ON GROUPS OF ROTATIONS IN MINKOWSKI SPACE I

By

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§ 1. **Introduction.** Let F_{n+1} be an $n+1$-dimensional Finsler space with the fundamental function $F(x^i, x'^i)$ $(i=1, 2, \cdots, n+1)$.[1] At every point with coordinate (x_0^i) in F_{n+1}, we obtain an $n+1$-dimensional Minkowski space M_{n+1}, whose indicatrix I_n is given by the end points of the vectors (X^i)'s at the origin (x_0^i) satisfying the equation

$$g_{ij}(x_0, X)X^iX^j=1 \quad \left(g_{ij}(x, X)=\frac{1}{2}\partial^2F(x, X)\Big/\partial X^i\partial X^j\right). \tag{1.1}$$

At any point (x_0^i), the quadratic form for any fixed vector X_0^i:

$$g_{ij}(x_0, X_0)X^iX^j=1 \tag{1.2}$$

defines a hyperquadric I_n^* which is in double contact with I_n at two points of coordinates (X_0^i) and $(-X_0^i)$ respectively.

L. Berwald [5],[2] E. Cartan [6] and many others regarded a Finsler space as a space of line-elements (x^i, x'^i). From this point of view, we can obtain for each line-element $(x_0^i, x_0'^i)$ an $n+1$-dimensional tangent Euclidean space $E_{n+1}(x_0^i, x_0'^i)$ whose indicatrix is a hyperquadric I_n^* determined by (1.2) by putting $X_0^i=x_0'^i$. Under this consideration the connection in F_{n+1} was established by defining a suitable correspondence between neighbouring tangent Euclidean spaces $E_{n+1}(x^i, x'^i)$ and $E_{n+1}(x^i+dx^i, x'^i+dx'^i)$.

Recently W. Barthel [1]-[4], A. Kawaguchi [9], D. Laugwitz [10] and H. Rund [11] reconstructed the foundation of the theory of Finsler space from the stand point that the Finsler space is a point space but is not a line-element space, that is to say, the tangent space at each point (x_0^i) in F_{n+1} should be regarded as a Minkowski space with an Indicatrix determined by $F(x_0^i, X^i)=1$. On account of this fact, in order to establish the theory of Finsler space, it becomes an important problem to study

1) For the sake of convenience, we suppose that the dimension of a Finsler space is $n+1$, because in the present paper we shall discuss mainly about the theory of transformations in an n-dimensional indicatrix I_n.

2) Numbers in brackets refer to the references at the end of the paper.

the properties of Minkowski space. However since the foundation of the theory of Minkowski space depends on the structure of the indicatrix, it is necessary for us to study the properties of indicatrix in Minkowski space. From this point of view, A. Kawaguchi [9] and T. Sumitomo [12] developed the reduction theorems of Finsler space, namely, they gave various conditions by which the Finsler space can be regarded essentially as a Riemannian space and also gave interesting geometrical meanings of them. One of those results showed us that a Lie group of *rotations* is intransitive on I_n and if it is transitive, the Minkowski space M_{n+1} is to be euclidean in essential and then the considered Finsler space F_{n+1} may be regarded as a Riemannian space, where the *rotation* means a centro-affine transformation in M_{n+1} which leaves invariant the indicatrix I_n.

The present author wishes to study more precisely the properties of a Lie group of rotations. As will be shown in § 3, I_n may be regarded as an n-dimensional compact Riemannian space whose metric tensor is naturally induced from the metric of M_{n+1} and it is remarkable that the Riemannian space admits a symmetric covariant tensor of order three. In the present paper we shall study, in an n-dimensional compact Riemannian space I_n, the properties of point transformations which are induced on I_n by rotations in M_{n+1}.

In § 2 the fundamental concepts of Minkowski space will be given and we shall give a definition of a rotation in M_{n+1}. § 3 devoted to show some characters of an n-dimensional compact Riemannian space I_n. In § 4, we will show that an infinitesimal point transformation $\bar{u}^\alpha = u^\alpha + \eta^\alpha(u)\delta t$ coincides with the transformation induced on I_n by an infinitesimal rotation in M_{n+1} when the condition $\mathfrak{L} g_{\alpha\beta} = 2\eta^\gamma A_{\alpha\beta\gamma}$ be satisfied, where $\mathfrak{L}$ denotes Lie differential with respect to the above stated infinitesimal transformation and $A_{\alpha\beta\gamma}$ is a component of a symmetric covariant tensor defined in I_n. Since Lie differential is a main tool in our discussion, § 5 devoted to give some formulas on Lie derivatives and several fundamental relations which will be useful in the following discussions.

When an indicatrix admits an intransitive group of rotation, it becomes an interesting problem to study the connection between structures of the indicatrix and those of the intransitive group. An example will be given in § 6 to show some those connections. The integrability conditions of the infinitesimal point transformation, induced on I_n by an infinitesimal rotation in M_{n+1}, are given in § 7. § 8 contains some theorems concerning with properties of infinitesimal transformations in I_n, and also several characters of Lie group of the transformations will be given at the end of § 8.

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§ 2. **Preliminaries and definition of rotation in M_{n+1}.** Minkowski space M_{n+1} is an $n+1$-dimensional affine space in which the distance between two points $P=(P^1, P^2, \cdots, P^{n+1})$ and $Q=(Q^1, Q^2, \cdots, Q^{n+1})$ be determined by $F(Q-P)$. In the present paper we put the following assumptions about the function $F(X) \equiv F(X^1, X^2, \cdots, X^{n+1})$:

(I) $F(X)>0$ for $X \neq 0$,

(II) $F(X)=F(-X)$,

(III) $F(\rho X)=\rho F(X)$ for $\rho>0$,

(IV) $F(X)+F(Y)>F(X+Y)$ for linearly independent vectors X and Y,

(V) $F(X)$ is continuous and continuously differentiable sufficiently many times,

(VI) regular, i.e. the matrix of

$$g_{jk}(X)=\frac{\partial^2 L}{\partial X^j \partial X^k} \quad (j, k=1, 2, \cdots, n+1)^{3)}$$

has rank $n+1$, where $L=\frac{1}{2}F^2$.

