A_n, z)=\omega(A_n, z)$ by *b*), $\omega_{n,l}(z)\Rightarrow \omega(A_n, z)$, where $\omega_{n,l}(z)$ is a harmonic function in $(R_l-R_0-G_{m,n})$ such that $\omega_{n,l}(z)=\omega(A_n, z)$ on $(\partial G_{n,m}\frown(R_l-R_0))+\partial R_0$ and $\frac{\partial}{\partial n}\omega_{n,l}(z)=0$ on $\partial R_l-G_{n,m}$.

$$D_{R_l-R_0-G_{n,m}}(\omega_{n,l}(z), \omega_{n+i,l}(z))=\int_{\partial G_{n,m}\frown(R_l-R_0)}\omega_{n,l}(z)\frac{\partial}{\partial n}\omega_{n+i,l}(z)\,ds$$
$$=\left(1-\frac{1}{m}\right)\int_{\partial G_{n,m}}\frac{\partial}{\partial n}\omega_{n+i,l}(z)\,ds=\left(1-\frac{1}{m}\right)\frac{\partial}{\partial n}\omega_{n+i,l}(z)\,ds$$
$$=\int_{\partial G_{n+i,m}}\omega_{n+i,l}(z)\frac{\partial}{\partial n}\omega_{n+i,l}(z)\,ds=D_{R_l-R_0-G_{n+i,m}}(\omega_{n+i,l}(z)).$$

Since $\omega_{n,l}(z)\Rightarrow\omega(A_n, z)$ and $\omega_{n+i,l}(z)\Rightarrow\omega(A_{n+i}, z)$,

$$D_{R-R_0-G_{n,m}}(\omega(A_n, z)),\ \omega(A_{n+i}, z))=D_{R-R_0-G_{n+i,m}}(\omega(A_{n+i}, z)).$$

Let $m\to\infty$. Then $D(\omega(A_{n+i}, z))=D(\omega(A_n, z),\ \omega(A_{n+i}, z))$ and $D(\omega(A_n, z)-\omega(A_{n+i}, z))=D(\omega(A_n, z))-D(\omega(A_{n+i}, z))$, whence $D(\omega(A_n, z))\downarrow\geqq 0$ as $n\to\infty$. Hence $\omega(A_n, z)\Rightarrow\omega(H, z)$. Now $\omega(A_n, z)$ has M.D.I. over $(R-R_0-A_{n,m})$ by *a*) and over $R-R_0-G_{n,m}$ by *b*). Hence by Lemma 1. *d*) $\omega(H, z)$ has M.D.I. over $R-R_0-A_{m,n}$ and over $R-R_0-G_{n,m}$. Thus we have *e* and *f*).

Proof of g). By *b*) $\omega(A_n, z)=\left(1-\frac{1}{m}\right)\omega(G_{n,m}, z)$ in $R-R_0-G_{n,m}$: $G_{n,m}=E\left[z\in\overline{R}:\ \omega(A_n, z)>1-\frac{1}{m}\right]$.

Let $m_1, m_2\cdots$ be a sequence such that $m_1<m_2<m_3\cdots$, $\lim_n m_n=\infty$. Then by $A_n\supset A_{n+1}\cdots$ and $m_n<m_{n+1}$

$$G_{n,m}=E\left[z\in\overline{R}:\omega(A_n, z)>1-\frac{1}{m_n}\right]\supset E\left[z\in\overline{R}:\omega(A_{n+1}, z)>1-\frac{1}{m_{n+1}}\right]=G_{n+1,m_{n+1}}.$$

For simplicity put $G_{n,m}=G'_n$. Then $\omega(A_n, z)=\left(1-\frac{1}{m_n}\right)\omega(G'_n, z)$ in $R-R_0-G'_n$. Hence

$$\omega(H, z)=\lim_n\omega(A_n, z)=\lim_n\omega(G'_n, z). \tag{53}$$

By $G'_n\supset G'_{n+1}\supset G'_{n+2},\cdots$, $\omega(G'_n, z)\Rightarrow$ a function which is equal to $\omega(H, z)$ by (53). Hence $\omega(H, z)$ is C.P. $\omega(\{G'_n\}, z)$ defined by a decreasing sequence of domains $\{G'_n\}$. Hence $\omega(H, z)$ has properties from P.C.1. to P.C.6. and we have (54) and (55). Hence

$$\text{If } \omega(H, z)>0,\ \sup_{z\in G'_n}(H, z)=1 \text{ for every } n. \tag{54}$$

$$\omega(H, z)=\left(1-\frac{1}{m}\right)\omega(G^*_m, z)\ \text{in}\ R-R_0-G^*_m: G^*_m=E\left[z\in R:\ \omega(H, z)>1-\frac{1}{m}\right]. \tag{55}$$

Next we show $\sup_{z \in H} \omega(H, z)=1$, if $\omega(H, z)>0$.

By c) $\omega(A_n, z)=1$ on A_n except an F_σ set of capacity zero, hence $\omega(A_n \frown CG'_n)=0$ and $A_n \frown CG'_n$ is an F_σ set of capacity zero which we denote by F'^n_σ. Hence $A_n \subset G'_n + F'^n_\sigma$ and

$$H=\bigcap_n A_n \subset (\bigcap_n G'_n + \sum_n F'^n_\sigma), \tag{56}$$

where G'_n is an open set by the semicontinuity of $\omega(A_n, z)$.

Put $CG^*_m = E\left[z \in \bar{R}:\ \omega(H, z) \leqq 1-\frac{1}{m}\right]$. Then by (53)

$$\omega(H \frown CG^*_m, z) \leqq \lim_n \omega(G'_n \frown CG^*_n, z) \leqq \omega(\{G'_n\}, z)=\omega(H, z)$$

and $$\sup_{z \in CG^*_m \frown G'_n} \omega(\{G'_n \frown CG^*_m\}, z) \leqq \sup_{z \in CG^*_n} \omega(H, z)<1-\frac{1}{m}<1,$$

where $\omega(\{G'_n \frown CG^*_m\}, z)$ is C.P. defined by sequence $\{G'_n \frown CG^*_m\}: n=1,2,\cdots$. Hence by P.C.2. $\omega(\{G'_n \frown CG^*_m\}, z)=0$ i.e. $\lim_{n=\infty} \lim \omega(G'_n \frown CG^*_m, z)=0$. (57)

Let $n_1, n_2, \cdots$ be a sequence such that $\int_{\partial R_0} \frac{\partial}{\partial n} \omega(G'_{n_i} \frown CG^*_m, z)\, ds \leqq \frac{1}{2^i}$. Then $\omega^*(z)=\sum_i^\infty \omega(G'_{n_i} \frown CG^*_m, z)$ is $\overline{\text{s}}$uperharmonic and $\omega^*(z)<\infty$ and

$$\sum^\infty \omega(G'_{n_i} \frown CG^*_m, z) \geqq i_0 \quad \text{in } \bigcap_{i=0}^{i_0} (G'_{n_i} \frown CG^*_m). \tag{58}$$

If $H \ni p: p \in B_S$, then $\omega(H, z) \geqq \omega(p, z)$, $\omega(H, p) \geqq \omega(p, p)=1$ by Theorem 10, b). In this case our assertion is trivial. Hence we can suppose that $H \frown B_S=0$. Let $p \in R-R_0+B_1-B_S$ be a point in $\bigcap^\infty G'_n$. Then $\sup_{z \in R} N(z, p)=\infty$ and $N(z, p)$ is N-minimal. Let $V_M(p)$ be a neighbourhood of p such that $V_M(p)=E[z \in R:\ N(z, p)>M]$ and $\partial V_M(p)$ is a regular niveau curve.

