ON THE GIBBS PHENOMENON

By

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1. Introduction.

The Gibbs phenomenon of the Fourier series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ was early found by W. Gibbs [1], see A. Zygmund [22]. And further H. Cramér [2], T. H. Grownall [3], B. Kuttner [4] [5] [6], O. Szász [9] [10] [11], M. Cheng [12], L. Lorch [13] L. Ching-Hsi [18], A. E. Livingston [19] and the author [20] investigated the Gibbs phenomenon of the same Fourier series for various kinds of means: Cesàro, Riesz, Euler, etc..

The Gibbs phenomenon of the Fourier series of a function which has a discontinuity point of the second kind was recently investigated by S. Izumi and M. Satô [14] [15] and B. Kuttner [7] [8]. The author [16] [17] proved some theorems concerning the Gibbs phenomenon of the Fourier series of this kind for Cesàro means. The object of the present paper is to study the Gibbs phenomenon of such Fourier series for Riesz, Borel, Euler and Hausdorff means.

2. B. Kuttner [6] proved the following

Theorem 1. *If $0<\lambda<2$, there is a function $r(\lambda)$ such that the Gibbs phenomenon vanishes for the means (R, n^{λ}, κ) of the Fourier series of a function having a simple discontinuity if $\kappa \geqq r(\lambda)$, but not if $\kappa<r(\lambda)$. The function $r(\lambda)$ is continuous and (strictly) increasing, and is, for all $\lambda<2$, less than the function $k(\lambda)$ defined in [5]. It tends to 0 as $\lambda \to 0$, equals Cramér's constant r_0 when $r=1$, see [2], and tends to infinity as $\lambda \to 2$. If $\lambda=2$, the Gibbs phenomenon persists for the means (R, n^{λ}, κ) however large κ may be.*

We shall extend this theorem to the discontinuity point of the second kind satisfying the following conditions, see [14] [15]. That is

Theorem 2. *Let $f(x)$ be an odd function about ξ, and suppose that*

$$f(x)=l\psi(x-\xi)+g(x),$$

where $\psi(x)$ is a periodic function with period 2π such that

$$\psi(x)=(\pi-x)/2 \qquad (0<x<2\pi),$$

and where

$$\limsup_{x\downarrow\xi} f(x)=l\pi/2, \qquad \liminf_{x\uparrow\xi} f(x)=-l\pi/2,$$

$$\liminf_{x \downarrow \xi} f(x) \geqq -l\pi/2, \quad \limsup_{x \uparrow \xi} f(x) \leqq l\pi/2.$$

Let the Fourier series of $f(x)$ be

$$f(x) \sim \sum_{n=1}^{\infty} \alpha_n \sin n(x-\xi)$$

where

$$\alpha_n = \frac{l}{n} + a(n), \quad \sum |\Delta\alpha_n| < \infty$$

$$a(t) = \frac{2}{\pi} \int_0^{\pi} g(u) \sin ut \, du$$

$$(1) \qquad \frac{1}{t} \int_0^t t\, a(t)\, dt = o(1)$$

$$(2) \qquad m \cdot \max_{0 \leqq t \leqq m} \sum_{j=1}^{\infty} |a(t+2jm) - a(t+(2j-1)m)| \to 0 \quad \text{as} \quad m \to \infty.$$

Then the same results as Theorem 1 *holds for the means* (R, n^{λ}, κ) *of* $\sum \alpha_n \sin n(x-\xi)$.

Theorem 2 is a extension of our previous result ([16] Theorem 4). In order to prove Theorem 2 we use the methods of H. Cramér and B. Kuttner ([2], [6])

L. Lorch [13] proved the following

Theorem 3. *Let $B_x(t)$ denote the x-th Borel exponential or integral means of the Fourier series*

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

Then, for given τ, $0 \leqq \tau \leqq \infty$,

$$\lim_{x \to \infty} B_x(t_x) = \int_0^{\tau} \frac{\sin v}{v} dv$$

where

$$t_x \to +0 \quad \text{and} \quad x t_x \to \tau.$$

Thus, the Borel means display the same Gibbs phenomenon and have the same Gibbs ratio as classic convergence.

On the other hand, S. Izumi and M. Sato [14] have the followings

Theorem 4. *There exists a function $f(t)$ which satifies*

$$(3) \qquad \int_0^h f(u)\, du = o(|h|), \quad \text{and}$$

$$(4) \qquad \int_0^h \{f(t+u) - f(t-u)\}\, du = o(|h|), \quad \text{uniformly in } t,$$

and further presents the Gibbs phenomenon at $t=0$ *for classic convergence.*

Here we shall prove

Theorem 5. *Under the same condition as Theorem 4 Borel means* $B_x(t)$ *of Fourier series of* $f(t)$ *do not present the Gibbs phenomenon at* $t=0$.

Further concerning the Gibbs phenomenon for Euler means of the Fourier series O. Szász [9] proved the following.

Theorem 6. *For the Euler means of the series* $\sum_1^n \frac{\sin nt}{n}$ *we have*

$$\lim \sigma_n(t_n)=\int_0^{r\tau} \frac{\sin v}{v}\, dv$$

as $nt_n \to \tau$, $nt_n^2 \to 0$.

The general Euler means of a sequence $\{s_n\}$ depend on a parameter r, and are defined as follows, see G. H. Hardy [21],

$$\sigma_{n,r}=\sigma_n=\sum_{\nu=0}^{n} \binom{n}{\nu} r^\nu (1-r)^{n-\nu} s_\nu \qquad (n=0, 1, 2, \cdots).$$

We assume $0<r\leqq 1$, in which case the summation method is regular. For $r=1$ the definition reduces to that of ordinary convergence.

We shall prove here the following

Theorem 7. *If*

$$\int_0^h f(u)\, du=o(|h|), \quad \textit{and} \tag{3}$$

$$\int_0^h \{f(t+u)-f(t-u\}\, du=o(|h|), \quad \textit{uniformly in } t, \tag{4}$$

then Euler means $\sigma_n(t)$ *of the Fourier series of* $f(t)$ *do not present the Gibbs phenomenon at* $t=0$ *for any order* r, $0<r<1$.

O. Szász [10] later proved the theorem for Hausdorff means which is a generalization of Theorem 6. That is

Theorem 8. *For the Hausdorff means of the series* $\sum_{n=1}^{\infty} \frac{\sin nt}{n}$ *we have*

$$\lim_{n\to\infty} h_n(t_n)=\int_0^1 \int_0^\tau \frac{\sin ry}{y}\, dy\, d\psi(r),$$

as $nt_n \to \tau \leqq \infty$. *And*

$$\limsup_{\substack{n\to\infty\\ t\to 0}} h_n(t)=\max_{\tau>0} \int_0^1 \{1-\psi(r)\} \frac{\sin \tau r}{r}\, dr.$$

Here we define the Hausdorff means of the sequence $\{s_n\}$ by

$$h_n=\sum_0^n \binom{n}{\nu} s_\nu \int_0^1 r^\nu(1-r)^{n-\nu}d\psi(r) \qquad (n=0, 1, 2, \cdots)$$

where $\psi(r)$ is of bounded variation in $0\leqq r\leqq 1$, see G. H. Hardy [21]. This transformation is regular if and only if

$$\int_0^1 d\psi(r)=\psi(1)-\psi(0)=1\,,$$

and if $\psi(r)$ is continuous at $r=0$. We may assume that $\psi(0)=0$; then the above conditions become

$$\psi(1)=1 \quad \text{and} \quad \psi(+0)=\psi(0)=0\,.$$

We shall prove here the following.

Theorem 9. *If $f(t)$ is bounded and*

$$(3) \qquad \int_0^h f(u)\,du=o(|h|)\,,$$

$$(4) \qquad \int_0^h \{f(t+u)-f(t-u)\}\,du=o(|h|) \text{ uniformly in } t, \text{ and further}$$

if $\psi(r)$ is continuous at $r=1$, then Hausdorff means $h_n(t)$ of the Fourier series of $f(t)$ do not present the Gibbs phenomenon at $t=0$.

3. Proof of Theorem 2.

For the proof, we need a lemma which is an extention of the Kuttner's.

Lemma. *Let*

$$\Psi(u, x)=\sum_{n=1}^u \left(1-\left(\frac{n}{u}\right)^\lambda\right)^\kappa a(n)\sin nx$$

$$\Phi(u, x)=\int_0^u \left(1-\left(\frac{t}{u}\right)^\lambda\right)^\kappa a(t)\sin tx\,dt$$

Then $\Psi(u, x)-\Phi(u, x)\to 0$, as $x\to 0$, $u\to\infty$ independently.