In M_{n+1} the indicatrix is given by the equation

$$F(X^1, X^2, \cdots, X^{n+1})=1, \tag{2.1}$$

namely, I_n is an n-dimensional hypersurface consisting of all points $X=(X^1, X^2, \cdots, X^{n+1})$ which satisfy (2.1), where X^i be regarded as coordinates of the end point of the radius vector at the origin $O=(0, 0, \cdots, 0)$ of the coordinate system.

Let us consider any centro-affine transformation A_0 in M_{n+1}, whose centre coincides with the origin $O=(0, 0, \cdots, 0)$, i.e.

$$A_0 : \bar{X}^i=a^i_j X^j \quad (a^i_j=\text{const. and det.} \, |a^i_j| \neq 0)\,. \tag{2.2}$$

If the relation $F(\bar{X})=F(X)$ holds good for any vectors X^i and $\bar{X}^i$, that is to say, the indicatrix I_n remains unaltered under a transformation A_0,

3) Throughout the present paper the Latin indices $i, j, k, \cdots$ are supposed to run over the range $1, 2, \cdots, n+1$.

we call the transformation the *rotation* in M_{n+1}. Denoting by

(2.3) $$\bar{X}^i=(\delta^i_j+\xi^i_j\delta t)X^j=X^i+\xi^i_jX^j\delta t$$

the infinitesimal transformation of the centro-affine transformation A_0, we find that $\xi^i_jX^j$ is a vector which gives a general infinitesimal transformation of A_0.

Now, putting $C_{ijk}(X)=\frac{1}{2}\frac{\partial g_{ij}}{\partial X^k}$, C_{ijk} is a component of a symmetric covariant tensor. Then, as we have

$$\left\{ {i \atop jk} \right\}=\frac{1}{2}g^{ih}\left(\frac{\partial g_{hk}}{\partial X^j}+\frac{\partial g_{jh}}{\partial X^k}-\frac{\partial g_{jk}}{\partial X^h}\right)=C^i_{\cdot jk},$$

we can define a covariant derivative of a tensor $T^i_{\cdot jk}$ by

(2.4) $$T^i_{\cdot jk|l}=\frac{\partial T^i_{\cdot jk}}{\partial X^l}+T^h_{\cdot jk}C^i_{\cdot hl}-T^i_{\cdot hk}C^h_{\cdot jl}-T^i_{\cdot jh}C^h_{\cdot kl}.$$

Making use of the covariant derivative, we can obtain the Lie derivative of a tensor $T^i_{\cdot jk}$ with respect to an infinitesimal transformation $\bar{X}^i=X^i+\xi^i(X)\delta t$ as follows:

(2.5) $$XT^i_{\cdot jk}=T^i_{\cdot jk|l}\xi^l-T^l_{\cdot jk}\xi^i_{|l}+T^i_{\cdot lk}\xi^l_{|j}+T^i_{\cdot jl}\xi^l_{|k}.$$

Accordingly, for a generating vector $\xi^i_jX^j$ we have from (2.4)

(2.6) $$(\xi^i_jX^j)_{|l}=\xi^i_l+\xi^h_jX^jC^i_{\cdot hl}.$$

From the definition, rotation in M_{n+1} is characterized by the equation $Xg_{ij}=0$. Making use of (2.5) and (2.6) we find that

(2.7) $$Xg_{ij}=g_{ki}\xi^k_j+g_{kj}\xi^k_i+2C_{ijk}\xi^k_lX^l=0$$

must be satisfied for an infinitesimal rotation.

The last equation is also obtained by the following way. By virtue of the assumptions (III), (IV) and (V), we can see that the covariant vector $X_j=g_{jk}(X)X^k$ represents a hyperplane which is parallel to a hyperplane tangent to the indicatrix I_n at the point (X^i). On the other hand if we put

(2.8) $$Y_k\equiv\frac{1}{F}\frac{\partial F}{\partial X^k}=\frac{1}{2L}g_{kj}X^j,$$

we find that the equation of the indicatrix I_n can be expressed by $X^kY_k=1$, where X^k denotes current coordinate. Then if the point (X^i) lies on the indicatrix, (2.8) gives us the relation

(2.9) $$Y_k=X_k=\frac{\partial F}{\partial X^k}.$$

According to the definition of rotation, $F(X)$ is an absolute invariant of an infinitesimal transformation (2.3). Then, by means of (2.9) it should be satisfied that

$$XF=\xi_l^k X^l X_k = g_{km}\xi_l^k X^l X^m = 0\,. \tag{2.10}$$

Since $g_{ij}(X^h)$ is homogeneous of degree zero with respect to X^h, $C_{ijk}(X)X^k=0$ holds good. Therefore, differentiating the relation (2.10) with respect to X^i and X^j, we obtain (2.7).

As $C_{ijk}(X)$ is a symmetric tensor, by means of (2.4)–(2.7) we can easily see that

$$XC^i_{\cdot jk}=\frac{1}{2}g^{ih}\{(Xg_{jh})_{|k}+(Xg_{hk})_{|j}-(Xg_{jk})_{|h}\}$$

Therefore, if $Xg_{hj}=0$ holds good, we have $XC_{ijk}=0$.

Theorem 2.1. *An infinitesimal rotation leaves invariant the symmetric tensor* C_{ijk}.

A set of all rotations forms a subgroup of a group of centro-affine transformations. If the indicatrix I_n admits an r-parameter Lie group G_r of the rotations, its point is a fixed point of G_r or lies on one of the family of affine W-submanifolds determined by G_r. According to the remarkable fact obtained by A. Kawaguchi [9], if G_r is transitive on the indicatrix, it is an affine W-hypersurface and because of our assumptions stated in § 2, the indicatrix must be a hyper-ellipsoid. Then, the considered Minkowski space is to be euclidean in essential. This is the fundamental conclusion for our following discussions.