Assume $\sup_{z \in H} \omega(H, z)<K<1-\frac{2}{m}$. Then $H \subset CG^*_{2m}$. Let $v_{n'_i}(p)=E\left[z \in \bar{R}:\ \delta(z, p)<\frac{1}{m'_n}\right]$ such that $v_{n'}(p) \subset G'_{n_i}$. Such $v_{n_i}(p)$ can be chosen, because G'_{n_i} is open by the semicontinuity of $\omega(A_n, z)$. By the definition of the value of a $\overline{\text{s}}$uperharmonic functions at a point in $R-R_0+B_1-B_S$

$$\omega(H, p)=\frac{1}{2\pi} \lim_{M \to \infty} \int_{\partial V_M(p) \frown v_{n_i}(p)} \omega(H, z) \frac{\partial}{\partial n} N(z, p)\, ds \leqq \left(1-\frac{2}{m}\right)$$

by Theorem 12, a). This implies

$$\underline{\lim}_{M=\infty} \int_{\partial V_M(p) \frown v_{n_i}(p) \frown CG^*_m} N(z, p)\, ds \geqq 2\pi\varepsilon_0>0 \text{ by } \lim_{M \to \infty} \int_{\partial V_M(p) \frown v_{n_i}(p)} \frac{\partial}{\partial n} N(z, p)\, ds=2\pi.$$

Now by (58) $\omega^*(z)\geqq i_0$ in $(G'_{n_{i_0}}\frown CG^*_m)\supset(v_{n\cdot_0}(p)\frown CG^*_m)$, whence

$$\omega^*(p)\geqq\frac{1}{2\pi}\int_{\partial V_M(p)\frown v_{n_i}(p)\frown CG^*_m}\omega^*(z)\frac{\partial}{\partial n}N(z,p)\,ds\geqq\varepsilon_0 i_0.$$

Let $i_0\to\infty$. Then $\omega^*(p)=\infty$. Now $\omega^*(z)$ is lower semicontinuous, hence $\omega(z)\to\infty$ as $z\to p\in(R-R_0+B_1-B_S)\frown(\bigcap^{\infty}G'_n)\frown CG^*_{2m}$.

On the other hand, $\sum F'^n_\sigma+B_0$ is an F_σ set of capacity zero, whence we can construct a superharmonic function $\omega^{**}(z)$ such that $\omega^{**}(z)\to\infty$ as $z\to p\in(\sum F'^n_\sigma+B_0)$. Put $\omega^{***}(z)=\omega^*(z)+\omega^{**}(z)$. Then $\varepsilon\omega^{***}(z)\to\infty$ as $\to z$ $p\in H\subset((\bigcap^{\infty}G_{n_i})\frown CG^*_{2m}+\sum F'^n_\sigma+B_0)$ by (56) and by $H\subset CG^*_{2m}$ for any given positive number ε. Put $\Delta_\varepsilon=E[z\in\overline{R}:\ \omega^{***}(z)\leqq 2]$. Then Δ_ε is closed and dist$(\Delta_\varepsilon, H)>d_\varepsilon>0$, because if it were not so $\varepsilon\omega^{***}(z)\leqq 2$ on at least a point of H by the lower semicontinuity of $\omega^{***}(z)$. Hence $\varepsilon\omega^{***}(z)\geqq\omega(H,z)$. Let $\varepsilon\to 0$. Then $\omega(H,z)=0$. This is a contradiction. Hence if $\omega(H,z)\neq 0$, $\sup_{z\in H}\omega(H,z)=1$.

11. Maximum principles.

Theorem 15. a) *Let $U(z)$ be a positive $\overline{\overline{\text{s}}}$uperharmonic function in $\overline{R}-R_0$ such that $U(z)=0$ on ∂R_0 which is represented by a canonical mass distribution μ such that $U(z)=\int N(z,p)\,d\mu(p)$. Let F be the kernel of μ. If $U(z)\leqq M$ at points on which the mass is distributed (this implies $U(z)\leqq M$ on F by the lower semicontinuity of $U(z)$) and if $\mu=0$ on B_S (set of singular points), then*

$$\int_{C_\lambda}U(z)\frac{\partial}{\partial n}U(z)\,ds\leqq 2\pi M\int d\mu,$$

where $C_\lambda=E[z\in R:\ U(z)=\lambda]$.

b). Let $U(z)$ be a positive $\overline{\text{s}}$uperharmonic function such that $U(z)=0$ on ∂R_0. Put $G_{M_i}=E[z\in R:\ U(z)>M_i]:\lim_i M_i=\infty$. Put $U'(z)=\lim_i {}_{G_{M_i}}U(z)$. *Then $U'(z)$ is represented by a canonical mass distribution μ on $\bigcap_i\overline{G}_{M_i}$. Let F be the kernel of μ. Then if $U'(z)>0$, $\sup_{z\in F}U'(z)=\infty$.*

c). Let $U(z)$ be a positive $\overline{\text{s}}$uperharmonic function in $\overline{R}-R_0$ with $U(z)=0$ on ∂R_0 and let μ be its canonical mass distribution whose kernel is F. If $U(z)>0$ and Cap$(F)=0$, $\sup_{z\in F}U(z)=\infty$.

d). Let $U(z)$ be a positive $\overline{\text{s}}$uperharmonic function in $\overline{R}-R_0$ with

$U(z)=$ on ∂R_0 and let μ be its canonical mass distribution whose kernel is F. If $U(z)\leqq M$ on F, then

$$U(z)\leqq M\omega(F,z).$$

Proof of a). Let $U_i(z)=\sum_{j=1}^{i}c_jN(z,p_j)$ such that $p_j\in(F\frown(R-R_0+B_1-B_S))$: $c_j>0$ and $\sum^{i}c_j=$total mass of μ. Put $V_\lambda(p_j)=E[z\in R:\ c_jN(z,p_j)>\lambda]$ such that $\partial V_\lambda(p_j)$ is regular. Such $V_\lambda(p_j)$ exists, since $\sup_{z\in R}N(z,p)=\infty$ for $p\in R-R_0+B_1-B_S$. Then $\sum^{i}c_jN(z,p_j)$ has M.D.I. over $R-R_0-\sum^{i}V_\lambda(p_j)$, whence $U_i(z)$ has M.D.I. over $R-R_0-\sum^{i}V_\lambda(p_j)$. Put $D_{\lambda,i}=E[z\in R:\ U_i(z)>\lambda]$. Then $D_{\lambda,i}\supset\sum^{i}V_\lambda(p_j)$ and $U_i(z)$ has M.D.I. over $R-R_0-D_{\lambda,i}$ i.e. $U_i(z)=\omega(D_{\lambda,i}\,z)$ in $R-R_0-D_{\lambda,i}$, whence we can find a domain $D_{\lambda',i}=E[z\in R:\ U_i(z)>\lambda']$ such that $\lambda>\lambda'>0$ and $\partial D_{\lambda',i}$ is a regular niveau curve of $U_i(z)$. Let ${}_{CD_{\lambda',i}}U(z)$ be a harmonic function in $CD_{\lambda',i}$ with ${}_{CD_{\lambda',i}}U(z)=U(z)$ on $\partial D_{\lambda',i}+\partial R_0$. Then

$$U(z)\geqq{}_{CD_{\lambda',i}}U(z)=\lim_{M=\infty}{}_{CD_{\lambda',i}}U^M(z)\ \text{ and }\ {}_{CD_{\lambda',i}}U^M(z)=\lim_{n}U_n^M(z),$$

where $U_n^M(z)$ is a harmonic function in $D_{\lambda',i}\frown(R_n-R_0)$ such that $U_n^M(z)=\min(M,U(z))$ on $\partial D_{\lambda',i}\frown(R_n-R_0)$ and $\frac{\partial}{\partial n}U_n^M(z)=0$ on $\partial R_n\frown D_{\lambda',i}$.

