PARTIALLY ORDERED ABELIAN SEMIGROUPS. III

ON THE REVERSIBLE PARTIAL ORDER DEFINED ON AN ABELIAN SEMIGROUP

By

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In their paper,[1] Ben Dushnik and E. W. Miller introduced the concept of the reversible partial order and expressed the theorem about this concept. In this Part III, I shall show that the same one is held in the partially ordered abelian semigroup by adding the certain condition.

Definition 1. A set S is said to be a *partially ordered abelian semigroup* (p.o. semigroup), when S is (I) an abelian semigroup (not necessarily contains the unit element), (II) a partially ordered set, and satisfies (III) the *homogeneity*: $a \geqq b$ implies $ac \geqq bc$ for any c of S.

A partial order which satisfies the condition (III) is called a *partial order defined on an abelian semigroup.*

Moreover, if a partial order defined on an abelian semigroup S is a linear order, then S is said to be a *linearly ordered abelian semigroup* (l.o. semigroup). (Definition 1, O.I.)

Definition 2. Let $\mathfrak{S}=\{P_\alpha\}$ be any set of partial orders, each defined on the same abelian semigroup S. We define the new partial order P on S as follows: For any two elements a, b, we put $a \geqq b$ in P if and only if $a \geqq b$ in every P_α of the set $\mathfrak{S}$. Indeed, P is again a partial order defined on S. This partial order P is said to be the *product* of the partial orders P_α or to be *realized* by the set $\mathfrak{S}$ of partial orders P_α. (Definition 9, O.I.)

By the *dimension* of a partial order P defided on an abelian semigroup S is meant the smallest cardinal number $\mathfrak{m}$ such that P is realized by $\mathfrak{m}$ linear orders defined on S.

Definition 3. Let P and Q be two partial orders defined on the same

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Partially ordered abelian semigroup. I. On the extension of the strong partial order defined on abelian semigroups. Journ. Fac. Sci., Hokkaido University, Series I, vol. XI (1951), pp. 181–189; this is referred to hereafter as "O.I."

1) Ben Dushnik and E. W. Miller: Partially ordered sets, Amer. Math. Journ. vol. 63 (1941), pp. 600–610.

abelian semigroup S, and suppose that *any* two distinct elements of S are comparable in *just one* of these partial orders; in such a case we shall say that P and Q are *conjugate* partial orders. A partial order will be called *reversible* if and only if it has a conjugate.[2)]

If P is a partial order defined on S, then the partial order obtained from P by inverting the sense of all ordered pairs will be called a *dual* order, which is denoted by P^*.

Theorem 1. *Let P and Q are conjugate partial orders defined on an abelian semigroup S. Then we can define a linear order L_1 on S such that $a>b$ in L_1 if and only if $a>b$ in either P or Q; denoted by $L_1=P+Q$. Similary $L_2=P+Q^*$ is a linear order defined on S.*

Proof. We shall prove only the transivity of L_1.

From $a>b$, $b>c$ in L_1, we can consider the following four cases: (i) $a>b$, $b>c$ in P, (ii) $a>b$, $b>c$ in Q, (iii) $a>b$ in P, $b>c$ in Q, (iv) $a>b$ in Q, $b>c$ in P.

In cases (i) and (ii), $a>c$ in L_1 is cleary. In case (iii), if $c>a$ in P or Q, then $c>b$ in P or $b>a$ in Q respectively, which is absured, therefore $a>c$ in P or Q and hence in L_1. Similary, in case (iv) $a>c$ in L_1 is held.

Theorem 2. *The following two properties of a partial order P defined on an abelian semigroup S are equivalent to each other:*

(1) *P is reversible.*

(2) *The dimension of P is 2.*

Proof. We shall show first that (1) implies (2). Suppose that the partial order P defined on S is reversible, and let Q be a partial order defined on S conjugate to P and Q^* be the dual order of Q. Then by Theorem 1, $L_1=P+Q$ and $L_2=P+Q^*$ are linear extensions of P and it is obvious that P is realized by linear orders L_1 and L_2.

Next we show that (2) implies (1). Let L_1 and L_2 be any two linear orders defined on S which together realize P. We define the other order Q as follows: $a>b$ in Q if and only if a and b are non-comparable in P and $a>b$ in L_1 (likewise $a>b$ in L_1 and $b>a$ in L_2). Then $a>b$ and $b>a$ in Q are contradictory. If $a>b$ and $b>c$ in Q, then we have $a>b>c$ in L_1 and $c>b>a$ in L_2, hence $a>c$ in Q. $a>b$ in Q implies that $ac\geqq bc$ in L_1 and $bc\geqq ac$ in L_2, i.e. $ac\geqq bc$ in Q. Therefore Q is a partial order defined on S. Evidently P and Q are conjugate.

Definieion 4. A *linear* extension L of a partial order P defined on

2) Cf. Ben Dushnik and E. W. Miller: l.c.

an abelian semigroup S will be called *separating* if and only if there exist three elements a, b and c in S such that $a>c$ in P, and b is not comparable with either a or c in P, while in L we have $a>b>c$.