§ 3. Indicatrix of Minkowski space. For the convenience of our discussion, following to the study of A. Kawaguchi [9] we shall give some fundamental properties of Riemannian space I_n. As indicatrix I_n is an n-dimensional hypersurface in an $n+1$-dimensional Minkowski space M_{n+1}, we may express the indicatrix I_n by $n+1$ equations involving n parameters as

$$X^i=X^i(u^\alpha)\,. \quad (i=1,2,\cdots,n+1\,;\ \alpha=1,2,\cdots,n)^{4)} \tag{3.1}$$

This means that we can regard I_n as an n-dimensional manifold with coordinate system (u^α). According to (2.9), if the point (X^i) lies on I_n, the equation of I_n is reduced to $X_iX^i=1$. Thus we get

4) Throughout the present paper, the Greek indices $\alpha, \beta, \gamma, \cdots$ take the values 1, 2, $\cdots, n$.

(3.2) $$X_i X^i_\alpha = 0 . \quad \left(X^i_\alpha \equiv \frac{\partial X^i}{\partial u^\alpha} , \ \mathrm{Rank}\left(\frac{\partial X^i}{\partial u^\alpha}\right) = n \right)$$

From the assumptions (I), (III), (V) and (VI), if we put $g_{\alpha\beta}(u) \equiv g_{ij}(X) X^i_\alpha X^j_\beta$, the indicatrix I_n may be regarded as an n-dimensional compact Riemannian space whose fundamental metric tensor is $g_{\alpha\beta}(u)$.

In consequence of (3.2), $n+1$ vectors X^i_α and X^i are linearly independent and they form an ennuple (X^i_α, X^i) at every point on I_n. As usual, putting

(3.3) $$\frac{1}{F} C_{ijk} = \frac{1}{2F} \frac{\partial g_{ij}}{\partial X^k} \equiv A_{ijk} ,$$

it is evident that $C_{ijk} = A_{ijk}$ holds good at every point on I_n. If we denote by $A_{\alpha\beta\gamma}$ the induced quantity of the symmetric tensor A_{ijk}, then it follows that

(3.4) $$A_{\alpha\beta\gamma} = A_{ijk} X^i_\alpha X^j_\beta X^k_\gamma , \quad A_{ijk} = A_{\alpha\beta\gamma} X^\alpha_i X^\beta_j X^\gamma_k ,$$

where $X_h^\delta = X_{h\varepsilon} g^{\varepsilon\delta}$ and $X_{h\varepsilon} = g_{hl} X^l_\varepsilon$.

Denoting by $X^i_{\alpha\beta}$ and $X_{i\alpha\beta}$ the covariant derivatives of X^i_α and $X_{i\alpha}$ with respect to the Christoffel symbols constructed by $g_{\alpha\beta}$, we have the following decompositions with respect to the ennuple (X^i_α, X^i):

(3.5) $$X^i_{\alpha\beta} = -A_{\alpha\beta}^{\cdot\cdot\delta} X^i_\delta - g_{\alpha\beta} X^i ,$$

(3.6) $$X_{i\alpha\beta} = -A_{\alpha\beta\delta} X_i^\delta - g_{\alpha\beta} X_i ,$$

where $A_{\alpha\beta}^{\cdot\cdot\delta} = A_{\alpha\beta\gamma} g^{\gamma\delta}$.

Making use of (3.5), by means of the Ricci's identity $X^i_{\beta\gamma\delta} - X^i_{\beta\delta\gamma} = X^i_\alpha R^{\alpha}_{\cdot\beta\gamma\delta}$, we can easily obtain

(3.7) $$R_{\alpha\beta\gamma\delta} = S_{\alpha\beta\gamma\delta} + (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) ,$$

where $R_{\alpha\beta\gamma\delta} = g_{\alpha\varepsilon} R^{\varepsilon}_{\cdot\beta\gamma\delta}$ and

$$R^{\alpha}_{\cdot\beta\gamma\delta} = \frac{\partial \left\{ {\alpha \atop \beta\delta} \right\}}{\partial u^\gamma} - \frac{\partial \left\{ {\alpha \atop \beta\gamma} \right\}}{\partial u^\delta} + \left\{ {\varepsilon \atop \beta\delta} \right\} \left\{ {\alpha \atop \varepsilon\gamma} \right\} - \left\{ {\varepsilon \atop \beta\gamma} \right\} \left\{ {\alpha \atop \varepsilon\delta} \right\} ,$$

$$S_{\alpha\beta\gamma\delta} = A_{\alpha\delta}^{\cdot\cdot\sigma} A_{\sigma\gamma\beta} - A_{\alpha\gamma}^{\cdot\cdot\sigma} A_{\sigma\delta\beta} .$$

It is remarkable that, by virtue of (3.3) and (3.4), $A_{\alpha\beta\gamma}$ is a component of a symmetric tensor in I_n and also it satisfies the relation

(3.8) $$A_{\alpha\beta\gamma;\delta} = A_{\alpha\beta\delta;\gamma} .$$ [5]

Summarizing the preceding results, since I_n is closed, we can regard the indicatrix I_n as an n-dimensional compact Riemannian space, provided

5) Throughout the present paper, semi-colon is used to represent covariant differentiation with respect to the Christoffel symbols made by $g_{\alpha\beta}$.

that there exists a symmetric tensor $A_{\alpha\beta\gamma}$ which satisfies the relation (3.8), and the curvature tensor has the form (3.7).

§4. **Infinitesimal rotation on I_n.** According to the definition given in §2, the rotation transforms any point on the indicatrix I_n onto the point on the same. Now, we shall seek the condition under which an infinitesimal point transformation $\bar{u}^\alpha = u^\alpha + \eta^\alpha(u)\delta t$ on I_n coincides with rotation in M_{n+1}.