Proof. Let us now use Euler's summation formula which reads as follows.

$$(5) \qquad \sum_{n=1}^u h(n)=\int_0^u h(t)\,dt+\int_0^u\left(t-[t]-\frac{1}{2}\right)h'(t)\,dt-\frac{1}{2}h(0)+\frac{1}{2}h(u)\,,$$

where $h(t)$ is continuously differentiable. In (5) we put

$$h(t)=\left(1-\left(\frac{t}{u}\right)^\lambda\right)^\kappa a(t)\sin tx$$

then $h(t)$ is continuously differentiable and $h(0)=h(u)=0$. Furthermore

$$h'(t)=-\kappa\lambda\frac{t^{\lambda-1}}{u^\lambda}\left(1-\left(\frac{t}{u}\right)^\lambda\right)^{\kappa-1}a(t)\sin tx+\left(1-\left(\frac{t}{u}\right)^\lambda\right)^\kappa a'(t)\sin tx$$
$$+\left(1-\left(\frac{t}{u}\right)^\lambda\right)^\kappa a(t)x\cos tx$$

Thus we get

$$\Psi(u,x)-\Phi(u,x)=\int_0^u P(t)h'(t)\,dt$$
$$=-\kappa\lambda\frac{1}{u^\lambda}\int_0^u P(t)t^{\lambda-1}\left(1-\left(\frac{t}{u}\right)^\lambda\right)^{\kappa-1}a(t)\sin tx\,dt$$
$$+\int_0^u P(t)\left(1-\left(\frac{t}{u}\right)^\lambda\right)^\kappa a'(t)\sin tx\,dt+x\int_0^u P(t)\left(1-\left(\frac{t}{u}\right)^\lambda\right)^\kappa a(t)\cos tx\,dt$$
$$=I_1+I_2+I_3,$$

say, where

$$P(t)=t-[t]-\frac{1}{2}.$$

We have first

$$I_1=-\kappa\lambda\frac{1}{u^\lambda}\int_0^u P(t)\,t^{\lambda-1}\left(1-\left(\frac{t}{u}\right)^\lambda\right)^{\kappa-1}a(t)\sin tx\,dt$$

and hence

$$|I_1|\leqq\frac{A}{u^\lambda u^{\lambda(\kappa-1)}}\int_0^u t^{\lambda-1}(u^\lambda-t^\lambda)^{\kappa-1}|a(t)|\,dt$$
$$=\frac{A}{u^{\lambda\kappa}}\int_0^u\frac{t^{\lambda-1}|a(t)|}{(u^\lambda-t^\lambda)^{1-\kappa}}dt=\frac{A}{u^{\lambda\kappa}}\left\{\int_0^v+\int_v^u\right\}.$$

We take v such that

$$|a(t)|<\varepsilon \quad \text{for } t>v,$$

which is possible by $a(t)=o(1)$ (as $t\to\infty$). Thus we get

$$|I_1|\leqq\frac{A}{u^{\lambda\kappa}}\int_0^v\frac{t^{\lambda-1}}{(u^\lambda-t^\lambda)^{1-\kappa}}dt+\frac{A\varepsilon}{u^{\lambda\kappa}}\int_v^u\frac{t^{\lambda-1}}{(u^\lambda-t^\lambda)^{1-\kappa}}dt$$
$$\leqq A\{u^{\lambda\kappa}-(u^\lambda-v^\lambda)^\kappa\}/u^{\lambda\kappa}+A\varepsilon(u^\lambda-v^\lambda)^\kappa/u^{\lambda\kappa}$$
$$\leqq A\varepsilon,$$

for sufficiently large u. Secondly, we have

$$I_2=\int_0^u P(t)\left(1-\left(\frac{t}{u}\right)^\lambda\right)^\kappa a'(t)\sin tx\,dt$$

By the second mean value theorem

$$I_2=\int_0^{\theta_u} P(t)a'(t)\sin tx\,dt \qquad (0<\theta_u<u)$$

and then by integration by parts,

$$I_2=\left[a'(t)\int_0^t P(v)\sin vx\,dv\right]_0^{\theta_u}-\int_0^{\theta_u} a''(t)\,dt\int_0^t P(v)\sin vx\,dv=I_{21}-I_{22}\,,$$

say.

Since
$$P(v)=\sum_{\nu=1}^{\infty}(\sin 2\pi\nu v)/\pi\nu\,,$$
we have

$$\int_0^t P(v)\sin vx\,dv=\int_0^t \sin vx\left(\sum_{\nu=1}^{\infty}\frac{\sin 2\pi\nu v}{\pi\nu}\right)dv$$
$$=\sum_{\nu=1}^{\infty}\frac{1}{\pi\nu}\int_0^t \sin vx\sin 2\pi\nu v\,dv\,,$$

where the change of order of summation and integration is legitimate, since the series $\sum \sin\nu v/\nu$ is boundedly convergent. The last sum is

$$\sum_{\nu=1}^{\infty}\frac{1}{2\pi\nu}\int_0^t\{\cos(x-2\pi\nu)v-\cos(x+2\pi\nu)v\}\,dv$$
$$=\sum_{\nu=1}^{\infty}\frac{1}{2\pi\nu}\left(\frac{\sin(x-2\pi\nu)t}{x-2\pi\nu}-\frac{\sin(x+2\pi\nu)t}{x+2\pi\nu}\right)$$
$$=\sum_{\nu=1}^{\infty}\frac{1}{2\pi\nu}\,\frac{(x+2\pi\nu)\sin(x-2\pi\nu)t-(x-2\pi\nu)\sin(x+2\pi\nu)t}{(x-2\pi\nu)(x+2\pi\nu)}$$
$$=\sum_{\nu=1}^{\infty}\left(\frac{x}{2\pi\nu}\,\frac{\sin(x-2\pi\nu)t-\sin(x+2\pi\nu)t}{(x-2\pi\nu)(x+2\pi\nu)}\right.$$
$$\left.+\frac{\sin(x-2\pi\nu)t+\sin(x+2\pi\nu)t}{(x-2\pi\nu)(x+2\pi\nu)}\right).$$

Accordingly we get

$$I_{22}=\int_0^{\theta_u} a''(t)\,dt\int_0^t P(v)\sin vx\,dv$$
$$=x\sum_{\nu=1}^{\infty}\int_0^{\theta_u} a''(t)\,\frac{\sin(x-2\pi\nu)t-\sin(x+2\pi\nu)t}{2\pi\nu(x-2\pi\nu)(x+2\pi\nu)}\,dt$$
$$+\sum_{\nu=1}^{\infty}\int_0^{\theta_u} a''(t)\,\frac{\sin(x-2\pi\nu)t+\sin(x+2\pi\nu)t}{(x-2\pi\nu)(x+2\pi\nu)}\,dt=J_1+J_2\,,$$

say.

Since

$$\int_0^\infty |a''(t)|\,dt<\infty\,, \qquad \int_0^{\theta_u} a''(t)\sin ut\,dt$$

is bounded, and hence

$$|J_1|\leqq Ax\sum\nu^{-3}$$

which is less than ε for small x. Concerning J_2 we write

$$J_2=\sum_{\nu=1}^{\infty}=\sum_{\nu=1}^{N}+\sum_{\nu=N+1}^{\infty}=J_{21}+J_{22}\,,$$

where N is taken such that $\sum_{\nu=N+1}^{\infty}\nu^{-2}<\varepsilon$. Then

$$|J_{22}|<A\varepsilon\,.$$

Since $a''(t)$ is absolutely integrable,

$$\left|\int_0^{\theta_u} a''(t)\{\sin(2\pi\nu+x)t-\sin(2\pi\nu-x)t\}\,dt\right|$$

$$\leqq 2\left|\int_0^M a''(t)\sin xt\cos 2\pi\nu t\,dt\right|+2\int_M^{\theta_u}|a''(t)|\,dt\,.$$

If M is taken such that $\int_M^\infty |a''(t)|\,dt<\varepsilon$, then, for such fixed M

$$\left|\int_0^M a''(t)\sin xt\cos 2\pi\nu t\,dt\right|\leqq xM\int_0^M |a''(t)|\,dt\leqq Ax\,,$$

which is less than ε for sufficiently small x.
Thus we have proved that

$$|I_{22}|=\left|\int_0^{\theta_u} a''(t)\,dt\int_0^t P(v)\sin vx\,dv\right|<A\varepsilon$$

Similary we get

$$|I_{21}|<A\varepsilon\,,$$

for sufficiently small x. Thus we get

$$|I_2|<A\varepsilon$$

for sufficiently small x. Finally

$$I_3=x\int_0^u P(t)\left(1-\left(\frac{t}{u}\right)^\lambda\right)^\kappa a(t)\cos tx\,dt$$

$$=\frac{x}{u^{\lambda\kappa}}\int_0^u P(t)(u^\lambda-t^\lambda)^\kappa a(t)\cos tx\,dt$$

and then by the second mean value theorem

$$I_3=x\int_0^{\xi_u} P(t)\,a(t)\cos tx\,dt \qquad (0<\xi_u<u)$$

$$=x\Big[a(t)\int_0^t P(v)\cos vx\,dv\Big]_0^{\xi_u}-x\int_0^{\xi_u} a'(t)\,dt\int_0^t P(v)\cos vx\,dv\,.$$

We have now

$$\int_0^t P(v)\cos vx\,dv=\int_0^t \cos vx\Big(\sum_{\nu=1}^{\infty}\frac{\sin 2\pi\nu v}{\pi\nu}\Big)dv$$

$$=\sum_{\nu=1}^{\infty}\frac{1}{\pi\nu}\int_0^t \cos vx\sin 2\pi\nu v\,dv$$

$$=\sum_{\nu=1}^{\infty}\frac{1}{2\pi\nu}\int_0^t \{\sin(x+2\pi\nu)v+\sin(2\pi\nu-x)v\}\,dv$$

$$=\sum_{\nu=1}^{\infty}\frac{1}{2\pi\nu}\Big(\frac{1-\cos(x+2\pi\nu)t}{x+2\pi\nu}-\frac{1-\cos(2\pi\nu-x)t}{2\pi\nu-x}\Big)$$

$$=O\Big(\sum\frac{1}{\nu^2}\Big)=O(1)\,.$$

Therefore

$$|I_3|\leqq Ax+Ax\int_0^{\xi_u}|a'(t)|\,dt\leqq Ax\,,$$

which is less than ε for sufficiently small x. Summing up above estimations, we get

$$|\Psi(u,x)-\Phi(u,x)|<A\varepsilon\,.$$

Thus the lemma is proved.