Let $N_n(z,p_j)$ be a harmonic function in $(D_{\lambda',i}-V_\lambda(p_j))\cap(R_n-R_0)$ such that $N_n(z,p_j)=N(z,p_j)$ on $\partial(D_{\lambda',i}-V_\lambda(p_j))\frown(R_n-R_0)$ and $\frac{\partial}{\partial n}N_n(z,p_j)=0$ on $\partial R_n\frown(D_{\lambda',i}-V_\lambda(p_j))$. Then $N_n(z,p_j)\Rightarrow N(z,p_j)$ in $D_{\lambda',i}-V_\lambda(p_j)$ and $U_i(z)=\sum^{i}c_jN_n(z,p_j)=\lambda'$ on $\partial D_{\lambda',i}$ and $U_n^M(z)\Rightarrow{}_{CD_{\lambda',i}}U^M(z)$ as $n\to\infty$ in $D_{\lambda'i}$. By the Green's formula

$$\int_{(\partial D_{\lambda',i}+\partial V_\lambda(p))\frown R_n}U_n^M(z)\frac{\partial}{\partial n}c_jN_n(z,p_j)\,ds=\int_{(\partial D_{\lambda',i}+\partial V_\lambda(p))\frown R_n}c_jN_n(z,p_j)\frac{\partial}{\partial n}U_n^M(z)\,ds:j=1,2,\cdots,i.$$

By $\int_{\partial V_\lambda(p_j)\frown R_n}c_jN_n(z,p_j)\frac{\partial}{\partial n}U_n^M(z)\,ds=\lambda\int_{\partial V_\lambda(p_j)\frown R_n}\frac{\partial}{\partial n}U_n^M(z)\,ds=\lambda\int_{\partial R_n\frown V_\lambda(p_j)}\frac{\partial}{\partial n}U_n^M(z)\,ds=0$ we have

$$\int_{\partial D_{\lambda',i}\frown R_n}U_n^M(z)\frac{\partial}{\partial n}c_jN_n(z,p_j)\,ds=\int_{\partial V_\lambda(p_j)\frown R_n}U_n^M(z)\frac{\partial}{\partial n}c_jN_n(z,p_j)\,ds$$
$$+\int_{\partial D_{\lambda',i}\frown R_n}c_jN_n(z,p_j)\frac{\partial}{\partial n}U_n^M(z)\,ds\,. \tag{59}$$

By summing up (59) for $j=1,2,\cdots,i$ and by

$$\sum^{i} \int_{\partial D_{\lambda',i}\frown R_n} c_j N_n(z, p_j)\frac{\partial}{\partial n} U_n^M(z)\, ds = \lambda' \int_{\partial D_{\lambda',i}\frown R_n} \frac{\partial}{\partial n} U_n^M(z)\, ds = \lambda' \int_{D_{\lambda',i}\frown \partial R_n} \frac{\partial}{\partial n} U_n^M(z)\, ds = 0,$$

we have

$$\int_{\partial D_{\lambda',i}\frown R_n} U_n^M(z)\frac{\partial}{\partial n}\sum_{j=1}^{i} c_j N_n(z, p_j)\, ds = \sum_{j=1}^{i} \int_{\partial V_\lambda(p_j)} U_n^M(z)\frac{\partial}{\partial n} c_j N_n(z, p_j)\, ds. \tag{60}$$

Put $U_{i,n}(z) = \sum^{i} c_j N_n(z, p_j)$. Then $U_{i,n}(z) \leqq i\lambda$ in $R_n - R_0 - \sum^{i} V_\lambda(p_j)$ by $c_j N_n(z, p_j) < \lambda$ in $R - R_0 - V_\lambda(p_j)$.

Consider $U_{i,n}(z)$ in $D_{\lambda'_i} - D_{\lambda_i}$. Then $U_{i,n}(z) = \min_{z\in(D_{\lambda'_i} - D_{\lambda_i})} U_{i,n}(z) = \lambda'$ on $\partial D_{\lambda',i}$, $\frac{\partial}{\partial n} U_{i,n}(z) \geqq 0$ on $\partial D_{\lambda',i}$ and $U_{in}(z) \leqq i\lambda$ on $\partial D_{\lambda,i}$. Then by the regularity of $\partial D_{\lambda',i}$ and by Theorem 3, *b*)

$$\int_{\partial D_{\lambda',i}\frown R_n} U_n^M(z)\frac{\partial}{\partial n}\sum^{i} c_j N_n(z, p_j)\, ds \to \int_{\partial D_{\lambda',i}} {}_{CD_{\lambda',i}}U^M(z)\frac{\partial}{\partial n}\sum^{i} c_j N(z, p_j)\, ds \text{ as } n\to\infty. \tag{61}$$

Similarly $c_j N_n(z, p_j) = \max_{z\in D_{\lambda',i}\frown V_\lambda(p)} c_j N_n(z, p_j)$ on $\partial V_\lambda(p_j)$, $\frac{\partial}{\partial n} N_n(z, p_j) \geqq 0$ on $\partial V_\lambda(p_j)$ and

$$\int_{\partial V_\lambda(p_j)\frown R_n} U_n^M(z)\frac{\partial}{\partial n} N_n(z, p_j)\, ds \to \int_{\partial V_\lambda(p_j)} {}_{CD_{\lambda',i}}U^M(z)\frac{\partial}{\partial n} N(z, p_j)\, ds \text{ as } n\to\infty. \tag{62}$$

In (60) let $n\to\infty$ and then $M\to\infty$. Then by (61) and (62)

$$\int_{\partial D_{\lambda',i}} {}_{CD_{\lambda',i}}U(z)\frac{\partial}{\partial n} U_i(z)\, ds = \sum_{j=1}^{i} \int_{\partial V_\lambda(p_j)} c_j U_{CD_{\lambda',i}}(z)\frac{\partial}{\partial n} N(z, p_j)\, ds. \tag{63}$$

Assume $U(z) \leqq M$ on $F\frown(R - R_0 + B_1)$. Then by the local superharmonicity of $U(z)$ at p_j

$$M \geqq U(p_j) = \lim_{L=\infty} \frac{1}{2\pi} \int_{\partial V_L(p_j)} U(z)\frac{\partial}{\partial n} N(z, p_j)\, ds \geqq \frac{1}{2\pi} \int_{\partial V_\lambda(p)} U(z)\frac{\partial}{\partial n} N(z, p_j)\, ds.$$

Hence by (63) and by ${}_{CD_{\lambda',i}}U(z) = U(z)$ on $\partial D_{\lambda',i}$, we have

$$\int_{\partial D_{\lambda',i}} U(z)\frac{\partial}{\partial n} U_i(z)\, ds = \int_{\partial D_{\lambda',i}} {}_{CD_{\lambda',i}}U(z)\frac{\partial}{\partial n} U_i(z)\, ds \leqq 2\pi(\sum c_i)\, M. \tag{64}$$

By the continuity of $N(z, p)$ for fixed z with respect to p, there exists a sequence of linear forms $\sum^{i} c_j N(z, p_j) = U_i(z)$ such that $\sum c_j =$ total mass of $U(z)$, $p_j \in F$ and $U_i(z) \to U(z)$ as $i\to\infty$ in $R_m - R_0$ uniformly for any given number m. Now $U_i(z) \to U(z)$ implies $\frac{\partial}{\partial n} U_i(z) \to \frac{\partial}{\partial n} U(z)$ in $R_m - R_0$ and $\partial D_{\lambda',i} = E[z\in R\colon U_i(z) = \lambda']$ tends to ${}_{\lambda'}C = E[z\in R\colon U(z) = \lambda']$. Then by Fatou's lemma

$$\int_{C_{\lambda'}\frown R_m} U(z)\frac{\partial}{\partial n}U(z)\,ds \leqq \underline{\lim_i} \int_{\partial D_{\lambda',i}\frown R_m} U(z)\frac{\partial}{\partial n}U_i(z)\,ds$$

$$\leqq \underline{\lim_i} \int_{\partial D_{\lambda',i}} U(z)\frac{\partial}{\partial n}U_i(z) \leqq 2\pi M\times(\text{total mass of } U(z)).$$

Let $m\to\infty$. Then

$$\int_{C_{\lambda'}} U(z)\frac{\partial}{\partial n}U(z)\,ds \leqq 2\pi M\times(\text{total mass of } U(z)).$$

Thus we have $a)$.

Proof of b). Put $G_{M_i}=E[z\in R:\ U(z)>M_i]: M_1<M_2,\cdots,\ \lim_i M_i=\infty$. Then G_{M_i} is open by the semicontinuity of $U(z)$ and $\omega(G_{M_i}, z)\leqq\frac{U(z)}{M_i}$. Let $M_i\to\infty$. Then $\lim\omega(G_{M_i}, z)=0$. Put $\lim_{i=\infty}{}_{G_{M_i}}U(z)=U'(z)$. Then by Theorem 6, $b)$

$$\lim_i {}_{G_{M_i}}U'(z)=U'(z).$$

If $\sup_{z\in R} U'(z)<M<\infty$, $U'(z)\leqq M\omega(G_{M_i}, z)\to 0$ as $M_i\to\infty$. In this case our assertion is trivial. We suppose $\sup_{z\in R} U'(z)=\infty$. Let μ be the canonical mass distribution of $U'(z)$. *We show that μ has no mass at any point of B_S*. Assume that $U'(z)$ has a mass m at $p\in B_S$. Then $U'(z)=m\omega(p,z)+U''(z)$ and $U''(z)$ is also supperharmonic in $\overline{R}-R_0$. Hence

$${}_{G_{M_i}}U'(z)=U'(z)={}_{G_{M_i}}(U'(z)-m\omega(p,z))+{}_{G_{M_i}}(m\omega(p,z)). \tag{65}$$

Now $m\omega(p,z)\leqq m$ in $\overline{R}-R_0$, whence ${}_{G_{M_i}}(m\omega(p,z))\leqq m\omega(G_{M_i}, z)$. Let $M_i\to\infty$. Then $\omega(G_{M_i}, z)=0$. Hence $\lim_{i=\infty}{}_{G_{M_i}}(m\omega(p,z))=0$ and

$$U'(z)\gneqq U'(z)-m\omega(p,z)\geqq\lim_i {}_{G_{M_i}}(U'(z)-m\omega(p,z))=\lim_i {}_{G_{M_i}}U'(z)=U'(z). \tag{66}$$

This is a contradiction. Hence $U'(z)$ has no mass at any point $p\in B_S$.