Theorem 3.[3] *Let P be a partial order defined on an abelian semigroup S which satisfies the condition (E).*[4] *Then the following three properties of a partial order P are equivalent to each other:*

(1) *P is reversible.*

(2) *The dimension of P is 2.*

(3) *There exists a linear extension of P which is non-separating.*

Proof. (1) and (2) are equivalent by Theorem 2.

We show now that (2) implies (3) without the condition (E). Let L_1 and L_2 be any two linear orders defined on S which together realize $P: P=L_1\times L_2$. If L_1 is separating, then there exist three elements a, b and c such that $a>c$ in P, $a>b>c$ in L_1 and b is not comparable with either a or c in P. Hence we have $c>b>a$ in L_2 which is impossible.

To show that (3) implies (1) we shall suppose that L is a non-separating linear extension of P. We define the other order Q as follows: $a>b$ in Q if and only if a and b are non-comparable in P and $a>b$ in L. Then clearly $a>b$ and $b>a$ in Q are contradictory. If $a>b$ and $b>c$ in Q, then we have $a>b>c$ in L and a and c are non-comparable in P, for otherwise $a>c$ in P would imply that L is separating contrary to the assumption, hence we have $a>c$ in Q. $a>b$ in Q implies that $ac\geqq bc$ in L. If $ac>bc$ in P, then by the condition (E) $a>b$ in P which is impossible. Hence $ac=bc$ or ac and bc are non-comparable in P, and hence $ac\geqq bc$ in Q. Therefore Q is a partial order defined on S. Clearly P and Q are conjugate.

Definition 5. Let S be a p.o. semigroup and P be the partial order defined on S. For any element a of S, we denote the set of all elements x such that $x\leqq a$ in P by $\bar{a}$. Then the correspondence $a\leftrightarrow\bar{a}$ is one-to-one. We put $\bar{a}\geqq\bar{b}$ if and only if $\bar{b}$ is a subset of $\bar{a}$, likewise $a\geqq b$ in P, then the family $\bar{S}=\{\bar{a}\}$ is become a partially ordered set. Next we define the product $\bar{a}\cdot\bar{b}=\overline{ab}$, then the family $\bar{S}$ is a commutative semigroup, moreover $\bar{S}$ become a p.o. semigroup. Clearly S and $\bar{S}$ are order-isomorphic.[5]

More generally, if there exists a one-to-one correspondence between

3) Ben Dushnik and E. W. Miller: l.c. Theorem 3.61.

4) Condition (E) (order cancellation law):

$$ac>bc \text{ in } P \text{ implies } a>b \text{ in } P.$$

5) See Definition 3, O.I.

the elements of the p.o. semigroup S and the family $\mathfrak{R}$ of subsets of the certain set R (a subset of R which corresponds with an element a of S, denote by $s(a)$), and $a \geqq b$ in P if and only if $s(a) \supseteqq s(b)$ (in the sense of set-inclusion), then by the defining the product $s(a) \cdot s(b) = s(ab)$, two p.o. semigroups S and $\mathfrak{R}$ are order-isomorphic.

Any family $\mathfrak{R}$ of the subsets of the set R which has the above properties will be called a *representation* of P.

Theorem 4. *Let P be a partial order defined on an abelian semigroup S which satisfies the condition (E). Then the following two proparties are equivalent to each other:*

(1) *P is reversible.*

(4) *There exists a representation of P by means of a family $\mathfrak{R} = \{I_a\}$ of closed intervals on some l.o. semigroup R, and let $I_a = [\alpha_1, \alpha_2]$, $I_b = [\beta_1, \beta_2]$, $I_{ac} = [\gamma_1, \gamma_2]$, $I_{bc} = [\delta_1, \delta_2]$, and if a and b are non-comparable in P, then $\alpha_1 < \beta_1$ (and $\alpha_2 < \beta_2$) implies $\gamma_1 \leqq \delta_1$ (and $\gamma_2 \leqq \delta_2$) or its dual.*

Proof. We shall show (1) implies (4). Let P be reversible, and hence the dimension of P is 2. Let A and B be any two linear orders defined on S which together realize P.

Let S' be a l.o. semigroup which is anti-order-isomorphic to the l.o. semigroup S in the linear order B, where the set S' is disjoint from S, and the linear order defined on S' is denoted by B'.

Let R be the union of S, S' and the new element 0 which belongs to neigther S nor S'.

We define the multiplication in R as follows:

$$0 \cdot 0 = 0,$$

$$x \cdot 0 = 0 \cdot x = 0 \qquad \text{for any } x \text{ in } S \text{ or } S',$$

$$a \cdot a' = a' \cdot a = 0 \qquad \text{for any } a \text{ in } S \text{ and } a' \text{ in } S',$$

and for any two elements x and y of $S(S')$ the product is the same as in $S(S')$.

Thus R becomes the abelian semigroup under the multiplication introduced above.