We shall now prove Theorem 2. Since

$$f(x)=\psi(x)+g(x) \qquad (l=1,\ \xi=0)$$

we have

$$\Psi_f(u,\theta)=\Psi_\psi(u,\theta)+\Psi_g(u,\theta)\,.$$

From Theorem 1 and lemma it is sufficient to prove that $\Phi_g(u,\theta)\to 0$ for all λ and κ as $u\to\infty$ and $x\to 0$ independently. Here

$$\Phi_g(u,x)=\int_0^u\Big(1-\Big(\frac{t}{u}\Big)^{\lambda}\Big)^{\kappa}a(t)\sin xt\,dt\,.$$

It is sufficient to prove for the case $\kappa=0$. We write

$$\int_0^u a(t) \sin xt \, dt = \int_0^{\pi/x} + \int_{\pi/x}^{2\pi/x} + \cdots + \int_{(2k+1)\pi/x}^{u}$$

$$= \int_0^{\pi/x} a(t) \sin xt \, dt + \sum_{j=1}^{k} \int_0^{\pi/x} (a(t+2j\pi/x) - a(t+(2j-1)\pi/x)) \sin xt \, dt$$

$$+ \int_{(2k+1)\pi/x}^{u} a(t) \sin xt \, dt = I_1 + I_2 + I_3 , \quad \text{say,}$$

where

$$(2k+1)\pi/x < u \leqq (2k+3)\pi/x .$$

We have

$$I_1 = o(1) , \quad I_3 = o(1)$$

from the assumption (1) by integration by parts, and $I_2 = o(1)$ from the assumption (2).

This completes the proof of Theorem 2.

4. Proof of Theorem 5.

It is sufficient to prove that

$$B_x(t) = e^{-x} \sum_{n=0}^{\infty} s_n(t) \frac{x^n}{n!}$$

tends to 0 as $x \to \infty$, where $s_n(t)$ is the nth partial sum of the Fourier series of $f(t)$, i.e.

$$s_n(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+u) \frac{\sin\left(n+\frac{1}{2}\right)u}{2 \sin \frac{u}{2}} du .$$

Hence

$$B_x(t) = \frac{e^{-x}}{\pi} \sum_{n=0}^{\infty} \frac{x^n}{n!} \int_{-\pi}^{\pi} f(t+u) \frac{\sin\left(n+\frac{1}{2}\right)u}{2 \sin \frac{u}{2}} du .$$

On the other hand we use the formula, see G. H. Hardy [21],

$$\sum_{n=0}^{\infty} \left[\sin\left(n+\frac{1}{2}\right)u\right] \frac{x^n}{n!} = \exp(x \cos u)\left[\sin\left(x \sin u + \frac{1}{2}u\right)\right] .$$

Consequently

$$B_x(t) = \frac{e^{-x}}{\pi} \int_{-\pi}^{\pi} f(t+u) \frac{1}{2 \sin \frac{u}{2}} [\exp(x \cos u)\left[\sin\left(x \sin u + \frac{1}{2}u\right)\right] du$$

$$= \frac{e^{-x}}{\pi} \left\{ \int_0^{\pi} + \int_{-\pi}^{0} \right\} = I + J , \quad \text{say.}$$

We shall prove $I=o(1)$, since $J=o(1)$ may be estimated quite similarly.

$$I=\frac{1}{\pi}\int_0^{\pi} f(t+u)\frac{1}{2\sin\frac{u}{2}}[\exp(x\overline{\cos u-1})]\left[\sin\left(x\sin u+\frac{1}{2}u\right)\right]du$$

$$=\frac{1}{\pi}\left\{\int_0^{\pi/x}+\int_{\pi/x}^{\delta}+\int_{\delta}^{\pi}\right\}=\frac{1}{\pi}\{I_1+I_2+I_3\}, \quad \text{say.}$$

where δ is a arbitrary positive number sufficiently small.

$$I_1=\int_0^{\pi/x} f(t+u)\frac{1}{u}\exp\left(-\frac{xu^2}{2}\right)\sin\left(x+\frac{1}{2}\right)u\,du+o(1)$$

$$=\int_0^{\xi} f(t+u)\frac{\sin\left(x+\frac{1}{2}\right)u}{u}du+o(1), \qquad 0\leqq\xi\leqq\frac{\pi}{x},$$

from the second mean value theorem. From the condition (3)

$$I_1=o(1) \quad \text{as } x\to\infty,$$

for sufficiently small t.

Next

$$I_3=\int_{\delta}^{\pi} f(t+u)\frac{1}{2\sin\frac{u}{2}}[\exp x\overline{\cos u-1}]\left[\sin\left(x\sin u+\frac{1}{2}u\right)\right]du,$$

$$|I_3|\leqq A\int_{\delta}^{\pi}\frac{|f(t+u)|}{u}\left[\exp\left(-2x\sin^2\frac{u}{2}\right)\right]du$$

$$\leqq\frac{A}{\delta}\int_{\delta}^{\pi}|f(t+u)|\exp\left[-\frac{2xu^2}{\pi^2}\right]du=o(1) \qquad \text{as } x\to\infty,$$

where A is a constant which may be different at each occurrence. Finally

$$I_2=\int_{\pi/x}^{\delta} f(t+u)\frac{1}{2\sin\frac{u}{2}}[\exp x\overline{\cos u-1}]\left[\sin\left(x\sin u+\frac{1}{2}u\right)\right]du$$

$$=\int_{\pi/x}^{\delta} f(t+u)\exp\left(-\frac{xu^2}{2}\right)\frac{\sin\left(x\sin u+\frac{1}{2}u\right)}{u}du+o(1)$$

$$=\int_{\pi/x}^{2\pi/x}+\int_{2\pi/x}^{4\pi/x}+\cdots+\int_{y\pi/x}^{\delta}, \text{ where } \quad y=\left[\frac{x\delta}{\pi}\right].$$

Since

$$\sin\left(u+\frac{2k\pi}{x}+\frac{\pi}{x}\right)=\sin\left(u+\frac{2k\pi}{x}\right)\cos\frac{\pi}{x}+\cos\left(u+\frac{2k\pi}{x}\right)\sin\frac{\pi}{x}$$

$$=\sin\left(u+\frac{2k\pi}{x}\right)+\frac{\pi}{x}-\frac{\pi}{x}\lambda\sin^2\frac{\delta}{2}+O\left(\frac{1}{x^2}\right),$$

where $\quad 0\leqq\lambda\leqq 1$.

We have

$$\sin\left\{x\sin\left(u+\frac{2k+1}{x}\pi\right)+\frac{1}{2}\left(u+\frac{2k+1}{x}\pi\right)\right\}$$

$$=\sin\left\{x\sin\left(u+\frac{2k\pi}{x}\right)+\frac{1}{2}\left(u+\frac{2k\pi}{x}\right)+\pi-\lambda\pi\sin^2\frac{\delta}{2}+O\left(\frac{1}{x}\right)\right\}$$

$$=-\sin\left\{x\sin\left(u+\frac{2k\pi}{x}\right)+\frac{1}{2}\left(u+\frac{2k\pi}{x}\right)-\lambda\pi\sin^2\frac{\delta}{2}+O\left(\frac{1}{x}\right)\right\}$$

$$=-\sin\left\{x\sin\left(u+\frac{2k\pi}{x}\right)+\frac{1}{2}\left(u+\frac{2k\pi}{x}\right)\right\}+\sigma\lambda\pi\sin^2\frac{\delta}{2}+O\left(\frac{1}{x}\right)$$

for sufficiently large x, where $\sigma=O(1)$,

$$I_2=\sum_{k=0}^{\left[\frac{y}{2}\right]}\int_{\pi/x}^{2\pi/x}\left[f\left(t+u+\frac{2k\pi}{x}\right)\exp\left(-\frac{x}{2}\left(u+\frac{2k\pi}{x}\right)^2\right)\frac{1}{u+2k\pi/x}\right.$$
$$\left.-f\left(t+u+\frac{(2k+1)\pi}{x}\right)\exp\left(-\frac{x}{2}\left(u+\frac{(2k+1)\pi}{x}\right)^2\right)\frac{1}{u+(2k+1)\pi/x}\right]\cdot$$
$$\sin\left\{x\sin\left(u+\frac{2k\pi}{x}\right)+\frac{1}{2}\left(u+\frac{2k\pi}{x}\right)\right\}du$$
$$+\sum_{k=0}^{\left[\frac{y}{2}\right]}\int_{\pi/x}^{2\pi/x}f\left(t+u+\frac{(2k+1)\pi}{x}\right)\exp\left(-\frac{x}{2}\left(u+\frac{(2k+1)\pi}{x}\right)^2\cdot\right.$$
$$\frac{O\left(\frac{1}{x}\right)+\sigma\lambda\pi\sin^2\frac{\delta}{2}}{u+(2k+1)\pi/x}du+o(1)$$