${}_{G_{M_i}}U'(z)=U'(z)$ is clear by definition. *We show*

$${}_{\widetilde{G}_{M_i}}U'(z)=U'(z) \textit{ for any } M_i<\infty,$$

where $\widetilde{G}_{M_i}=E[z\in R:\ U'(z)>M_i]$.

By $U(z)\geqq U'(z)$, $\lim_{i=\infty}\omega(\widetilde{G}_{M_i}, z)=0$. Let $U_{\widetilde{M}_i, M_j}(z)$ be a harmonic function in $R-R_0-(\widetilde{G}_{M_i}+G_{M_j}): M_i<M_j$ such that $U_{\widetilde{M}_i, M_j}(z)=U'(z)$ on $\partial R_0+\partial(\widetilde{G}_{M_i}+G_{M_j})$ and $U_{\widetilde{M}_i, M_j}(z)$ has M.D.I. over $R-R_0-(\widetilde{G}_{M_i}+G_{M_j})$. Then by $(\widetilde{G}_{M_i}+G_{M_j})\supset G_{M_j}$

$$U'(z)\geqq U_{\widetilde{M}_i, M_j}(z)\geqq {}_{G_{M_j}}U'(z)=U'(z). \tag{67}$$

Let $U^*_{\tilde{M}_i, M_j}(z)$ be a harmonic function in $R-R_0-(\tilde{G}_{M_i}+G_{M_j}): M_i<M_j$ such that $U^*_{\tilde{M}_i, M_j}(z)=U'(z)$ on $\partial R_0+\partial\tilde{G}_{M_i}-G_{M_j}$ and $\frac{\partial}{\partial n}U^*_{\tilde{M}_i, M_j}(z)=0$ on $\partial G_{M_j}-\tilde{G}_{M_i}$ and $U^*_{\tilde{M}_i, M_j}(z)$ has M.D.I. over $R-R_0-(\tilde{G}_{M_i}+G_{M_j})$.
Then $U^*_{\tilde{M}_i, M_j}(z)\Rightarrow{}_{\tilde{G}_{M_i}}U'(z)$ as $j\to\infty$.
On the other hand both $U_{\tilde{M}_i, M_j}(z)$ and $U^*_{\tilde{M}_i, M_j}(z)$ has M.D.I. over $R-R_0-(\tilde{G}_{M_i}+G_{M_j})$. Hence by the maximum principle

$$|U_{\tilde{M}_i, M_j}(z)-U^*_{\tilde{M}_i, M_j}(z)|\leqq M_i\omega(G_{M_j}, z)\to 0 \text{ as } j\to\infty.$$

Hence by (67)

$$U'(z)={}_{\tilde{G}_{M_i}}U(z)=\lim_{j=\infty} U^*_{\tilde{M}_i, M_j}(z)=\lim_{j=\infty} U_{\tilde{M}_i, M_j}(z)=\lim_{j=\infty} {}_{G_{M_j}}U(z)=U'(z).$$

Thus we have $$ {}_{\tilde{G}_{M_i}}U'(z)=U'(z). \tag{68}$$

Assume $\sup_{z\in F} U'(z)\leqq M$. Now $\mu=0$ on B_S. Hence by *a*) $\int_{C_\lambda} U'(z)\frac{\partial}{\partial n}U'(z)\,ds \leqq 2\pi M\int d\mu$. On the other hand, by (68) $U'(z)=M\omega(\tilde{G}_M, z)$, in $R-R_0-\tilde{G}_M$ hence $\partial\tilde{G}_{M_i}$ is a regular niveau curve for almost every number M_i and

$$\int_{C_{M_i}}\frac{\partial}{\partial n}U'(z)\,ds=\int_{\partial R_0}\frac{\partial}{\partial n}U'(z)\,ds \quad\text{and}\quad \lim_{M_i=\infty}\int_{C_{M_i}} U'(z)\frac{\partial}{\partial n}U(z)\,ds=\infty.$$

This is a contradiction. Hence we have *b*).

Proof of c). If $U(z)>0$, clearly $\sup_{z\in R} U(z)=\infty$. Now by the assumption, since $\mathrm{Cap}(F)=0$, μ has no mass at any point of B_S. We show ${}_{CD_\lambda}U(z)=U(z)$, where $D_\lambda=E[z\in R: U(z)<\lambda]$. Since by Theorem 13. *a*) $U(z)={}_FU(z)$. Now $U(z)$ has M.D.I. $\leqq 2\pi\lambda\times$(total mass of μ) on $D_\lambda-F_m$, $F_m =E\left[z\in\bar{R}: \delta(z, F)\leqq\frac{1}{m}\right]$. Hence $U_{m,n}(z)\Rightarrow U_m(z)$ as $n\to\infty$ and $U_m(z)\Rightarrow U(z)$ as $m\to\infty$, where $U_{m,n}(z)$ is a harmonic function in $(R_n-R_0)\frown(D_\lambda-F_m)$ such that $U_{m,n}(z)=U(z)$ on $(\partial(D_\lambda\frown CF_m)\frown(R-R_n))+\partial R_0$ and $\frac{\partial}{\partial n}U_{m,n}(z)=0$ on $\partial R_n\frown(D_\lambda-F_m)$. Let $U'_{m,n}(z)$ be a harmonic function in $(R_n-R_0)\frown(D_\lambda-F_m)$ such that $U'_{m,n}(z)=U(z)$ on $((\partial D_\lambda\frown CF_m)\frown(R_n-R_0))+\partial R_0$ and $\frac{\partial}{\partial n}U'_{m,n}(z)=0$ on $(\partial F_m\frown D_\lambda)+(\partial R_n\frown(D_\lambda-F_m))$. Then $U'_{m,n}(z)\Rightarrow U'_m(z)$ and $U'_m(z)\Rightarrow{}_{CD_\lambda}U(z)$, because ${}_{CD_\lambda}U(z)$ has M.D.I. over D_λ. Now

$U_{m,n}(z)=U'_{m,n}(z)$ on $((\partial D_\lambda-F_m)\frown(R_n-R_0)+\partial R_0)$,

$|U_{m,n}(z)-U'_{m\,n}(z)|<\lambda$ on $\partial E_m\frown D_\lambda$ by $U_{m,n}(z)$ and $U'_{m,n}(z)<\lambda$ in $D'_\lambda-F_m$

and $\frac{\partial}{\partial n}U_{m,n}(z)=\frac{\partial}{\partial n}U'_{m,n}(z)=0$ on $\partial R_n\frown D'_\lambda\frown CF_m$.

Hence by the $\overset{*}{\text{m}}$aximum principle

$$|U_{m,n}(z)-U'_{m,n}(z)|<\lambda\omega_n(F_m, z),$$

where $\omega_n(F_m, z)$ is a harmonic function in $R_n-R_0-F_m$ such that $\omega_n(F_m, z)=1$ on F_m, $\omega_n(F_m, z)=0$ on ∂R_0 and $\frac{\partial}{\partial n}\omega_n(F_m, z)=0$ on ∂R_n-F_m.

Let $n\to\infty$ and then $m\to\infty$. Then

$$|U(z)-{}_{CD_\lambda}U(z)|\leqq\lambda\omega(F, z)=0 \text{ by Cap}(F)=0.$$

Hence $U(z)$ has M.D.I. over D_λ and ${}_{CD_\lambda}U(z)=U(z)$, whence $U(z)=\omega(CD_\lambda, z)$ in D_λ. Hence for almost all λ $\int\limits_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds=\int\limits_{\partial D_\lambda}\frac{\partial}{\partial n}U(z)\,ds$ and $\varlimsup_{\lambda=\infty}\int\limits_{\partial D_\lambda}U(z)\frac{\partial}{\partial n}U(z)\,ds=\infty$.