In consequence of (2.10), it follows that a direction of the generating vector $\xi^i_j X^j$ of an infinitesimal rotation (2.3) is contained in a tangent hyperplane of I_n at a point (X^i). Hence we may put a component of the vector $\xi^i_j X^j$ such that

$$\xi^i_j X^j = \eta^\alpha(u) X^i_\alpha \,. \tag{4.1}$$

Differentiating the equation $\xi^i_j X^j X_i = 0$ with respect to u^α, it becomes

$$X_{i\alpha}\xi^i_j X^j + X_i \xi^i_j X^j_\alpha = 0 \,. \tag{4.2}$$

Moreover if we covariantly differentiate (4.2) with respect to u^β, we get

$$X_{i\alpha\beta}\xi^i_j X^j + X_i \xi^i_j X^j_{\alpha\beta} + X_{i\alpha}\xi^i_j X^j_\beta + X_{i\beta}\xi^i_j X^j_\alpha = 0 \,.$$

However, substituting (3.5) and (3.6), in consequence of (2.10) and (4.2) it is easily seen that $X_{i\alpha\beta}\xi^i_j X^j + X_i \xi^i_j X^j_{\alpha\beta} = 0$ holds good. Therefore, we can reduce the above expression in the form

$$X_{i\alpha}(\xi^i_j X^j)_{;\beta} + X_{i\beta}(\xi^i_j X^j)_{;\alpha} = 0 \,. \tag{4.3}$$

Making use of (4.1), we have from (4.3)

$$X_{i\alpha}\eta^\gamma_{;\beta} X^i_\gamma + X_{i\alpha}\eta^\gamma X^i_{\gamma\beta} + X_{i\beta}\eta^\gamma_{;\alpha} X^i_\gamma + X_{i\beta}\eta^\gamma X^i_{\gamma\alpha} = 0 \,. \tag{4.4}$$

On the other hand, by means of $g_{\alpha\beta}(u) = g_{ij}(X) X^i_\alpha X^j_\beta$ and (3.5), it is easily verified that

$$X_{i\alpha} X^i_\beta = g_{\alpha\beta} \,, \quad X_{i\alpha} X^i_{\beta\gamma} = -A_{\alpha\beta\gamma} \,.$$

In consequence of these relations, we get from (4.4) $\eta_{\alpha;\beta} + \eta_{\beta;\alpha} = 2\eta^\gamma A_{\alpha\beta\gamma}$. Therefore we have

Theorem 4.1. *If an infinitesimal point transformation $\bar{u}^\alpha = u^\alpha + \eta^\alpha(u)\delta t$ on I_n coincides with a rotation in M_{n+1}, it must be satisfied that*

$$\eta_{\alpha;\beta} + \eta_{\beta;\alpha} = 2\eta^\gamma A_{\alpha\beta\gamma} \,. \tag{4.5}$$

When an infinitesimal point transformation on I_n satisfies (4.5), we call it the *infinitesimal rotation on I_n*.

§ 5. **Fundamental relations and formulas on Lie derivatives.** In the following we give some fundamental formulas on Lie derivatives, which will be useful in the discussions of the present paper, and derive several relations for an infinitesimal rotation on I_n.

Denoting by $\mathfrak{L}T^{\cdot\lambda}_{\cdot\mu\nu}$ the Lie derivatives of an arbitrary tensor $T^{\cdot\lambda}_{\cdot\mu\nu}$ with respect to an infinitesimal transformation $\overline{u}^\alpha=u^\alpha+\eta^\alpha(u)\delta t$, it is given by the form

$$\mathfrak{L}T^{\cdot\lambda}_{\cdot\mu\nu}=T^{\cdot\lambda}_{\mu\nu;\alpha}\eta^\alpha-T^{\cdot\alpha}_{\cdot\mu\nu}\eta^\lambda_{;\alpha}+T^{\cdot\lambda}_{\cdot\alpha\nu}\eta^\alpha_{;\mu}+T^{\cdot\lambda}_{\cdot\mu\alpha}\eta^\alpha_{;\nu}\,. \tag{5.1}$$

In general, we have

$$\mathfrak{L}\left\{{}^{\alpha}_{\beta\gamma}\right\}=\frac{1}{2}g^{\alpha\delta}[(\mathfrak{L}g_{\delta\beta})_{;\gamma}+(\mathfrak{L}g_{\delta\gamma})_{;\beta}-(\mathfrak{L}g_{\beta\gamma})_{;\delta}]\,, \tag{5.2}$$

$$\mathfrak{L}\left\{{}^{\alpha}_{\beta\gamma}\right\}=\eta^\alpha_{;\beta;\gamma}-R^{\cdot\alpha}_{\cdot\beta\gamma\delta}\eta^\delta\,, \tag{5.3}$$

$$\mathfrak{L}R^{\cdot\alpha}_{\cdot\beta\gamma\delta}=(\mathfrak{L}\left\{{}^{\alpha}_{\beta\delta}\right\})_{;\gamma}-(\mathfrak{L}\left\{{}^{\alpha}_{\beta\gamma}\right\})_{;\delta}\,. \tag{5.4}$$

In consequence of Theorem 4.1, making use of the expression by Lie derivatives, an infinitesimal rotation on I_n is characterized by the condition

$$\mathfrak{L}g_{\alpha\beta}=\eta_{\alpha;\beta}+\eta_{\beta;\alpha}=2\eta^\gamma A_{\alpha\beta\gamma}\,. \tag{5.5}$$

Also, from (5.5) we get for an infinitesimal rotation on I_n

$$\mathfrak{L}g^{\alpha\beta}=-2\eta^\gamma A^{\alpha\beta}_{\cdot\cdot\gamma}\,, \tag{5.6}$$

and substituting (5.5) into (5.2) we have

$$\mathfrak{L}\left\{{}^{\alpha}_{\beta\gamma}\right\}=A^{\cdot\alpha}_{\cdot\beta\gamma;\delta}\eta^\delta+\eta^\delta_{;\lambda}(\delta^\lambda_\gamma A^{\cdot\alpha}_{\cdot\beta\delta}+\delta^\lambda_\beta A^{\cdot\alpha}_{\cdot\gamma\delta}-g^{\alpha\lambda}A_{\beta\gamma\delta})\,. \tag{5.7}$$

§ 6. **Special case.** When an indicatrix admits an intransitive group of rotations, we can derive some properties of the indicatrix in connection with structures of the group. In the following, as an example we shall consider the special 2-dimensional indicatrix I_2 in M_3, and derive its properties by means of the group structures.