$=I_{21}+I_{22}+o(1)$, say. Here,

$$I_{22}=\sum_{k=0}^{\left[\frac{y}{2}\right]}\left[\exp\left(-\frac{x}{2}\left(\frac{(2k+2)^2\pi^2}{x^2}\right)\right]\frac{o\left(\frac{1}{x}\right)}{2k+2}$$
$$+\sum_{k=0}^{\left[\frac{y}{2}\right]}\left[\exp\left(-\frac{x}{2}\frac{(2k+2)^2\pi^2}{x^2}\right)\right]\frac{o(1)}{2k+2}\sigma\lambda\pi\sin^2\frac{\delta}{2}=o(1)$$

from the condition (3).

$$I_{21}=\sum_{k=0}^{\left[\frac{y}{2}\right]}\int_{\pi/x}^{2\pi/x}\left\{f\left(t+u+\frac{2k\pi}{x}\right)\exp\left(-\frac{x}{2}(u+2k\pi/x)^2\right)-f(t+u+(2k+1)\pi/x)\cdot\right.$$
$$\left.\exp\left(-\frac{x}{2}(u+(2k+1)\pi/x)\right)\right\}\frac{\sin\left\{x\sin\left(u+\frac{2k\pi}{x}\right)+\frac{1}{2}\left(u+\frac{2k\pi}{x}\right)\right\}}{u+2k\pi/x}du$$

$$+\sum_{k=0}^{\left[\frac{y}{2}\right]}\int_{\pi/x}^{2\pi/x} f(t+u+(2k+1)\pi/x)\left\{\exp\left(-\frac{x(u+(2k+1)\pi/x)^2}{2}\right)\right\}\cdot$$

$$\left\{\frac{1}{u+2k\pi/x}-\frac{1}{u+(2k+1)\pi/x}\right\}du$$

$=K+L$, say.

When u varies from $\frac{\pi}{x}$ to $\frac{2\pi}{x}$, $\sin\left\{x\sin\left(u+\frac{2k\pi}{x}\right)+\frac{1}{2}\left(u+\frac{2k\pi}{x}\right)\right\}$ varies from $\sin\left\{x\sin\left(\frac{2k+1}{x}\pi\right)+\frac{1}{2}\frac{2k+1}{x}\pi\right\}$ to $\sin\left\{x\sin\left(\frac{2k+2}{x}\pi\right)+\frac{1}{2}\frac{2k+2}{x}\pi\right\}$.

On the other hand, since

$$\sin\frac{2k+2}{x}\pi-\sin\frac{2k+1}{x}\pi=-\sin\frac{2k+1}{x}\pi\cdot 2\sin^2\frac{\pi}{2x}+\cos\frac{2k+1}{x}\pi\sin\frac{\pi}{x},$$

we have

$$\left\{x\sin\left(\frac{2k+1}{x}\pi\right)+\frac{1}{2}\frac{2k+1}{x}\pi\right\}-\left\{x\sin\left(\frac{2k+2}{x}\pi\right)+\frac{1}{2}\frac{2k+2}{x}\pi\right\}$$

$$=x\left(\frac{\pi}{x}-2\lambda\frac{\pi}{2}\sin^2\frac{\delta}{2}+O\left(\frac{1}{x^2}\right)\right)-\frac{\pi}{2x}$$

$$=\pi-2\lambda\pi\sin^2\frac{\delta}{2}+O\left(\frac{1}{x}\right),\quad \text{where}\quad 0\leqq\lambda\leqq 1.$$

This may be $\pi-\varepsilon$ for sufficiently small δ and for sufficiently large x.

Hence when u varies from $\frac{\pi}{x}$ to $\frac{2\pi}{x}$, $\sin\left\{x\sin\left(u+\frac{2k\pi}{x}\right)+\frac{1}{2}\left(u+\frac{2k\pi}{x}\right)\right\}$ has at most one extremum for sufficiently large x. Hence we put

$$K=\sum_{k=0}^{\left[\frac{y}{2}\right]}\int_{\pi/x}^{2\pi/x}\frac{1}{u+2k\pi/x}\left\{f(t+u+2k\pi/x)-f(t+u+(2k+1)\pi/x)\right\}\cdot$$

$$\exp\left(-\frac{x(u+2k\pi/x)^2}{2}\right)\sin\left\{x\sin\left(u+\frac{2k\pi}{x}\right)+\frac{1}{2}\left(u+\frac{2k\pi}{x}\right)\right\}du$$

$$+\sum_{k=0}^{\left[\frac{y}{2}\right]}\int_{\pi/x}^{2\pi/x}\frac{1}{u+2k\pi/x}f\left(t+u+\frac{(2k+1)\pi}{x}\right)\left\{\exp\left(-\frac{x(u+2k\pi/x)^2}{2}\right)-\right.$$

$$\left.\exp\left(-\frac{x(u+(2k+1)\pi/x)^2}{2}\right)\right\}\sin\left\{x\sin\left(u+\frac{2k\pi}{x}\right)+\frac{1}{2}\left(u+\frac{2k\pi}{x}\right)\right\}du$$

$=K_1+K_2$, say.

From the second mean value theorem

$$K_1=\sum_{k=0}^{[\frac{y}{2}]}\frac{\exp\left(-\frac{x((2k+1)\pi/x)^2}{2}\right)}{\frac{2k+1}{x}\pi}\cdot$$

$$\left[\varepsilon_k\int_{\eta_k}^{\xi_k}\{f(t+u+2k\pi/x)-f(t+u+(2k+1)\pi/x)\}\,du\right.$$

$$\left.+\varepsilon_k'\int_{\eta_k}^{\xi'_k}\{f(t+u+2k\pi/x)-f(t+u+(2k+1)\pi/x)\}\,du\right]$$

where $\frac{\pi}{x}\leqq\eta_k<\xi_k\leqq\frac{2\pi}{x}$, $\frac{\pi}{x}\leqq\eta_k'<\xi_k'\leqq\frac{2\pi}{x}$, and $-1\leqq\varepsilon_k,\ \varepsilon_k'\leqq 1$.

From (4)

$$K_1=o\left(\sum_{k=0}^{[\frac{y}{2}]}\frac{\exp\left(-\frac{x((2k+1)\pi/x)^2}{2}\right)}{2k+1}\right)=o(1)\,.$$

Similarly

$$K_2=o\left(\sum_{k=0}^{[\frac{y}{2}]}\frac{1}{\frac{2k+1}{x}\pi}\left\{\exp\left(-\frac{x((2k+1)\pi/x)^2}{2}\right)-\exp\left(-\frac{x((2k+2)\pi/x)^2}{2}\right)\right\}\right)$$

$$=o(1)\,,$$

from the second mean value theorem and condition (3).
We get $L=o(1)$ similarly.

Thus the theorem is proved for the exponential means.

For the xth integral means $B_x^*(t)$ we know

$$B_x(t)=e^{-x}\sum a_n\frac{x^n}{n!}+B_x^*(t)\,,$$

where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\cos n(\theta-t)\,d\theta.$$

From the Riemann-Lebesgue theorem $a_n\to 0$, as $n\to\infty$ uniformly in t. Hence

$$\lim_{x\to\infty}B_x(t)=\lim_{x\to\infty}B_x^*(t)\,.$$

Consequently Theorem 5 holds also for the Borel integral means.

5. Proof of Theorem 7.

Let $s_n(t)$ be the nth partial sum of the Fourier series of $f(t)$. Then the Euler means of order r of the sequence $\{s_n(t)\}$ is

$$\sigma_{n,r}(t)=\sum_{\nu=0}^{n}\binom{n}{\nu}r^{\nu}(1-r)^{n-\nu}s_{\nu}(t)$$

$$=\sum_{\nu=0}^{n}\binom{n}{\nu}r^{\nu}(1-r)^{n-\nu}\left\{\frac{1}{\pi}\int_{-\pi}^{\pi}f(t+u)\frac{\sin\left(\nu+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du\right\}$$

$$=\frac{1}{\pi}\int_{-u}^{u}\frac{f(t+u)}{2\sin\frac{u}{2}}\sum_{\nu=0}^{n}\binom{n}{\nu}r^{\nu}(1-r)^{n-\nu}\sin\left(\nu+\frac{1}{2}\right)u\,du,$$

where

$$\sum_{\nu=0}^{n}\binom{n}{\nu}r^{\nu}(1-r)^{n-\nu}\sin\left(\nu+\frac{1}{2}\right)u$$

$$=\mathfrak{I}\sum_{\nu=0}^{n}\binom{n}{\nu}r^{\nu}(1-r)^{n-\nu}e^{i\left(\nu+\frac{1}{2}\right)u}=\mathfrak{I}e^{i\frac{1}{2}u}(1-r+re^{iu})^{n}.$$

Here we use the Szász lemma [9]. Let

$$(1-r+re^{iu})=\rho e^{i\alpha}\quad(\rho\geqq 0),\quad\text{then}$$

$$(6)\qquad\begin{cases}\rho\cos\alpha=1-r+r\cos u\\ \rho\sin\alpha=r\sin u.\end{cases}$$