Thus by $a)$ $\qquad\sup_{z\in F}U(z)=\infty.$

Proof of d). Put $G_{M_i}=E[z\in R\colon U(z)>M_i]$, $M_1<M_2,\cdots$, $\lim_i M_i=\infty$. Then G_{M_i} is open and $\omega(G_{M_i}, z)\leqq\frac{U(z)}{M_i}$. Let $M_i\to\infty$. Then $\lim_{i=\infty}\omega(G_{M_i}, z)=0$. Put $\lim_i {}_{G_{M_i}}U(z)=U'(z)$. Then by Theorem 6. $b)$ $U(z)-U'(z)$ is $\overline{\text{s}}$uperharmonic in $\overline{R}-R_0$ and $\lim_i {}_{G_{M_i}}(U'(z))=U'(z)$. Now the total mass of ${}_{G_{M_i}}U(z)\leqq\frac{1}{2\pi}\int\limits_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds$. Hence we can find an weak limit μ' of the distribution μ'_i of $\{{}_{G_{M_i}}U(z)\}$ on $\bigcap\limits_i^\infty G_{M_i}$. Now $\bigcap^\infty G_{M_i}$ is of capacity zero, but we don't know that $\bigcap\overline{G}_{M_i}$ is of capacity zero or not. Let μ'^* be the canonical distribution of μ'. Then by ${}_{\overline{G}_{M_i}}U'(z)=U'(z)$ by $\overline{G}_{M_i}\supset G_{M_i}$, μ'^* has no mass outside of $\bigcap\limits_i^\infty\overline{G}_{M_i}$ and μ'^* is contained in $\bigcap\limits_i\overline{G}_{M_i}$. Also $U(z)-U'(z)$ has a canonical mass distribution $\mu''^*\geqq 0$. Then $\mu'^*+\mu''^*$ is a canonical distribution of $U(z)=U'(z)+(U(z)-U'(z))$. By Theorem 13. $d)$ the kernel of $\mu'^*+\mu''^*$ is contained in F which is the kernel of the distribution μ^* of $U(z)$, whence the kernel F' of μ'^* is contained in F. *We show* $U'(z)=0$. In fact, by the assumption $\sup_{z\in F'}U'(z)\leqq\sup_{z\in F}U(z)\leqq M$. *Hence by b)*

$$U'(z)=0. \tag{69}$$

$G_{M+a}=E[z\in\overline{R}\colon U(z)>M+a]: a>0$ is open and $F_n=E\left[z\in\overline{R}: \delta(z, F)\leqq\frac{1}{n}\right]$

is closed and $G_{M+n}\frown F_n$ is an F_σ set. Now $U(z)\geqq M+a$ in G_{M+a} implies $U(z)\geqq(M+a)\omega(G_{M+a}\frown F_n, z)$.

Let $n\to\infty$. Then $U(z)\geqq(M+a)\omega(G_{M+a}\frown F, z)$, where $F\frown G_{M+a}=\lim_n(F_n\frown G_{M+a})$ is an $F_{\sigma\delta}$ set.

Assume $\omega(G_{M+a}\frown F, z)>0$. Then by Theorem 14. g) $\sup_{z\in G_{M+a}\frown F}\omega(G_{M+a}\frown F, z)=1$ on $G_{M+a}\frown F$.

Hence by $U(z)\geqq(M+a)\omega(G_{M+a}\frown F, z)$, $\sup_{z\in F}U(z)\geqq M+a$ on F. This contradicts $U(z)\leqq M$ on F. *Hence*

$$\omega(G_{M+a}\frown F, z)=0. \tag{70}$$

By (69) there exists a number N_0 such that ${}_{G_N}U(z_0)<\varepsilon$ for any given point z_0 and any given positive number $\varepsilon>0$ for $N\geqq N_0$.

$U(z)={}_{F_n}U(z)\leqq a+M$ on $\partial F_n\frown C(F_n\frown G_{M+a})$, $U(z)\leqq N$ on $\partial F_n\frown CG_N$ and $U(z)={}_{G_N}U(z)$ on $\partial F_n\frown G_N$. Hence by the $\overset{*}{\text{m}}$aximum principle

$$U(z)={}_{F_n}U(z)\leqq(a+M)\omega(F_n, z))+N\omega(G_{M+a}\frown F_n, z)+{}_{G_N}U(z).$$

Let $n\to\infty$. Then by (70)

$${}_FU(z)\leqq(a+M)\omega(F, z)+\varepsilon \quad\text{for } z=z_0.$$

Let $\varepsilon\to 0$ and $a\to 0$. Then

$$U(z)={}_FU(z)\leqq M\omega(F, z).$$

Thus we have *d*).

12. Mass distributions. In the sequel we consider Problem of Equilibrium. It is important to summarize the properties of the space and the kernel $N(p, q)$.

1). The space $\overline{R}-R_0$ is composed of a Riemann surface $R-R_0$ and its ideal boundary $B=B_1+B_0$, where B_1 and B_0 are the sets of N-minimal points and of N-non minimal points respectively and B_0 is an F_σ set of capacity zero. On B_0 we cannot distribute any true mass. A distribution μ on B_0 may be called a pseudo distribution in the sense that μ can be replaced (by Theorem 8) by a canonical distribution on $R-R_0+B_1$ without any change of $U(z)=\int N(z, p)\,d\mu(p)$.

2). The kernel $N(p, q)$ satisfies the following conditions:

a) $N(p, q)=N(q, p): p$ and $q\in\overline{R}-R_0$.

b) $N(p, q)$ is harmonic with respect to p in $R-R_0$ for fixed $q\in\overline{R}-R_0$, whence $N(p, q)$ is continuous in wider sense ($N(p, q)$ may be infinite at q) with respect to p for fixed $q\in\overline{R}-R_0$ and $N(p, q)$ is continuous (with respect to δ-metric) in $\bar{R}-R_0$ with respect to $q\in\bar{R}-R_0$ for fixed $p\in R-R_0$.

c) $N(p,q)$ is lower semicontinuous in $\bar{R}-R_0$ for fixed $q\in\bar{R}-R_0$. But it cannot be verified that $N(p,q)$ is lower semicontinuous in $\bar{R}-R_0$ in both arguments p and q in $\bar{R}-R_0$.

d) The potential $U(z)=\int N(z,p)\,d\mu(p):\mu(p)\geqq 0$ (in the following we call $U(z)$ the potential of a distribution μ) is superharmonic in $\bar{R}-R_0$, superharmonic locally at any point of $R-R_0+B_1$ and lower semicontinuous in $\bar{R}-R_0$.

3) Maximum principle is valid. Let $U(z)$ be the potential of a positive canonical mass distribution μ. If $U(z)\leqq M$ on the kernel F of μ, $U(z)\leqq M\omega(F,z)$ in $\bar{R}-R_0$, where $\omega(F,z)$ is C.P. of F.

4) Function theoretic Equilibrium Problem can be solved: let F be a closed set of positive capacity. Then C.P. $\omega(F,z)=1$ on F except at most an F_σ set of capacity zero and $\omega(F,z)$ can be represented by a cannonical mass distribution μ whose kernel is contained in F.

Energy Integral $I(\mu)$ of a mass distribution μ on $\bar{R}-R_0$ is defined as

$$I(\mu)=\iint N(p,q)\,d\mu(p)\,d\mu(q)=\int U(p)\,d\mu(p).$$

$\overset{*}{C}$*apacity* (*potential theoretic*) of a closed set in $R-\bar{R}_0$ is defined by $\dfrac{1}{\inf\limits_{\mu\in ca} I(\mu)}$, where $\inf\limits_{\mu\in ca} I(\mu)$ is the infinimum of Energy Integrals of all positive canonical mass distribution on F of mass unity. If $F\frown(R-R_0+B_1)=0$, we define $\overset{*}{\mathrm{C}}\mathrm{ap}(F)=0$.

Problem of Equilibrium.