For convenience' sake, let us denote coordinates of a point in M_3 by (x, y, z) in stead of (x^1, x^2, x^3). If the finite equations of a centro-affine transformation in M_3 are given by

$$\begin{cases} \overline{x}=a_{11}x+a_{12}y+a_{13}z \\ \overline{y}=a_{21}x+a_{22}y+a_{23}z \\ \overline{z}=a_{31}x+a_{32}y+a_{33}z \end{cases} \qquad \det.|a_{ij}|\neq 0\,, \tag{6.1}$$

all transformations of the type (6.1) form a 9-parameter Lie group G_9 and its symbols are given by

$$xf_x,\quad yf_x,\quad zf_x,\qquad xf_y,\quad yf_y,\quad zf_y,$$
$$xf_z,\quad yf_z,\quad zf_z\quad \left(\text{where } f_x\equiv\frac{\partial f}{\partial x},\ f_y\equiv\frac{\partial f}{\partial y},\ f_z\equiv\frac{\partial f}{\partial z}\right).$$

Now, we shall suppose that an indicatrix I_2 is a revolutional surface and equation of it has a form

$$F(x^2+y^2, z) = 1\,. \tag{6.2}$$

Since we have from (6.2)

$$2F_v x dx + 2F_v y dy + F_z dz = 0 \quad (\text{where } v = x^2+y^2)\,,$$

the indicatrix I_2 admits a transformation determined by a generating vector with component $(y, -x, 0)$, namely admits a one-parameter group of rotations with the symbol $Xf=yf_x-xf_y$, which is one of subgroups contained in the above G_9. In this case, if we denote by $P(0, 0, z_0)$ and $Q(0, 0, -z_0)$ two points determined by intersection of I_2 and z-axis, then P and Q are fixed points of rotations.

On the other hand, except fixed points P and Q, we may represent I_2 by three equations involving two parameters u^1 and u^2 such that

$$x=f(u^2)\cos u^1\,. \quad y=f(u^2)\sin u^1\,, \quad z=u^2\,,$$

where we put $x^2+y^2=v(z)=f^2(z)$ and u^1 denotes angle between radius vector $(x, y, 0)$ and x-axis. Then we have

$$\begin{aligned} &X_1^{\bar 1}\equiv\frac{\partial x}{\partial u^1}=-y\,, && X_1^{\bar 2}\equiv\frac{\partial y}{\partial u^1}-x\,, && X_1^{\bar 3}\equiv\frac{\partial z}{\partial u^1}=0\,,\\ &X_2^{\bar 1}\equiv\frac{\partial x}{\partial u^2}=f'\cos u^1\,, && X_2^{\bar 2}\equiv\frac{\partial y}{\partial u^2}=f'\sin u^1\,, && X_2^{\bar 3}\equiv\frac{\partial z}{\partial u^2}=1\,,\end{aligned} \tag{6.3}$$

where $f'=\dfrac{df}{du^2}$.

Putting $\frac{1}{2}F^2(x^2+y^2, z)=L(x^2+y^2, z)$ and if we denote by $g_{\bar i \bar j}$ the components of fundamental metric tensor in M_3, we have

$$\begin{aligned} &g_{\bar 1\bar 1}\equiv L_{xx}=2L_v+4x^2L_{vv}, && g_{\bar 1\bar 2}\equiv L_{xy}=4xyL_{vv}, && g_{\bar 1\bar 3}\equiv L_{xz}=2xL_{vz},\\ &g_{\bar 2\bar 2}\equiv L_{yy}=2L_v+4y^2L_{vv}, && g_{\bar 2\bar 3}\equiv L_{yz}=2yL_{vz}, && g_{\bar 3\bar 3}=L_{zz},\end{aligned} \tag{6.4}$$

Also, from the definition $A_{\bar i\bar j\bar k}=\dfrac{1}{2}\dfrac{\partial g_{\bar i\bar j}}{\partial x^k}$ it follows that

$$\begin{aligned} &A_{\bar 1\bar 1\bar 1}=6xL_{vv}+4x^2L_{vvv}, && A_{\bar 1\bar 1\bar 2}=2yL_{vv}+4x^2yL_{vvv}, &&\\ &A_{\bar 1\bar 1\bar 3}=L_{vz}+4x^2L_{vvz}, && A_{\bar 1\bar 2\bar 2}=2xL_{vv}+4xy^2L_{vvv}, &&\\ &A_{\bar 2\bar 2\bar 2}=2yL_{vv}+4y^3L_{vvv}, && A_{\bar 1\bar 2\bar 3}=2xyL_{vvz}, && A_{\bar 2\bar 2\bar 3}=L_{vz}+4y^2L_{vvz},\\ &A_{\bar 1\bar 3\bar 3}=xL_{vzz}, && A_{\bar 2\bar 3\bar 3}=yL_{vzz}, && A_{\bar 3\bar 3\bar 3}=\tfrac{1}{2}L_{zzz}.\end{aligned} \tag{6.5}$$

Making use of (4.3), (4.4) and (4.5), we can obtain induced components $g_{\alpha\beta}$ and $A_{\alpha\beta\gamma}$ $(\alpha, \beta, \gamma=1, 2)$ of $g_{\bar{i}\bar{j}}$ and $A_{\bar{i}\bar{j}\bar{k}}$ respectively with respect to the reference system (6.3). After direct calculations we find that