Further

$$(7)\qquad\rho^2=(1-r)^2+r^2+2r(1-r)\cos u=1-2r(1-r)(1-\cos u)$$

$$=1-4r(1-r)\sin^2\frac{u}{2}\leqq 1,\quad\text{whence}$$

$$1-\rho^2=4r(1-r)\sin^2\frac{u}{2}>0\quad\text{for}\quad 0<\delta<u<\pi,\quad\text{or}$$

$$0\leqq\rho<1.$$

Now we get

$$\sigma_{n,r}(t)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t+u)\frac{\rho^n\sin\left(n\alpha+\frac{u}{2}\right)}{2\sin\frac{u}{2}}du.$$

It is sufficient to prove

$$\int_{0}^{\pi}f(t+u)\frac{\rho^n\sin\left(n\alpha+\frac{u}{2}\right)}{2\sin\frac{u}{2}}du=o(1)$$

from (3) and (4). $\int_{-\pi}^{0}=o(1)$ may be estimated quite similarly.

$$I=\int_0^{\pi} f(t+u)\frac{\rho^n \sin\left(n\alpha+\dfrac{u}{2}\right)}{2\sin\dfrac{u}{2}}du=\int_0^{\pi/nr}+\int_{\pi/nr}^{\delta}+\int_{\delta}^{\pi}=I_1+I_2+I_3, \quad \text{say,}$$

where δ is a positive constant arbitrary small. On the other hand

$$1-\rho^n=(1-\rho)\sum_0^{n-1}\rho^\nu<n(1-\rho), \quad \text{and}$$

$$1-\rho^2=4r(1-r)\sin^2\frac{u}{2}<r(1-r)u^2,$$

so that

$$1-\rho<r(1-r)u^2\leqq\frac{u^2}{4}.$$

It follows that

$$1-\rho^n<nu^2/4, \quad \text{or}$$

$$1-\rho^n=\lambda nu^2, \qquad 0<\lambda<\frac{1}{4}.$$

Now

$$I_1=\int_0^{\pi/nr} f(t+u)\frac{\rho^n\sin\left(n\alpha+\dfrac{u}{2}\right)}{2\sin\dfrac{u}{2}}du$$

$$=\frac{1}{2}\int_0^{\pi/nr} f(t+u)\rho^n\cot\frac{u}{2}\sin n\alpha\, du+\frac{1}{2}\int_0^{\pi/nr} f(t+u)\rho^n\cos n\alpha\, du$$

$$=\frac{1}{2}\int_0^{\pi/nr} f(t+u)\rho^n\cot\frac{u}{2}\sin n\alpha\, du+o(1)$$

$$=\frac{1}{2}\int_0^{\pi/nr} f(t+u)\cot\frac{u}{2}\sin n\alpha\, du-\frac{1}{2}\int_0^{\pi/nr} f(t+u)\lambda nu^2\cot\frac{u}{2}\sin n\alpha\, du+o(1)$$

$$=\frac{1}{2}I_{11}-\frac{1}{2}I_{12}+o(1), \quad \text{say.}$$

Next, from (7)

$$\rho^2\geqq(1-r)^2+r^2\geqq r^2, \quad \text{so} \quad \rho\geqq r$$

and now from (6)

$$r\sin u=\rho\sin\alpha\geqq r\sin\alpha,$$

hence

$$u\geqq\alpha \quad \text{for small } u.$$

It is known that.

$$0<u-\sin u<u^3$$

And from

$$\rho\alpha-ru=\rho(\alpha-\sin\alpha)-r(u-\sin u), \quad \text{we get}$$
$$|\rho\alpha-ru|<\alpha^3+u^3<2u^3,$$
$$|\alpha-ru|\leqq|\rho\alpha-ru|+(1-\rho)\alpha<2u^3+u^3=3u^3, \quad \text{or}$$
$$\alpha=ru+\mu u^3, \quad |\mu|<3.$$

We now have

$$\begin{aligned}
I_{11}&=\int_0^{\pi/nr} f(t+u)\cot\frac{u}{2}\sin n(ru+\mu u^3)\,du\\
&=\int_0^{\pi/nr} f(t+u)\left\{\frac{2}{u}+\text{continuous function}\right\}\sin n(ru+\mu u^3)\,du\\
&=2\int_0^{\pi/nr} f(t+u)\frac{\sin n(ru+\mu u^3)}{u}\,du+o(1)\\
&=\int_0^{\pi/nr} f(t+u)\frac{\sin(nru+n\cdot o(u^2))}{u}\,du+o(1)\\
&=2\int_0^{\pi/nr} f(t+u)\frac{\sin nru}{u}\,du+o\left\{\int_0^{\pi/nr}|f(t+u)|u\cdot n\cdot du\right\}+o(1)\\
&=2\int_0^{\pi/nr} f(t+u)\frac{\sin nru}{u}\,du+o(1)\\
&=o(1).
\end{aligned}$$

from (3) for sufficiently small t.

$$\begin{aligned}
I_{12}&=\int_0^{\pi/nr} f(t+u)\lambda nu^2\cot\frac{u}{2}\sin(nru+n\mu u^3)\,du\\
&=2\int_0^{\pi/nr} f(t+u)\lambda nu\sin(nru+n\mu u^3)\,du+o(1)\\
&=o(1).
\end{aligned}$$

Consequently $\qquad I_1=o(1)$.

Next

$$I_3=\int_\delta^\pi f(t+u)\frac{\rho^n\sin\left(n\alpha+\dfrac{u}{2}\right)}{2\sin\dfrac{u}{2}}\,du,$$

and then

$$|I_3| \leqq \frac{1}{2}\int_\delta^\pi |f(t+u)| \frac{\pi}{u}\rho^n\, du .$$

Since $0 \leqq \rho < \sigma < 1$ for $\delta < u < \pi$, $I_3 = o(1)$.
Finally

$$I_2 = \int_{\pi/nr}^{\delta} f(t+u) \frac{\rho^n \sin\left(n\alpha + \dfrac{u}{2}\right)}{2\sin\dfrac{u}{2}} du$$

$$= \frac{1}{2}\int_{\pi/nr}^{\delta} f(t+u)\rho^n \sin n\alpha \cot\frac{u}{2} du + \frac{1}{2}\int_{\pi/nr}^{\delta} f(t+u)\rho^n \cos n\alpha\, du$$

$$= \int_{\pi/nr}^{\delta} f(t+u)\,\rho^n \frac{\sin n\alpha}{u} + \varepsilon$$

for sufficiently small δ, where $|\varepsilon|$ is arbitrary small. It is sufficient to prove

$$I_2^* = \int_{\pi/nr}^{\delta} f(t+u)\,\rho^n \frac{\sin n\alpha}{u} du = o(1) ,$$

where

$$\alpha = ru + \mu u^3 , \quad |\mu| < 3 .$$

$$I_2^* = \int_{\pi/nr}^{2\pi/nr} + \int_{2\pi/nr}^{3\pi/nr} + \cdots + \int_{y\pi/nr}^{\delta} , \quad \text{where } y = \left[\frac{nr\delta}{\pi}\right].$$

$$I_2^* = \sum_{k=0}^{\left[\frac{y}{2}-1\right]} \int_{\pi/nr}^{2\pi/nr} \left\{ \frac{f(t+u+2k\pi/nr)}{u+2k\pi/nr} \rho^n(u+2k\pi/nr) \sin(nru + n\mu(u+2k\pi/nr)^3) \right.$$
$$- \frac{f(t+u+(2k+1)\pi/nr)}{u+(2k+1)\pi/nr} \rho^n(u+(2k+1)\pi/nr) \cdot$$
$$\left. \sin(nru + n\mu(u+(2k+1)\pi/nr)^3) \right\} du + o(1) .$$

$$= \sum \int_{\pi/nr}^{2\pi/nr} \frac{f(t+u+2k\pi/nr)}{u+2k\pi/nr} \rho^n(u+2k\pi/nr) \{\sin(nru + n\mu(u+2k\pi/nr)^3)$$
$$- \sin(nru + n\mu(u+(2k+1)\pi/nr)^3\}\, du$$

$$+ \sum \int_{\pi/nr}^{2\pi/nr} \left\{ \frac{f(t+u+2k\pi/nr)}{u+2k\pi/nr} \rho^n(u+2k\pi/nr) - \frac{f(t+u+(2k+1)\pi/nr)}{u+(2k+1)\pi/nr} \cdot \right.$$
$$\left. \rho^n(u+(2k+1)\pi/nr) \right\} \sin(nru + n\mu(u+(2k+1)\pi/nr)^3)\, du + o(1)$$

$$= I_{21} + I_{22} + o(1) , \quad \text{say.}$$

$$I_{21}=\sum_{k=0}^{[\frac{y}{2}-1]}\int_{\pi/nr}^{2\pi/nr}\frac{f(t+u+2k\pi/nr)}{u+2k\pi/nr}\rho^n(u+2k\pi/nr)\{\sin(nru+n\mu(u+2k\pi/nr)^3)$$
$$-\sin(nru+n\mu(u+(2k+1)\pi/nr)^3)\}\,du\,.$$