Theorem 16. *a) Let μ be a positive mass distribution and let μ^* be its canonical mass distribution. Then $I(\mu)=I(\mu^*)$ and $I(\mu)$ does not depend on a choice of particular distribution.*

b) Let F be a closed set such that $\overset{*}{\mathrm{C}}\mathrm{ap}(F)>0$. *Then* $\overset{*}{\mathrm{C}}\mathrm{ap}(F)<14$ $\mathrm{Cap}(F)$. *If $F\subset B_0$,* $\mathrm{Cap}(F)=0$ *and* $\overset{*}{\mathrm{C}}\mathrm{ap}(F)=0$ *by definition. Hence by Theorem* 13. *a)* $\overset{*}{\mathrm{C}}\mathrm{ap}(F)>0$ *if and only if* $\mathrm{Cap}(F)>0$.

c) Let F be a closed set of positive capacity (clearly of positive $\overset{*}{c}$apacity by b)). Let $\{\mu_n\}$ be a minimizing sequence of positive canonical mass distributions on F of mass unity such that $I(\mu_n)\downarrow\inf\limits_{\mu\in ca} I(\mu)$. Let μ be an weak limit of $\{\mu_n\}$. Then μ is also a positive canonical mass distri-

bution on F of mass unity.

d) Let $V=\inf_{\mu\in ca} I(\mu)$. *Then there exists a canonical mass distribution* μ *such that* $I(\mu)=V$.

e) Let F be a closed set in $\overline{R}-R_0$ *of positive* $\overset{*}{\text{c}}$*apacity. Let* μ *be a positive canonical mass distribution on F such that* $I(\mu)=V$. *Then the potential $U(z)$ of μ satisfies the following conditions:*

1) $U(z)\geqq V$ *on F except at most a set of capacity zero.*

2) $U(z)=V\omega(F,z)$ *and* $I(\mu)=D(\omega(F,z))=V$.

3) $\overset{*}{\mathrm{Cap}}(F)=\mathrm{Cap}(F)$.

Proof of a). Suppose p and q are points in $\overline{R}-R_0$. Then $N(z,p)=\int N(z,p_\alpha)\,d\mu_p(p_\alpha)$ and $N(z,q)=\int N(z,q_\beta)\,d\mu_q(q_\beta)$, where $\mu_p(p_\alpha)$ and $\mu_q(q_\beta)$ are canonical mass distributions of $N(z,p)$ and $N(z,q)$ respectively. Then

$$\begin{aligned}I(\mu)&=\iint N(p,q)\,d\mu(p)\,d\mu(q)=\iiint N(p_\alpha,q)\,d\mu_p(p_\alpha)\,d\mu(p)\,d\mu(q)\\&=\iiiint N(p_\alpha,q_\beta)\,d\mu_p(p_\alpha)\,d\mu_q(q_\beta)\,d\mu(p)\,d\mu(q)\\&=\iint N(p_\alpha,q_\beta)\int d\mu_p(p_\alpha)\,d\mu(p)\int d\mu_q(q_\beta)\,d\mu(q)\\&=\iint N(p_\alpha,q_\beta)\,d\mu(p_\alpha)\,d\mu(q_\beta)=I(\mu^*).\end{aligned}$$

For other distributions we have the same value, hence $I(\mu)$ does not depend on particular distributions. Thus we have *a*).

Proof of b). Let V be the infinimum of all positive canonical mass distributions on F of positive capacity of mass unity. Let $\{\mu_n\}$ be a minimizing sequence of canonical distributions of mass unity on F such that $I(\mu_n)=V+\varepsilon_n:\varepsilon_n\downarrow 0$. Put $\varepsilon_0=\frac{V}{10}$. Let n_0 be a number such that $\varepsilon_n\leqq\frac{\varepsilon_0}{10}$ for $n\geqq n_0$. Let $\mathfrak{M}'_n$ be the mass of the restriction μ'_n of μ_n on the set $E[z\in F:\ U_n(z)\leqq V+\varepsilon_0]$, where $U_n(z)$ is the potential of μ_0. Then since $I(\mu_n)=V+\varepsilon_n$,

$$\mathfrak{M}'_n\geqq 1-\frac{V+\varepsilon_n}{V+\frac{\varepsilon_0}{2}}>\frac{1}{13},$$

because if the set $E[z\in F:\ U_n(z)>V+\varepsilon_0]$ has mass $>\dfrac{V+\varepsilon_n}{V+\frac{\varepsilon_0}{2}}$,

$$V+\varepsilon_n=I(\mu_n)\geqq\int U_n(z)\,d\mu_n'(p)>(V+\varepsilon_0)\times\left(\frac{V+\varepsilon_n}{V+\frac{\varepsilon_0}{2}}\right)>(V+\varepsilon_n).$$

Let $U_n'(z)$ be the potential of μ_n'. Then $U_n'(z)\leqq V+\varepsilon_0$ on the kernel $F'(\subset F)$ of μ_n'. Hence by the maximum principle $U_n'(z)\leqq(V+\varepsilon_0)\omega(F',z)\leqq(V+\varepsilon_0)$ $\omega(F,z)$, whence

$$V+\varepsilon_0\geqq\frac{\int_{\partial R_0}\frac{\partial}{\partial n}U_n'(z)\,ds}{\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F,z)\,ds}\geqq\frac{\frac{1}{13}}{\mathrm{Cap}(F)}.\quad \text{Hence by } \frac{V}{10}=\varepsilon_0$$

$$V\geqq\frac{10}{143\,\mathrm{Cap}(F)}\quad\text{and}\quad \overset{*}{\mathrm{Cap}}(F)=\frac{1}{V}<14\,\mathrm{Cap}(F).$$

Conversely if $\mathrm{Cap}(F)>0$, then $\omega(F,z)={}_F\omega(F,z)=\int_{F\frown(R-R_0+B_1)}N(z,p)\,d\mu^*(p)$ by Theorem 13. *c*). Now $\omega(F,z)\leqq1$ on F and the total mass of μ^* is given by $\mathfrak{M}=\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F,z)\,ds$. Hence $\frac{\mu^*}{\mathfrak{M}}$ is a canonical distribution on F of mass unity and its Energy Integral $\leqq\frac{1}{\mathfrak{M}}$, whence $\inf_{\mu\in ca}I(\mu)=V\leqq\frac{1}{\mathfrak{M}}$ and $\overset{*}{\mathrm{Cap}}(F)$ $\geqq\mathfrak{M}=\mathrm{Cap}(F)$. If $F\frown(R-R_0+B_1)=0$, i.e. $F\subset B_0$, $\overset{*}{\mathrm{Cap}}(F)=0$ by definition and $\mathrm{Cap}(F)=0$ by the fact that B_0 is an F_σ set of capacity zero. Thus $\overset{*}{\mathrm{Cap}}(F)>0$ if and only if $\mathrm{Cap}(F)>0$.

Proof of c). $R_0=\bigcup_{m=1}^{\infty}\Gamma_m$ (see the proof of Theorem 7. *a*)), where Γ_m is closed and of capacity zero. Let $\Gamma_{m,i}=E\left[z\in\overline{R}:\ \delta(\Gamma_m,z)\leqq\frac{1}{i}\right]$. Then $\Gamma_{m,i}$ is closed and $\mathrm{Cap}(\Gamma_{m,i})\to0$ as $i\to\infty$ for every m. Hence for any given number l there exists a number $i(m,l)$ such that $\mathrm{Cap}(\Gamma_{m,i})<\frac{1}{14\times2^{m+l}V}$, where $\frac{1}{V}=\overset{*}{\mathrm{Cap}}(F)$. Let $\{\mu_n\}$ be a minimizing sequence such that $I(\mu_n)=V+\varepsilon_n:\lim_n\varepsilon_n=0$ and $\varepsilon_n<\frac{V}{20}$. Let $\mathfrak{M}_n$ be the mass of the restriction μ_n' of μ on $\Gamma_{m,i}$. Then by $\overset{*}{\mathrm{Cap}}(\Gamma_{m,i})<14\,\mathrm{Cap}(\Gamma_{m,i})$ $\mathfrak{M}_n<\frac{\sqrt{2}}{2^{\frac{m+l}{2}}}$, because if $\mathfrak{M}_n\geqq\frac{\sqrt{2}}{2^{\frac{m+l}{2}}}$, $\frac{21}{20}V\geqq V+\varepsilon_n=I(\mu_n)\geqq I(\mu_n')\geqq\frac{\mathfrak{M}_n^2}{\mathrm{Cap}(\Gamma_{m,i})}>2V$. Put $O_{m,2i}=E\left[z\in\overline{R}:\ \delta(\Gamma_{m,i}\,z)<\frac{2}{2i}\right]$. Then $O_{m,2i}$ is open and $E\Big[z\in\overline{R}:\ \delta(\Gamma_m,z)$