(6.6) $\quad g_{12}=g_{\bar{i}\bar{j}}X_1^{\bar{i}}X_1^{\bar{j}}=0,\ A_{111}=A_{\bar{i}\bar{j}\bar{k}}X_1^{\bar{i}}X_1^{\bar{j}}X_1^{\bar{k}}=0,\ A_{122}=A_{\bar{i}\bar{j}\bar{k}}X_1^{\bar{i}}X_2^{\bar{j}}X_2^{\bar{k}}=0.$

As component of a generating vector of one-parameter group with symbol $Xf=yf_x-xf_y$ is $(y, -x, 0)$, denoting by (η^1, η^2) its induced component with respect to the reference system (6.3), it follows that

$$y=\eta^1X_1^{\bar{1}}+\eta^2X_2^{\bar{1}},\quad -x=\eta^1X_1^{\bar{2}}+\eta^2X_2^{\bar{2}},\quad 0=\eta^1X_1^{\bar{3}}+\eta^2X_2^{\bar{3}}.$$

Therefore, substituting (6.3) into the above relations, we find that $\eta^\alpha=-\delta_1^\alpha$ holds good except at fixed points P and Q.

Since η^α generates an infinitesimal rotation on I_2, in consequence of (4.5), it should be satisfied that $\eta_{\alpha;\beta}+\eta_{\beta;\alpha}=2\eta^\gamma A_{\alpha\beta\gamma}$. However this last relation can be rewritten as follows:

$$\eta^\gamma\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}+g_{\alpha\gamma}\frac{\partial\eta^\gamma}{\partial u^\beta}+g_{\beta\gamma}\frac{\partial\eta^\gamma}{\partial u^\alpha}=2\eta^\gamma A_{\alpha\beta\gamma}\quad(\alpha, \beta, \gamma=1, 2)\,. \tag{6.7}$$

Substituting $\eta^\alpha=-\delta_1^\alpha$ into (6.7) it follows that $\partial g_{\alpha\beta}/\partial u^1=2A_{\alpha\beta1}$. Then, in consequence of (6.6) we find that g_{11} and g_{22} do not contain a variable u^1. This shows us that in our example an infinitesimal rotation on I_2 coincides with a motion with respect to the metric in I_2 [8].

Theorem 6.1. *In a 3-dimensional Minkowski space, if I_2 is a revolutional surface, an infinitesimal rotation on I_2 coincides with a motion with respect to the induced Riemannian metric on I_2, and intersecting points on I_2 with its axis of rotation are fixed points of the transformation.*

Moreover, as $g_{12}=0$ it should be satisfied that $A_{112}=0$. Therefore, by means of this relation and (6.6) we find that $S_{1212}=0$ holds good at every point except at P and Q.

In the followings, we shall study about properties of groups of infinitesimal rotations on I_n under assumptions that I_n has no singular point.

§7. The determination of groups of infinitesimal rotations on I_n.

We shall consider that under what conditions the equation

$$\mathfrak{L}g_{\alpha\beta}=\eta_{\alpha;\beta}+\eta_{\beta;\alpha}=2\eta^\gamma A_{\alpha\beta\gamma} \tag{7.1}$$

admits one or more solutions.

By means of (5.3) and (5.6), it follows that

$$\eta^{\alpha}_{;\beta;\gamma}=\eta^{\delta}(A^{\alpha}_{\cdot\beta\gamma;\delta}+R^{\ \alpha}_{\cdot\beta\gamma\delta})+\eta^{\delta}_{;\lambda}(\delta^{\lambda}_{\gamma}A^{\alpha}_{\cdot\beta\delta}+\delta^{\lambda}_{\beta}A^{\alpha}_{\cdot\gamma\delta}-g^{\alpha\lambda}A_{\beta\gamma\delta})\,. \tag{7.2}$$

On the other hand, by means of the definition of covariant differential we have

$$\frac{\partial\eta^{\alpha}}{\partial u^{\beta}}=-\eta^{\delta}\{^{\ \alpha}_{\delta\beta}\}+\eta^{\alpha}_{;\beta}\,, \tag{7.3}$$

$$\frac{\partial\eta^{\alpha}_{;\beta}}{\partial u^{\gamma}}=-\eta^{\delta}_{;\beta}\{^{\ \alpha}_{\delta\gamma}\}+\eta^{\alpha}_{;\delta}\{^{\ \delta}_{\beta\gamma}\}+\eta^{\alpha}_{;\beta;\gamma}\,. \tag{7.4}$$

Substituting (7.2) into the last term in right hand side of the second equation, we can see that both $\partial\eta^{\alpha}/\partial u^{\beta}$ and $\partial\eta^{\alpha}_{;\beta}/\partial u^{\gamma}$ are expressed by linear forms with respect to η^{α} and $\eta^{\alpha}_{;\beta}$, and do not contain higher order derivatives of these unknown functions. In consequence of the above observations we have

Theorem 7.1. *In order that a Riemannian space I_n admits an infinitesimal rotation on I_n, it is necessary and sufficient that the system of linear partial equations*

$$\frac{\partial\eta^{\alpha}}{\partial u^{\beta}}=-\eta^{\delta}\{^{\ \alpha}_{\delta\beta}\}+\eta^{\alpha}_{;\beta} \tag{7.5}$$

$$\frac{\partial\eta^{\alpha}_{;\beta}}{\partial u^{\gamma}}=\eta^{\delta}(A^{\alpha}_{\cdot\beta\gamma;\delta}+R^{\ \alpha}_{\cdot\beta\gamma\delta})+\eta^{\delta}_{;\lambda}(\delta^{\lambda}_{\gamma}A^{\alpha}_{\cdot\beta\delta}+\delta^{\lambda}_{\beta}A^{\alpha}_{\cdot\gamma\delta}-g^{\alpha\lambda}A_{\beta\gamma\delta}-\delta^{\lambda}_{\beta}\{^{\ \alpha}_{\delta\gamma}\}+\delta^{\alpha}_{\delta}\{^{\ \lambda}_{\beta\gamma}\}) \tag{7.6}$$

admits solutions η^{α} and $\eta^{\alpha}_{;\beta}$ under the condition

$$\eta_{\alpha;\beta}+\eta_{\beta;\alpha}=2\eta^{\gamma}A_{\alpha\beta\gamma}\quad(\eta_{\alpha;\beta}=g_{\alpha\gamma}\eta^{\gamma}_{;\beta})\,. \tag{7.7}$$