Here

$$\sin(nru+n\mu(u+2k\pi/nr)^3)-\sin(nru+n\mu(n+(2k+1)\pi/nr)^3)$$
$$=2\sin\frac{1}{2}\{n\mu(u+2k\pi/nr)^3-n\mu(n+(2k+1)\pi/nr)^3\}\cdot$$
$$\cos\frac{1}{2}\{2nru+n\mu(u+2k\pi/nr)^3+n\mu(u+(2k+1)\pi/nr)^3\},$$

further

$$\frac{r}{\pi}\{n\mu(u+2k\pi/nr)^3-n\mu(u+(2k+1)\pi/nr)^3\}$$
$$=\frac{1}{x}\{\mu(u+2kx)^3-\mu(u+(2k+1)x)^3\},\quad \text{where}\quad x=\frac{\pi}{nr},$$
$$=\frac{1}{x}\{\mu(u+2kx)^3-\mu u^3\}+\frac{1}{x}\{\mu u^3-\mu(u+(2k+1)x)^3\}=P+Q,\quad \text{say.}$$

If n tends to ∞, or x tends to 0, we get

$$\lim_{x\to 0}P=\lim_{x\to 0}\frac{\mu(u+2kx)^3-\mu u^3}{x}=2k(\mu'(u)u^3+3\mu(u)u^2),$$

similarly

$$\lim_{x\to 0}Q=-(2k+1)(\mu'(u)u^3+3\mu(u)u^2),$$

whence

$$\lim_{x\to 0}(P+Q)=-(\mu'(u)u^3+3\mu(u)u^2).$$

On the other hand from (6)

$$\mu=\frac{1}{u^3}\cot^{-1}\frac{1-2r\sin^2\frac{u}{2}}{r\sin u}-\frac{r}{u^2},\quad \text{and so}$$

$$\mu'=\frac{3}{u^4}\cot^{-1}\frac{1-2r\sin^2\frac{u}{2}}{r\sin u}+\frac{1}{u^3}\,\frac{r^2+r\cos u-r^2\cos u}{\rho^2}+\frac{2r}{u^3}$$
$$=o\left(\frac{1}{u^3}\right)\quad \text{as } u\to 0.$$

Consequently

$$\sin(nru+n\mu(u+2k\pi/nr)^3)-\sin(nru+n\mu(u+(2k+1)\pi/nr)^3)\cdot$$

tends to 0 as $n\to\infty$, where $\dfrac{\pi}{nr}\leqq u\leqq\dfrac{2\pi}{nr}$.

We now have from the second mean value theorem

$$I_{21}=\sum_{k=0}^{[\frac{y}{2}-1]}\int_{\pi/nr}^{2\pi/nr}\frac{f(t+u+2k\pi/nr)}{u+2k\pi/nr}\rho^n(u+2k\pi/nr)\{\sin(nru+n\mu(u+2k\pi/nr)^3)$$
$$-\sin(nru+n\mu(u+(2k+1)\pi/nr)^3)\}\,du$$

$$=\sum_{k=0}^{[\frac{y}{2}-1]}\frac{\rho^n\left(\dfrac{2k+1}{nr}\pi\right)}{\dfrac{2k+1}{nr}\pi}\int_{\pi/nr}^{\xi_k}f(t+u+2k\pi/nr)\{\sin(nru+n\mu(u+2k\pi/nr)^3)$$
$$-\sin(nru+n\mu(u+(2k+1)\pi/nr)^3)\}\,du$$

$$=o(1),\quad\text{where }\frac{\pi}{nr}<\xi_k\leqq\frac{2\pi}{nr}.$$

Here in I_{22}

$$\frac{f(t+u+2k\pi/nr)}{u+2k\pi/nr}\rho^n(u+2k\pi/nr)-\frac{f(t+u+(2k+1)\pi/nr)}{u+(2k+1)\pi/nr}\rho^n(u+(2k+1)\pi/nr)$$

$$=\frac{f(t+u+2k\pi/nr)\rho^n(u+2k\pi/nr)-f(t+u+(2k+1)\pi/nr)\rho^n(u+(2k+1)\pi/nr)}{u+2k\pi/nr}$$
$$+\frac{\pi f(t+u+(2k+1)\pi/nr)\rho^n(u+(2k+1)\pi/nr)}{nr(u+2k\pi/nr)(u+(2k+1)\pi/nr)}$$

$$=\frac{f(t+u+2k\pi/nr)-f(t+u+(2k+1)\pi/nr)}{u+2k\pi/nr}\rho^n(u+2k\pi/nr)$$
$$+\frac{f(t+u+(2k+1)\pi/nr)}{u+2k\pi/nr}\{\rho^n(u+2k\pi/nr)-\rho^n(u+(2k+1)\pi/nr)\}$$
$$+\frac{\pi\cdot f(t+u+(2k+1)\pi/nr)\rho^n(u+(2k+1)\pi/nr)}{nr(u+2k\pi/nr)(u+(2k+1)\pi/nr)}=J_k+K_k+L_k,\quad\text{say.}$$

We put

$$I_{22}=\sum_{k=0}^{[\frac{y}{2}-1]}\int_{\pi/nr}^{2\pi/nr}J_k\sin\{nru+n\mu(u+(2k+1)\pi/nr)^3\}\,du$$
$$+\sum_{k=0}^{[\frac{y}{2}-1]}\int_{\pi/nr}^{2\pi/nr}K_k\sin\{nru+n\mu(u+(2k+1)\pi/nr)^3\}\,du$$
$$+\sum_{k=0}^{[\frac{y}{2}-1]}\int_{\pi/nr}^{2\pi/nr}L_k\sin\{nru+n\mu(u+(2k+1)\pi/nr)^3\}\,du$$

$=I_{221}+I_{222}+I_{223}$. Here

$$I_{221}=\sum_{k=0}^{[\frac{y}{2}-1]}\int_{\pi/nr}^{2\pi/nr}\frac{f(t+u+2k\pi/nr)-f(t+u+(2k+1)\pi/nr)}{u+2k\pi/nr}\cdot$$
$$\rho^n(u+2k\pi/nr)\sin(nru+n\mu(u+(2k+1)\pi/nr)^3)\,du$$

$$=o(1)$$

from the second mean value theorem and (4).

$$I_{222}=\sum_{k=0}^{[\frac{y}{2}-1]}\int_{\pi/nr}^{2\pi/nr}\frac{f(t+u+(2k+1)\pi/nr)}{u+2k\pi/nr}\{\rho^n(u+2k\pi/nr)$$

$$-\rho^n(u+(2k+1)\pi/nr)\}\sin(nru+n\mu(u+(2k+1)\pi/nr)^3)\,du$$

$$=\sum\frac{nr\varepsilon_k}{(2k+1)\pi}\left\{\rho^n\left(\frac{2k+2}{nr}\pi\right)-\rho^n\left(\frac{2k+3}{nr}\pi\right)\right\}\left\{\int_{\xi_k}^{\eta_k}f(t+u+(2k+1)\pi/nr)\cdot\right.$$

$$\sin(nru+n\mu(u+(2k+1)\pi/nr)^3)\,du$$

$$=o(1),\quad \text{where } \pi/nr\leqq\xi_k<\eta_k\leqq 2\pi/nr,\ 0<\varepsilon_k<1,$$

from the second mean value theorem. Similarly

$$I_{223}=\sum_{k=0}^{[\frac{y}{2}-1]}\int_{\pi/nr}^{2\pi/nr}\frac{\pi f(t+u+(2k+1)\pi/nr)\rho^n(u+(2k+1)\pi/nr)}{nr(u+2k\pi/nr)(u+(2k+1)\pi/nr)}\cdot$$

$$\sin(nru+n\mu(u+(2k+1)\pi/nr)^3)\,du$$

$$=\sum\frac{nr\rho^n((2k+2)\pi/nr)}{(2k+1)(2k+2)\pi^2}\int_{\pi/nr}^{\xi_k}f(t+u+(2k+1)\pi/nr)\cdot$$

$$\sin(nru+n\mu(u+(2k+1)\pi/nr)^3)\,du$$

$$=o(1).$$

This completes the proof of Theorem 7.