Let us now define the order-relation L in R as follows:

$$x > y \text{ in } L \ (x, y \in S) \quad \text{if and only if } x > y \text{ in } A,$$

$$x > y \text{ in } L \ (x, y \in S') \quad \text{if and only if } x > y \text{ in } B',$$

and we put

$$a > 0 > a' \text{ in } L \ (a \in S,\ a' \in S').$$

Then R becomes a l.o. semigroup.

For each a in S denote by a' the image of a in S', and denote by I_a the closed interval $[a', a]$ of R.

We will show that the family $\mathfrak{R}=\{I_a\}$ of all such intervals is a representation of P. Suppose first that $a>b$ in P. Then $a>b$ in A and $a'<b'$ in B', so that we have $a'<b'<b<a$ in L. This means that I_b is a proper subset of I_a.

Let $I_a=[a', a]$, $I_b=[b', b]$, $I_{ac}=[a'c', ac]$, $I_{bc}=[b'c', bc]$. If a and b are non-comparable in P, then from $a>b$ $(a'>b')$ in L we have $ac\geqq bc$ $(a'c'\geqq b'c')$ in L or its dual.

We prove that (4) implies (1). Suppose that P is a partial order which is represented by a family $\mathfrak{R}$ of intervals taken from some l.o. semigroup R, whose linear order is denoted by L. For each a in S, denote by I_a the interval of the family $\mathfrak{R}$ which corresponds to a. We notice first that if a and b are distinct elements of S which are not comparable in P, then I_a and I_b cannot have the same left (right)-hand end-point.

Suppose that $I_a=[\alpha_1, \alpha_2]$, $I_b=[\beta_1, \beta_2]$, $I_c=[\gamma_1, \gamma_2], \cdots$.

We define a new partial order Q defined on S as follows:

(i) a and b are not comparable in P,

(ii) $\alpha_1<\beta_1$ (and $\alpha_2<\beta_2$) in L.

It is easy to see that Q is the partial order defined on the set S. We shall now prove the homogeneity. Let $a>b$ in Q and $I_{ac}=[\lambda_1, \lambda_2]$, $I_{bc}=[\mu_1, \mu_2]$. If ac and bc are distinct and comparable in P, then by the condition (E) a and b are comparable in P which is impossible. If ac and bc are non-comparable in P, then $\lambda_1<\mu$, $\lambda_2<\mu_2$ and hence $ac>bc$ in Q.

Example 1. Let S_1 be an abelian semigroup generated by two elements a and b with the relation

$$a^m b^n = ab^n \quad \text{for any positive integers } m \text{ and } n.$$

By putting the order-relation

$$P: \quad \begin{cases} a^{m+1}>a^m \\ b^n>ab^n \end{cases} \quad \text{for any positive integer } n$$

S_1 becomes a p.o. semigroup, and the partial order P is reversible. Its conjugate order Q is as follows:

$$Q: \quad \begin{cases} a^m>b^n>b^{n+1}>ab^{n+1} \\ a^m>ab^n>b^{n+1} \end{cases} \quad \text{for any positive integers } m \text{ and } n.$$

The linear orders which together realize P are

$$a^{m+1} > a^m > b^n > ab^n > b^{n+1} > ab^{n+1}$$

and

$$b^{n+1} > ab^{n+1} > b^n > ab^n > a^{m+1} > a^m$$

for any positive integers m and n.

Example 2. Let S_2 be an abelian semigroup generated by two elements a and b with the relation

$$a^m b^n = b^n \quad \text{for any positive integers } m \text{ and } n.$$

By putting the two order-relations A and B

A: $\quad a^{m+1} > a^m > b^n > b^{n+1}$

B: $\quad b^{n+1} > b^n > a^{m+1} > a^m$

for any positive integers m and n,

S_2 becomes a l.o. semigroup in the orders A and B respectively. Let P be the partial order which is the product of A and B, that is

P: $\quad a^{m+1} > a^m$.

Then P has the conjugate order Q such that

Q: $\quad a^m > b^n > b^{n+1}$.

Example 3. Let S_3' and S_3'' be free abelian semigroups generated by elements a and b respectively. And by defining the order-relations

$$a^{m+1} > a^m \quad (m \geqq 1), \qquad b^{n+1} > b^n \quad (n > 1),$$

S_3' and S_3'' becomes a l.o. semigroup and a p.o. semigroup respectively. Let S_3 be the direct product of S_3' and S_3''. Then S_3 becomes a p.o. semigroup by introducing the following order-relation P:

$$a^i b^j > a^m b^n$$

if and only if

$$a^i > a^m \quad \text{or} \quad a^i = a^m \quad \text{and} \quad b^j > b^n.$$

Since ab^2 and ab are non-comparable in P in spite of $(ab^2)(ab) = a^2b^3 > a^2b^2 = (ab)(ab)$, P does not satisfy the condition (E).

Now, in S_3'' we define the another order-relation:

$$b^{n+1} > b^n \quad (n \geqq 1),$$

then we get the non-separating linear extension of P.

But we cannot realize the partial order P by two linear orders.

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