Now, we shall seek the integrability conditions of the mixed system of partial differential equations (7.5)-(7.7). Differentiating (7.3) with respect to u^{γ} it follows that

$$\frac{\partial^2\eta^{\alpha}}{\partial u^{\beta}\partial u^{\gamma}}=-\frac{\partial\eta^{\delta}}{\partial u^{\gamma}}\{^{\ \alpha}_{\delta\beta}\}-\eta^{\delta}\frac{\partial\{^{\ \alpha}_{\delta\beta}\}}{\partial u^{\gamma}}+\frac{\partial\eta^{\alpha}_{;\beta}}{\partial u^{\gamma}}.$$

If we substitute the relations (7.3) and (7.4) into the right hand side of the above expression, we obtain

$$\frac{\partial^2\eta^{\alpha}}{\partial u^{[\beta}\partial u^{\gamma]}}=2\Big[\eta^{\delta}R^{\ \alpha}_{\cdot\delta\beta\gamma}+\eta^{\delta}_{;\beta}\{^{\ \alpha}_{\delta\gamma}\}-\eta^{\delta}_{;\gamma}\{^{\ \alpha}_{\delta\beta}\}+\frac{\partial\eta^{\alpha}_{;\beta}}{\partial u^{\gamma}}-\frac{\partial\eta^{\alpha}_{;\gamma}}{\partial u^{\beta}}\Big].\text{[6]}$$

According to the Ricci's identity, we may put $\eta^{\alpha}_{;\beta;\gamma}-\eta^{\alpha}_{;\gamma;\beta}=-\eta^{\delta}R^{\ \alpha}_{\cdot\delta\beta\gamma}$. Therefore $\partial^2\eta^{\alpha}/\partial u^{[\beta}\partial u^{\gamma]}=0$ holds good identically. Next, we shall consider about (7.6).

Covariantly differentiating (7.2) with respect to u^{ε}, we obtain

6) We use the notation [ ] for the expression of skew-symmetric part such that $T_{[\alpha\beta]}=\frac{1}{2}(T_{\alpha\beta}-T_{\beta\alpha})$.

$$\eta^{\alpha}_{;\beta;\gamma;\varepsilon}=\eta^{\delta}(A^{\alpha}_{.\beta\gamma;\delta;\varepsilon}+R^{\alpha}_{.\beta\gamma\delta;\varepsilon})+\eta^{\delta}_{;\varepsilon}(A^{\alpha}_{.\beta\gamma;\delta}+R^{\alpha}_{.\beta\gamma\delta})$$
$$+\eta^{\delta}_{;\lambda}(\delta^{\lambda}_{\gamma}A^{\alpha}_{.\beta\delta;\varepsilon}+\delta^{\lambda}_{\beta}A^{\alpha}_{.\gamma\delta;\varepsilon}-g^{\alpha\lambda}A_{\beta\gamma\delta;\varepsilon})$$
$$+\eta^{\delta}_{;\lambda;\varepsilon}(\delta^{\lambda}_{\gamma}A^{\alpha}_{.\beta\delta}+\delta^{\lambda}_{\beta}A^{\alpha}_{.\gamma\delta}-g^{\alpha\lambda}A_{\beta\gamma\delta})\,.$$

In stead of $\eta^{\delta}_{;\lambda;\varepsilon}$ in the above equation, we substitute the term obtained from (7.2) by changing indices α, β and γ by δ, λ and ε respectively. Then it follows that

$$\eta^{\alpha}_{;\beta;\gamma;\varepsilon}=\eta^{\delta}P^{\alpha}_{.\beta\gamma\delta\varepsilon}+\eta^{\delta}_{;\lambda}Q^{\alpha\lambda}_{..\beta\gamma\delta\varepsilon}\,, \tag{7.8}$$

where

$$P^{\alpha}_{.\beta\gamma\delta\varepsilon}\equiv A^{\alpha}_{.\beta\gamma;\delta;\varepsilon}+R^{\alpha}_{.\beta\gamma\delta;\varepsilon}+A^{\alpha}_{.\beta\omega}(A^{\omega}_{.\gamma\varepsilon;\delta}+R^{\omega}_{.\gamma\varepsilon\delta})$$
$$+A^{\alpha}_{.\omega\gamma}(A^{\omega}_{.\beta\varepsilon;\delta}+R^{\omega}_{.\beta\varepsilon\delta})-A^{\omega}_{.\beta\gamma}(A^{\alpha}_{.\omega\varepsilon;\delta}-R^{\alpha}_{.\omega\varepsilon\delta})\,.$$
$$Q^{\alpha\lambda}_{..\beta\gamma\delta\varepsilon}\equiv\delta^{\lambda}_{\varepsilon}(A^{\alpha}_{.\beta\gamma;\delta}+R^{\alpha}_{.\beta\gamma\delta})+(\delta^{\lambda}_{\gamma}A^{\alpha}_{.\beta\delta;\varepsilon}+\delta^{\lambda}_{\beta}A^{\alpha}_{.\delta\gamma;\varepsilon}-g^{\alpha\lambda}A_{\delta\beta\gamma;\varepsilon})$$
$$+A^{\alpha}_{.\beta\omega}(\delta^{\lambda}_{\gamma}A^{\omega}_{.\varepsilon\delta}+\delta^{\lambda}_{\varepsilon}A^{\omega}_{.\gamma\delta}-g^{\omega\lambda}A_{\gamma\varepsilon\delta})+A^{\alpha}_{.\omega\gamma}(\delta^{\lambda}_{\varepsilon}A^{\omega}_{.\beta\delta}+\delta^{\lambda}_{\beta}A^{\omega}_{.\varepsilon\delta}-g^{\omega\lambda}A_{\beta\varepsilon\delta})$$
$$-A^{\omega}_{.\beta\gamma}(\delta^{\lambda}_{\varepsilon}A^{\alpha}_{.\delta\omega}-\delta^{\lambda}_{\omega}A^{\alpha}_{.\varepsilon\delta}+g^{\alpha\lambda}A_{\omega\varepsilon\delta})\,.$$