6. Proof of Theorem 9.

Let $s_n(t)$ be the nth partial sum of the Fourier series of $f(t)$, then

$$s_n(t)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t+u)\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du.$$

Hence the Hausdorff mean of sequence $\{s_n(t)\}$ is

$$h_n(t)=\int_0^1\sum_{\nu=0}^{n}\binom{n}{\nu}s_\nu(t)r^\nu(1-r)^{n-\nu}d\psi(r)$$

$$=\frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi}\frac{f(t+u)}{\sin\frac{u}{2}}\sum_{\nu=0}^{n}\binom{n}{\nu}r^\nu(1-r)^{n-\nu}\sin\left(\nu+\frac{1}{2}\right)u\,du\,d\psi(r)$$

$$=\frac{1}{2\pi}\mathfrak{I}\int_0^1\int_{-\pi}^{\pi}\frac{f(t+u)}{\sin\frac{u}{2}}(1-r+re^{iu})^ne^{iu/2}\,du\,d\psi(r)$$

where $\mathfrak{I}$ means the imaginary part. Here we put

$$(1-r+re^{iu})=\rho e^{i\alpha}$$

then we get from the Szász lemma [9]

$$\rho^2=1-4r(1-r)\sin^2\frac{u}{2},$$

$$1-\rho^n=\lambda nu^2 \quad \left(0<\lambda<\frac{1}{4}\right)$$

and $\alpha=ru+\mu u^3$, where $|\mu|<3$. (See the proof of Theorem 7.) Hence

$$h_n(t)=\frac{1}{2\pi}\Im\int_0^1\int_{-\pi}^{\pi}\frac{f(t+u)}{\sin\frac{u}{2}}\rho^n e^{i\left(n\alpha+\frac{u}{2}\right)}\,du\,d\psi(r)$$

$$=\frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi}\frac{f(t+u)\rho^n\sin\left(n\alpha+\frac{u}{2}\right)}{\sin\frac{u}{2}}\,du\,d\psi(r)$$

$$=\frac{1}{2\pi}\left\{\int_0^1\int_0^{\pi}+\int_0^1\int_{-\pi}^{0}\right\}du\,d\psi(r)$$

$$=\frac{1}{2\pi}(I+J),\quad \text{say.}$$

It is sufficient to prove

$$I=\int_0^1\int_0^{\pi}\frac{f(t+u)\rho^n\sin\left(n\alpha+\frac{u}{2}\right)}{\sin\frac{u}{2}}\,du\,d\psi(r)=o(1)$$

as $n\to\infty$, since $J=o(1)$ may be estimated quite similarly. Here

$$I=\int_0^1\int_0^{\pi}\frac{f(t+u)\rho^n\sin\left(n\alpha+\frac{u}{2}\right)}{\sin\frac{u}{2}}\,du\,d\psi(r)$$

$$=\left\{\int_0^1\int_0^{\pi/n}+\int_0^1\int_{\pi/n}^{\delta}+\int_0^1\int_{\delta}^{\pi}\right\}du\,d\psi(r)=I_1+I_2+I_3,\quad \text{say,}$$

where δ is a positive constant arbitrary small.

$$I_1=\int_0^1\int_0^{\pi/n}\frac{f(t+u)\rho^n\sin\left(n\alpha+\frac{u}{2}\right)}{\sin\frac{u}{2}}\,du\,d\psi(r)$$

$$=\int_0^1\int_0^{\pi/n} f(t+u)\rho^n \sin n\alpha \cot\frac{u}{2}\,du\,d\psi(r)$$

$$+\int_0^1\int_0^{\pi/n} f(t+u)\rho^n \cos n\alpha\,du\,d\psi(r)=I_{11}+I_{12}, \quad \text{say}.$$

We get further

$$I_{11}=\int_0^1\int_0^{\pi/n} f(t+u)\rho^n \sin n\alpha \cot\frac{u}{2}\,du\,d\psi(r)$$

$$=\int_0^1\int_0^{\pi/n} f(t+u)\sin n\alpha \cot\frac{u}{2}\,du\,d\psi(r)$$

$$-\int_0^1\int_0^{\pi/n} f(t+u)\lambda nu^2 \sin n\alpha \cot\frac{u}{2}\,du\,d\psi(r)=I_{111}-I_{112}, \quad \text{say},$$

where $0<\lambda<\dfrac{1}{4}$. In I_{111} we get

$$\int_0^{\pi/n} f(t+u)\sin n\alpha \cot\frac{u}{2}\,du=\int_0^{\pi/n} f(t+u)\sin n\alpha\Big(\frac{2}{u}+\text{continous function}\Big)du$$

$$=2\int_0^{\pi/n} f(t+u)\frac{\sin n\alpha}{u}\,du+o(1), \quad \text{uniformly in } r\ (0\leqq r\leqq 1),$$

as $n\to\infty$. While

$$\int_0^{\pi/n} f(t+u)\frac{\sin n\alpha}{u}\,du=\int_0^{\pi/n}\frac{f(t+u)\sin n(ru+\mu u^3)}{u}\,du$$

$$=\int_0^{\pi/n}\frac{f(t+u)\sin n(ru+o(u^2))}{u}\,du$$

$$=\int_0^{\pi/n}\frac{f(t+u)\sin nru}{u}\,du+o\Big\{\int_0^{\pi/n}|f(t+u)|\,un\,du\Big\}=o(1)$$

uniformly in r $(0\leqq r\leqq 1)$ from the condition (3) and the integration by parts. From the above estimations we get

$$I_{111}=\int_0^1\int_0^{\pi/n} f(t+u)\sin n\alpha \cot\frac{u}{2}\,du\,d\psi(r)=o(1).$$

Next in I_{112} we get similarly as in I_{111}

$$\int_0^{\pi/n} f(t+u)\lambda nu^2 \sin n\alpha \cot\frac{u}{2}\,du$$

$$=2\int_0^{\pi/n} f(t+u)\lambda nu \sin n\alpha\,du+o(1)=o(1)$$

uniformly in r $(0\leqq r\leqq 1)$. Hence

$$I_{112}=\int_0^1\int_0^{\pi/n} f(t+u)\lambda nu^2 \sin n\alpha \cot\frac{u}{2}\,du\,d\psi(r)=o(1)\,,$$

so that

$$I_{11}=I_{111}-I_{112}=o(1)\,.$$

Next we estimate

$$I_{12}=\int_0^1\int_0^{\pi/n} f(t+u)\rho^n \cos n\alpha\,du\,d\psi(r)\,.$$

Since in I_{12}

$$\left|\int_0^{\pi/n} f(t+u)\rho^n \cos n\alpha\,du\right|\leqq\left|\int_0^{\pi/n}|f(t+u)|\,du=o(1)\right.$$

uniformly in r $(0\leqq r\leqq 1)$, so we get

$$I_{12}=o(1)\,.$$

Next

$$I_3=\int_0^1\int_\delta^\pi \frac{f(t+u)\rho^n\sin\left(n\alpha+\dfrac{u}{2}\right)}{\sin\dfrac{u}{2}}\,du\,d\psi(r)$$

$$=\left\{\int_0^{1/\log n}\int_\delta^\pi+\int_{1/\log n}^{1-1/\log n}\int_\delta^\pi+\int_{1-1/\log n}^1\int_\delta^\pi\right\}du\,d\psi(r)=I_{31}+I_{32}+I_{33}\,,\quad \text{say.}$$

Here

$$|I_{31}|\leqq\left|\int_0^{1/\log n}\int_\delta^\pi \frac{f(t+u)\rho^n\sin\left(n\alpha+\dfrac{u}{2}\right)}{\sin\dfrac{u}{2}}\,du\,d\psi(r)\right|$$

$$\leqq\left\{\text{total variation of } \psi(r) \text{ in } 0\leqq r\leqq\frac{1}{\log n}\right\}\cdot$$

$$\max_{0\leqq r\leqq\frac{1}{\log n}}\left|\int_\delta^\pi \frac{f(t+u)\rho^n\sin\left(n\alpha+\dfrac{u}{2}\right)}{\sin\dfrac{u}{2}}\,du\right|$$

$$\leqq o(1)\cdot\frac{1}{\sin\dfrac{\delta}{2}}\int_\delta^\pi|f(t+u)|\,du=o(1),$$

since $\psi(r)$ continuous at $r=0$. Similarly we get $I_{33}=o(1)$ from the continuity of $\psi(r)$ at $r=1$.

Next we estimate

$$I_{32}=\int_{1/\log n}^{1-1/\log n}\int_{\delta}^{\pi}\frac{f(t+u)\rho^{n}\sin\left(n\alpha+\dfrac{u}{2}\right)}{\sin\dfrac{u}{2}}du\,d\psi(r)$$

Since

$$\rho^{n}=\left\{1-4r(1-r)\sin^{2}\frac{u}{2}\right\}^{\frac{n}{2}}$$

we get

$$|I_{32}|\leqq\left\{1-4\frac{1}{\log n}\left(1-\frac{1}{\log n}\right)\sin^{2}\frac{\delta}{2}\right\}^{\frac{n}{2}}\int_{1/\log n}^{1-1/\log n}|d\psi(r)|\cdot$$

$$\max_{\frac{1}{\log n}\leqq r\leqq 1-\frac{1}{\log n}}\left|\int_{\delta}^{\pi}\frac{f(t+u)\sin\left(n\alpha+\dfrac{u}{2}\right)}{\sin\dfrac{u}{2}}du\right|$$

$$\leqq A\left\{1-4\frac{1}{\log n}\left(1-\frac{1}{\log n}\right)\sin^{2}\frac{\delta}{2}\right\}^{\frac{n}{2}},$$

where A is a constant. Since

$$\lim_{n\to\infty}\left\{1-4\frac{1}{\log n}\left(1-\frac{1}{\log n}\right)\sin^{2}\frac{\delta}{2}\right\}^{\frac{n}{2}}=0,$$

we get $I_{32}=o(1)$.

Next we use the Szász lemma [10]; that is $\mu=O(r)$ for sufficiently small u.