$\leq\frac{1}{3i}\Big]=\Gamma_{m,3i}\subset O_{m,2i}\subset\Gamma_{m,i}$. Hence the mass of μ_n on $O_{m,2i}$ is smaller than $\frac{\sqrt{2}}{2^{\frac{m+l}{2}}}$. Let μ be an weak limit of $\{\mu_n\}$. Then it is known that the mass of μ on any open set G (closed set F) is smaller (larger) than $\underline{\lim}_n$ (mass of Γ_n on G) ($\overline{\lim}_n$ (mass of μ_n on F)). Hence the mass of μ on $\bigcup_{m=1}^{\infty}O_{m,2i}$ is smaller than $\sum_{m=1}^{\infty}\frac{\sqrt{2}}{2^{\frac{m+l}{2}}}=\frac{2+\sqrt{2}}{2^{\frac{l}{2}}}$. Put $A_l=\bigcup_{m=1}^{\infty}\Gamma_{m,3i}\Big(\subset\Big(\bigcup_{m=1}^{\infty}O_{m,2i}\Big)\Big)$. Then the mass of μ on $A_l\leq\frac{2+\sqrt{2}}{2^{\frac{l}{2}}}$. Now $B_0\subset\Big(\bigcap_{l=1}^{\infty}A_l\Big)$. Let $l\to\infty$. Then μ has no mass on B_0 i.e. μ is *a canonical distribution*. Clearly by the closedness of F μ has mass unity on μ.

Proof of d). Since it cannot be proved that $N(p,q)$ is lower semicontinuous in $\overline{R}-R_0$ in both arguments p and q in $\overline{R}-R_0$, it is not so clear that $I(\mu)\leq\underline{\lim}_n I(\mu_n):\mu=\lim_n\mu_n$. Let V be the infinimum of all canonical mass distributions on a closed set F of positive $\overset{*}{\text{c}}$apacity (Cap $(F)>0$ by *b*)) of mass unity. Let $V>\alpha>0$ and let μ be a canonical positive mass distribution of mass unity on F. Let μ' be the restriction of μ on the closed set $E[z\in F: U(z)\leq V-\alpha]:U(z)$ is the potential of μ and let $1-\mathfrak{M}$ be the mass of $\mu':\mathfrak{M}\geq 0$. Then $\frac{\mu'}{1-\mathfrak{M}}$ is a canonical distribution on F of mass unity and its potential $\hat{U}(z)=\int N(z,p)\Big(\frac{1}{1-\mathfrak{M}}\Big)d\mu'(p)\leq\frac{U(z)}{1-\mathfrak{M}}$ on the kernel of μ'. Assume $\mathfrak{M}<\frac{\alpha}{V}$. Then

$$I\Big(\frac{\mu'}{1-\mathfrak{M}}\Big)=\int U'(z)\Big(\frac{1}{1-\mathfrak{M}}\Big)d\mu'(p)\leq\int\frac{U(z)}{1-\mathfrak{M}}d\mu'(p)\leq\frac{V-\alpha}{1-\mathfrak{M}}<V.$$

This contradicts the definition of V. Hence the mass $\mathfrak{M}$ of any canonical distribution of mass unity on $E[z\in F: U(z)>V-\alpha]$ satisfies

$$\mathfrak{M}\geq\frac{\alpha}{V}. \tag{71}$$

Let $\{\mu_n\}$ be a minimizing sequence of canonical mass distributions on F of mass unity such that $I(\mu_n)=V+\varepsilon_n:\varepsilon_n\downarrow 0$. Then for any given positive number $\alpha<\min(1,V)$ there exists a number n such that $\varepsilon_n<\min\Big(1,V\frac{\alpha^4}{V^2A(\alpha)}\Big):A(\alpha)=1+2V+\frac{\alpha}{V}$. Suppose $\varepsilon_n<\frac{\alpha^4}{V^2A(\alpha)}$. Then the potential $U_n(z)$ of μ_n satisfies the following conditions:

1) *The capacity of the closed set* $F_n^{2\alpha}=E[z\in F: U(z)\leqq V-2\alpha]<\frac{\alpha}{V}$ (72)

2) *The mass of* μ_n *on* $F_n^{2\alpha}<\sqrt{28\alpha}$. (73)

We shall prove 1) and 2). Let μ'_n be the restriction of μ_n on $E[z\in F: U_n(z) > V-\alpha]: U_n(z)=\int N(z,p)\,d\mu_n(p)$. Then the mas $\mathfrak{M}_n$ of $\mu'_n>\frac{\alpha}{V}>0$ by (71). Put $\delta_n=\frac{\mathfrak{M}_n}{\frac{\alpha}{V}}$. Then $\delta_n>1$ and the mass of $\frac{\mu'_n}{\delta_n}=\frac{\alpha}{V}$. Let μ^* be the canonical mass distribution of $\frac{\alpha}{VC_n^{2\alpha}}\omega(F_n^{2\alpha},z)$: $C_n^{2\alpha}$ is the capacity of $F_n^{2\alpha}$. Then the kernel of μ^* is contained in F and the mass of $\mu^*=\frac{\alpha}{V}$. Let σ be a distribution on F such that $\sigma=\mu^*$ on $F_n^{2\alpha}$, $\sigma=0$ on $E[z\in F: V-\alpha>U_n(z)\geqq V-2]$ and $\sigma=-\frac{\mu'_n}{\delta_n}$ on $E[z\in F: U_n(z)>V-\alpha]$. Put $U_\alpha(z)=\int N(z,p)\,d\sigma$. Then $|U_\sigma(z)|\leqq\int N(z,p)(d\mu_n(p)+d\mu^*(p))=U_n(z)+\frac{1}{VC_n^{2\alpha}}\omega(F_n^{2\alpha},z)$. Henec $I(\sigma)$ $\leqq\int|U_\sigma(z)||d\sigma|\leqq\left(U_n(z)+\frac{d}{VC_n^{2\alpha}}\right)(d\mu_n(p)+d\mu^*(p))\leqq I(\mu_n)+\frac{\alpha}{VC_n^{2\alpha}}+\frac{\alpha}{V}(V-2\alpha)$ $+\frac{1}{C_n^{2\alpha}}\left(\frac{\alpha}{V}\right)^2\leqq V+\varepsilon_n+\frac{1}{C_n^{2\alpha}}\left(\frac{\alpha}{V}+\frac{\alpha^2}{V^2}\right)+\alpha<2V+\frac{1}{C_n^{2\alpha}}\left(\frac{\alpha}{V}+\frac{\alpha^2}{V^2}\right)$, because $\omega(F,z)\leqq 1$ on $\overline{R}-R_0$ and $U_n(z)\leqq V-2\alpha$ on the kernel of μ^*.

Assum $C_n^{2\alpha}>\frac{\alpha}{V}$. Then $I(\sigma)\leqq 2V+1+\frac{\alpha}{V}\leqq A(\alpha)$. Now $\mu_n+h\sigma$ is a positive canonical distribution on F of mass unity for $0\leqq h<\delta_n$ $(\delta_n>1)$. Hence

$$I(\mu_n+h\sigma)=V+\eta_n:\eta_n\geqq 0 \text{ and } \eta_n-\varepsilon_n=I(\mu_n+h\sigma)-I(\mu_n)$$
$$=2h\left(\frac{\alpha}{V}\right)((V-2\alpha)-(V-\alpha))+h^2I(\sigma)=2h\left(-\frac{\alpha^2}{V}+hI(\sigma)\right), \text{ whence}$$
$$\eta_n=h\left(hI(\sigma)-\frac{\alpha^2}{V}\right)+\varepsilon_n-\frac{h\alpha^2}{V}.$$

Put $h=\frac{\alpha^2}{VA(\alpha)}$. Then $h<\frac{\alpha^2}{V}<1$ and $\mu_n+h\sigma$ is a positive canonical distribution on F of mass unity. Now by $I(\sigma)\leqq A(\alpha)$ and $\varepsilon_n<\frac{\alpha^4}{V^2A(\alpha)}$ we have $\eta_n<0$. This contradicts that $\eta_n\geqq 0$. Hence $C_n^{2\alpha}\leqq\frac{\alpha}{V}$.

Next by c) $\overset{*}{\text{Cap}}(F_n^{2\alpha})<14\,C_n^{2\alpha}\leqq\frac{14}{V}$. Let μ'_n be the restriction of μ_n on $F_n^{2\alpha}$. Assume mass $\mathfrak{M}_n$ of $\mu'_n>\sqrt{28\alpha}$. Then by $C_n^{2\alpha}\leqq\frac{\alpha}{V}$ we have

$$I(\mu_n) \geqq I(\mu_n') \geqq \frac{\mathfrak{M}_n^2}{\mathrm{Cap}\,(F_n^{2\alpha})} > \frac{\mathfrak{M}_n^2}{14\,C_n^{2\alpha}} > 2V.$$

This contradicts that $I(\mu_n)=V+\varepsilon_n<2V$. Hence the mass of $\mu_n'<\sqrt{28\alpha}$. Thus we have *1*) and *2*).

Let $\alpha_1>\alpha_2>\cdots$ be a sequence such that $2^n\sqrt{\alpha_n}\downarrow 0$ as $n\to\infty$. Let $m(n)$ be the least integer satisfying $\varepsilon_m<\frac{\alpha_\alpha^4}{VA(\alpha_n)}$. We make $\mu_{m(n)}$ correspond to α_n and denote it by μ_n newly. Then we have subsequence $\{\mu_n\}$ of former $\{\mu_n\}$ such that

$$I(\mu_n)=V+\varepsilon_n \text{ and } \varepsilon_n<\frac{\alpha_n^4}{V^2A(\alpha_n)}:\ 2^n\sqrt{\alpha_n}\downarrow 0 \text{ as } n\to\infty.$$

Let $\mathfrak{M}_n'$ and $\mathfrak{M}_n''$ be the masses of μ_n on the set $E[z\in F: U_n(z)>V+2^n\sqrt{\alpha_n}]$ and on $E[z\in F: U_n(z)\leqq V-2\alpha_n]$ respectively, where $U_n(z)$ is the potential of μ_n. Then $\mathfrak{M}_n''<\sqrt{28\alpha_n}$. Consider $I(\mu_n)$. Then

$$V+\varepsilon_n=I(\mu_n)\geqq\int U_n(z)\,d\mu_n(p)\geqq(V+2^n\sqrt{\alpha_n})\mathfrak{M}_n''+(V-2\alpha)(1-\mathfrak{M}_n'-\mathfrak{M}_n'').$$

Whence $\varepsilon_n>\mathfrak{M}_n''(2^n\alpha_n+2\alpha_n)-2\alpha_n+(2\alpha_n-V)\sqrt{28\alpha_n}$. Hence

$$\mathfrak{M}_n'<\frac{\varepsilon_n+2\alpha_n-(2\alpha_n-V)\sqrt{28\alpha_n}}{2^n\sqrt{\alpha_n}+2\alpha_n}<\frac{6V}{2^n}.$$

Hence the mass of μ_n on $E[z\in F:\ V+2^n\sqrt{\alpha_n}>U_n(z)]>1-\frac{6V}{2^n}$. Let μ_n' be the restriction of μ_n on $E[z\in F:\ U_n(z)<V+2^n\sqrt{\alpha_n}]$ and put $\mu_n^*=\frac{\mu_n'}{1-\mathfrak{M}_n'}$. Then μ_n^* is also a canonical distribution on F of mass unity and $I(\mu_n^*)\leqq\frac{1}{1-\mathfrak{M}_n'}\int U_n(z)d\mu_n'(p)\leqq\frac{V+2^n\sqrt{\alpha_n}}{1-\frac{6V}{2^n}}=V+\zeta_n:\zeta_n\downarrow 0$ an $n\to\infty$. Hence $\{\mu_n^*\}$ is also a minimizing sequence of canonicnl mass distributions on F of mass unity. On the other hand, $U_n^*(z)=\int N(z,p)\,d\mu^*(p)\leqq V+\zeta_n$ on the kernel of μ_n^*. Hence by the maximum principle $U_n^*(z)\leqq V+\zeta_n$ in $\overline{R}-R_0$. Since the total mass of $\{\mu_n^*\}$ is unity and $N(z,p)$ is continuous in $\overline{R}-R_0$ with respect to p for $z\in R-R_0$. Hence there exists a subsequence $\{\mu_{n'}^*\}$ of $\{\mu_n^*\}$ and an weak limit μ^* of $\{\mu_{n'}^*\}$ such that $V=\lim_{n'=\infty}(V+\zeta_{n'})\geqq\overline{\lim_{n'=\infty}}\,U_{n'}^*(z)=U^*(z)=\int N(z,p)\,d\mu^*(p):z\in R-R_0$. Further by the semicontinuity of $U^*(z)$ $U^*(z)\leqq V$ in $\overline{R}-R_0$, whence $I(\mu^*)\leqq V$. On the other hand, since μ^* is also canonical by (*c*), $I(\mu^*)\geqq V$. Thus μ^* is the required canonical mass

distribution.

Proof of e). Suppose μ is a canonical distribution on F such that $I(\mu)=V$. Put $I(\mu)=V+\varepsilon$. Then $\varepsilon=0<\frac{\alpha^4}{VA(\alpha)}$ for any given positive number α. Hence by (72) *the potential $U(z)$ of $\mu\geqq V$ on F except at most a set of capacity zero.* Assume $U(z)>V$ at least one point p of the kernel of μ. Then by the lower semicontinuity $U(z)>V+\varepsilon:\varepsilon>0$ in a neighbourhood $\upsilon(p)$ of p and the mass $\mathfrak{M}$ of μ in $\upsilon(p)>0$. Whence $I(\mu)\geqq\mathfrak{M}(V+\varepsilon)+(1-\mathfrak{M})V>V$. This is a contradiction. Hence $U(z)=V$ on the kernel $F'(\subset F)$ of μ. Whence by the maximum principle $U(z)\leqq V$ in $\bar{R}-R_0$. Now $U(z)$ is harmonic in $R-R_0-F$, since $\mu=0$ on CF. Hence

$$U(z)\leqq V\omega(F',z)\leqq V\omega(F,z).$$

Inverse inequality is proved as follows: put $CG_{V-\varepsilon}=E[z\in\bar{R}: U(z)\leqq V-\varepsilon]$. Then $CG_{V-\varepsilon}\frown F$ is closed and of capacity zero ($\overset{*}{\text{c}}$apacity zero), whence we can construct a superharmonic function $\omega^{**}(z)$ such that $\omega^{**}(z)$ is continuous in $R-R_0$ and $\omega^{**}(z)\to\infty$ as $z\to p\in((CG_{V-\varepsilon}\frown F)+B_0)$. Put $U^*(z)=\alpha\omega^{**}(z)+U(z):\alpha>0$. Then $U^*(z)\geqq V$ on F. Put $CG^*_{V-\varepsilon}=E[z\in\bar{R}: U^*(z)\leqq V-\varepsilon]$. Then by the lower semicontinuity of $U^*(z)$ dist $(CG_{V-\varepsilon}, F)>\delta_0>0$. Let $F_m=E\left[z\in\bar{R}: \delta(z,F)\leqq\frac{1}{m}\right]:\frac{1}{m}<\delta_0$. Let $G^*_{V-\varepsilon}=E[z\in\bar{R}: U^*(z)>V-\varepsilon]$. Then $G^*_{V-\varepsilon}\supset F_m$. Now $U^*(z)$ is superharmonic in $\bar{R}-R_0$, hence

$$U^*(z)\geqq(V-\varepsilon)\omega(G^*_{V-\varepsilon},z)\geqq(V-\varepsilon)\omega(F_m,z)\geqq(V-\varepsilon)\omega(F,z),$$

Let $\alpha\to0$ and then $\varepsilon\to0$. Then $U(z)\geqq\omega(F,z)$. Thus $U(z)=\omega(F,z)$.

Next by $\int_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds=V\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F,z)\,ds$ we have at once

$$D(V\omega(F,z))=V^2\frac{1}{\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F,z)\,ds}=V \quad\text{and}$$

$$\text{Cap}\,(F)=\int\frac{\partial}{\partial n}\omega(F,z)\,ds=\frac{1}{V}=\overset{*}{\text{C}}\text{ap}\,(F).$$

Department of Mathematics
Hokkaido University

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