Finally

$$I_{2}=\int_{0}^{1}\int_{\pi/n}^{\delta}\frac{f(t+u)\rho^{n}\sin\left(n\alpha+\dfrac{u}{2}\right)}{\sin\dfrac{u}{2}}du\,d\psi(r)$$

$$=\left\{\int_{0}^{\pi/n\delta}\int_{\pi/n}^{\delta}+\int_{\pi/n\delta}^{1}\int_{\pi/n}^{\pi/nr}+\int_{\pi/n\delta}^{1}\int_{\pi/nr}^{\delta}\right\}du\,d\psi(r)=I_{21}+I_{22}+I_{23}, \quad \text{say.}$$

Here

$$I_{21}=\int_{0}^{\pi/n\delta}\int_{\pi/n}^{\delta}\frac{f(t+u)\rho^{n}\sin\left(n\alpha+\dfrac{u}{2}\right)}{\sin\dfrac{u}{2}}du\,d\psi(r)$$

$$=2\int_{0}^{\pi/n\delta}\int_{\pi/n}^{\delta}\frac{f(t+u)\rho^{n}\sin n\alpha}{u}du\,d\psi(r)+o(1)$$

Hence

$$|I_{21}| \leqq 2\int_0^{\pi/n\delta} |d\psi(r)| \max_{0\leqq r\leqq \pi/n\delta} \int_0^{\delta} \frac{|f(t+u)|\, n(ru+O(ru^3))|}{u} du + o(1)$$

$$\leqq 2\int_0^{\pi/n\delta} |d\psi(r)| \cdot \text{constant} = o(1) ,$$

from the continuity of $\psi(r)$ at $r=0$. Next

$$I_{22} = \int_{\pi/n\delta}^{1} \int_{\pi/n}^{\pi/nr} \frac{f(t+u)\rho^n \sin\left(n\alpha + \dfrac{u}{2}\right)}{\sin \dfrac{u}{2}} du\, d\psi(r)$$

$$= \int_{\pi/n\delta}^{1} \int_{\pi/n}^{\pi/nr} f(t+u)\rho^n \sin n\alpha \cot \frac{u}{2} du\, d\psi(r)$$

$$+ \int_{\pi/n\delta}^{1} \int_{\pi/n}^{\pi/nr} f(t+u)\rho^n \cos n\alpha\, du\, d\psi(r) = I_{221} + I_{222} , \quad \text{say.}$$

Here $|I_{222}| < \varepsilon$, where ε is a positive constant which is arbitrary small according as δ is small.

$$I_{221} = 2\int_{\pi/n\delta}^{1} \int_{\pi/n}^{\pi/nr} \frac{f(t+u)\rho^n \sin n\alpha}{u} du\, d\psi(r) + \varepsilon'$$

where $|\varepsilon'|$ is arbitrary small. Here we put

$$\int_{\pi/n\delta}^{1} \int_{\pi/n}^{\pi/nr} \frac{f(t+u)\rho^n \sin n\alpha}{u} du\, d\psi(r)$$

$$= \left\{ \int_{\pi/n\delta}^{\sigma} \int_{\pi/n}^{\pi/nr} + \int_{\sigma}^{1} \int_{\pi/n}^{\pi/nr} \right\} du\, d\psi(r) = I_{2211} + I_{2212} , \quad \text{say,}$$

where σ is a positive constant arbitrary small. Since $\alpha = O(ru)$ as $r \to 0$ and $u \to 0$,

$$|I_{2211}| \leqq \int_{\pi/nr}^{\sigma} \int_{\pi/n}^{\pi/nr} \frac{|f(t+u)\rho^n \sin n\alpha|}{u} du\, |d\psi(r)|$$

$$\leqq \int_{\pi/n\delta}^{\sigma} \int_{\pi/n}^{\pi/nr} K \cdot nr\, du\, |d\psi(r)| \leqq K\pi \int_0^{\sigma} |d\psi(r)| < \varepsilon$$

for sufficiently small δ and σ, where K is an appropriate constant. Next

$$I_{2212} = \int_{\sigma}^{1} \int_{\pi/n}^{\pi/nr} \frac{f(t+u)\rho^n \sin n\alpha}{u} du\, d\psi(r)$$

$$= \int_{\sigma}^{1} \int_{\pi/n}^{\pi/nr} \frac{f(t+u) \sin n\alpha}{u} du\, d\psi(r) + \varepsilon ,$$

since $\rho^n=1-\lambda nu^2$. On the other hand

$$\int_{\pi/n}^{\pi/nr}\frac{f(t+u)\sin n\alpha}{u}du=o(1)$$

uniformly in r ($\sigma\leqq r\leqq 1$) from the condition (3) and the integration by parts. Therefore $|I_{2212}|$ becomes arbitrary small acording as δ is small and $n\to\infty$. Therefore from the above estimations $|I_{22}|$ may be arbitrary small according as δ is small and $n\to\infty$. Next

$$I_{23}=\int_{\pi/n\delta}^{1}\int_{\pi/nr}^{\delta}\frac{f(t+u)\rho^n\sin\left(n\alpha+\dfrac{u}{2}\right)}{\sin\dfrac{u}{2}}du\,d\psi(r)$$

$$=2\int_{\pi/n\delta}^{1}\int_{\pi/nr}^{\delta}\frac{f(t+u)\rho^n\sin n\alpha}{u}du\,d\psi(r)+\varepsilon$$

for sufficiently sufficiently small δ, where $|\varepsilon|$ is arbitrary small. Here

$$\int_{\pi/n\delta}^{1}\int_{\pi/nr}^{\delta}\frac{f(t+u)\rho^n\sin n\alpha}{u}du\,d\psi(r)$$

$$=\left\{\int_{\pi/n\delta}^{\sigma}\int_{\pi/nr}^{\delta}+\int_{\sigma}^{1}\int_{\pi/nr}^{\delta}\right\}du\,d\psi(r)=I_{231}+I_{232}\,,\quad\text{say.}$$

Here

$$|I_{231}|\leqq\left|\int_{\pi/n\delta}^{\sigma}\int_{\pi/nr}^{\delta}\frac{f(t+u)\rho^n\sin n\alpha}{u}du\,d\psi(r)\right|=o(1)$$

from the condition (4) similarly as the proof of Theorem 7. Finally in I_{232}

$$\int_{\pi/nr}^{\delta}\frac{f(t+u)\rho^n\sin n\alpha}{u}du=o(1)$$

uniformly in r ($\sigma\leqq r\leqq 1-\sigma'$) from the condition (4) similarly as the proof of Theorem 7, and so $I_{232}=o(1)$ from the condition of $\psi(r)$. From the above estimations we get $I_{23}=o(1)$.

This completes the proof of Theorem 9.

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References

[1] W. GIBBS: Fourier series, Nature 59 (1899), p. 606.

[2] H. CRAMÉR: Études sur la sommation des séries de Fourier, Arkiv fur Math., Arkiv fur Math., 13 (1919), pp. 1–21.

[3] T. H. GRONWALL: Zur Gibbschen Erscheinung, Ann. of Math., 31 (1930), pp. 233–240.

[4] B. KUTTNER: Note on the Riesz means of a Fourier series, Journ. of London Math. Soc., 18 (1943), pp. 148–154.

[5] B. KUTTNER: On the Riesz means of a Fouries series (II), Journ. of London Math. Soc., 19 (1944), pp. 77–84.

[6] B. KUTTNER: On the Gibbs phenomenon for Riesz means, Journ. of London Math. Soc., 19 (1944), pp. 153–161.

[7] B. KUTTNER: Note on the Gibbs phenomenon, Journ. of London Math. Soc., 20 (1945), pp. 136–139.

[8] B. KUTTNER: A Further note on the Gibbs phenomenon, Journ. of London Math. Soc., 22 (1947), pp. 295–298.

[9] O. SZÁSZ: On the Gibbs phenomenon for Euler means, Acta Sci. Math., 12 (1950), pp. 107–111.

[10] O. SZÁSZ: Gibbs' phenomenon for Hausdorff means, Trans. Amer. Math. Soc., 69 (1950), pp. 440–456.

[11] O. SZÁSZ: On the Gibbs phenomenon for a class of linear transform, Institut Math. de l'Acad. Serbe des Sci., 4 (1952), pp. 135–144.

[12] M. CHENG: The Gibbs phenomenon and Bochner's summation method (I), Duke Math. Journ., 17 (1950), pp. 83–90.

[13] L. LORCH: The Gibbs phenomenon for Borel means means, Proc. Amer. Math. Soc., 8 (1957), pp. 81–84.

[14] S. IZUMI and M. SATÔ: Fourier series x, Rogosinski's lemma, Kodai Math. Sem. Report 8 (1956), pp. 164–180.

[15] S. IZUMI and M. SATÔ: The Gibbs phenomenon of partial sums and Cesàro means of Fourier series, Proc. Japan Acad. 33 (1957), pp. 284–292.

[16] K. ISHIGURO: Fourier series XI, Gibbs' phenomenon, Kodai Math. Sem. Report 8 (1956), pp. 181–188. Correction to this paper: ibid. 9 (1957), pp. 191–192.

[17] K. ISHIGURO: Fourier series XV, Gibbs' phenomenon, Proc. Japan Aead. 33 (1957), pp. 119–123.

[18] L. CHING-HSI: On the Gibbs phenomenon for the Riemann summation (R, 1) of Fourier series, Acta Math. Sinica 6 (1956), pp. 418–425.

[19] A. E. LIVINGSTON: Some Hausdorff means which exhibit Gibbs' phenomenon, Pacific Journ. 3 (1953) pp. 407–415.

[20] K. ISHIGURO: Gibbs' phenomenon for circle means, to be pressed.

[21] G. H. HARDY: Divergent series, Oxford (1949).

[22] A. ZYGMUND: Trigonometric series I, Cambridge (1959).