By means of the Ricci's identity we have

$$\eta^{\alpha}_{;\beta;\gamma;\varepsilon}-\eta^{\alpha}_{;\beta;\varepsilon;\gamma}=\eta^{\delta}_{;\lambda}(-\delta^{\lambda}_{\beta}R^{\alpha}_{.\delta\gamma\varepsilon}+\delta^{\alpha}_{\delta}R^{\lambda}_{.\beta\gamma\varepsilon})\,. \tag{7.9}$$

On the other hand, from (7.8) it follows that

$$\eta^{\alpha}_{;\beta;\gamma;\varepsilon}-\eta^{\alpha}_{;\beta;\varepsilon;\gamma}=\eta^{\delta}(P^{\alpha}_{.\beta\gamma\delta\varepsilon}-P^{\alpha}_{.\beta\varepsilon\delta\gamma})+\eta^{\delta}_{;\lambda}(Q^{\alpha\lambda}_{..\beta\gamma\delta\varepsilon}-Q^{\alpha\lambda}_{..\beta\varepsilon\delta\gamma})\,. \tag{7.10}$$

In consequence of (7.9) and (7.10), the integrability conditions are written in the form

$$\eta^{\delta}(P^{\alpha}_{.\beta\gamma\delta\varepsilon}-P^{\alpha}_{.\beta\varepsilon\delta\gamma})+\eta^{\delta}_{;\lambda}(Q^{\alpha\lambda}_{..\beta\gamma\delta\varepsilon}-Q^{\alpha\lambda}_{..\beta\varepsilon\delta\gamma}+\delta^{\lambda}_{\beta}R^{\alpha}_{.\delta\gamma\varepsilon}-\delta^{\alpha}_{\delta}R^{\lambda}_{.\beta\gamma\varepsilon})=0\,. \tag{7.11}$$

Making use of (3.8) and the Ricci's identity we find that

$$A^{\alpha}_{.\beta\gamma;\delta;\varepsilon}-A^{\alpha}_{.\beta\varepsilon;\delta;\gamma}=A^{\alpha}_{.\beta\delta;\gamma;\varepsilon}-A^{\alpha}_{.\beta\delta;\varepsilon;\gamma}=-A^{\omega}_{.\beta\delta}R^{\alpha}_{.\omega\gamma\varepsilon}+A^{\alpha}_{.\omega\delta}R^{\omega}_{.\beta\gamma\varepsilon}+A^{\alpha}_{.\beta\omega}R^{\omega}_{.\delta\gamma\varepsilon}\,,$$

and therefore, we obtain

$$\eta^{\delta}(P^{\alpha}_{.\beta\gamma\delta\varepsilon}-P^{\alpha}_{.\beta\varepsilon\delta\gamma})=\eta^{\delta}\{-A^{\omega}_{.\beta\delta}R^{\alpha}_{.\omega\gamma\varepsilon}+A^{\alpha}_{.\omega\delta}R^{\omega}_{.\beta\gamma\varepsilon}+(R^{\alpha}_{.\beta\gamma\varepsilon}-S^{\alpha}_{.\beta\gamma\varepsilon})_{;\delta}$$
$$+2A^{\alpha}_{.\omega[\gamma}R^{\omega}_{.|\beta|\varepsilon]\delta}+2A^{\omega}_{.\beta[\varepsilon}R^{\alpha}_{|\omega\delta|\gamma]}\}\,. \tag{7.12}$$

$$\eta^{\delta}_{;\lambda}(Q^{\alpha\lambda}_{..\beta\gamma\delta\varepsilon}-Q^{\alpha\lambda}_{..\beta\varepsilon\delta\gamma}+\delta^{\lambda}_{\beta}R^{\alpha}_{.\delta\gamma\varepsilon}-\delta^{\alpha}_{\delta}R^{\lambda}_{.\beta\gamma\varepsilon})$$
$$=\eta^{\delta}_{;\lambda}\{2\delta^{\lambda}_{[\varepsilon}(R^{\alpha}_{.|\beta|\gamma]\delta}-S^{\alpha}_{.|\beta|\gamma]\delta})+\delta^{\lambda}_{\beta}(R^{\alpha}_{.\delta\gamma\varepsilon}-S^{\alpha}_{.\delta\gamma\varepsilon})-\delta^{\alpha}_{\delta}(R^{\lambda}_{.\beta\gamma\varepsilon}-S^{\lambda}_{.\beta\gamma\varepsilon}) \tag{7.13}$$
$$-2g^{\omega\lambda}A_{\beta\delta[\varepsilon}A^{\alpha}_{.\gamma]\omega}+2A^{\alpha}_{.\delta[\varepsilon}A^{\lambda}_{.\gamma]\beta}\}-\eta^{\delta}_{;\lambda}(\delta^{\alpha}_{\delta}S^{\lambda}_{.\beta\gamma\varepsilon}+g^{\alpha\lambda}S_{\beta\delta\varepsilon\gamma})\,.$$

On the other hand, from the relation $\eta_{\nu;\lambda}+\eta_{\lambda;\nu}=2\eta^{\sigma}A_{\nu\lambda\sigma}$, it follows that

$$\eta^{\delta}_{;\lambda}g^{\alpha\lambda}=-\eta^{\alpha}_{;\nu}g^{\nu\delta}+2\eta^{\sigma}A^{\alpha\delta}_{..\sigma} \tag{7.14}$$

Substituting (7.14) into the last term in right hand side of (7.13), by means of the identity $S_{\alpha\beta\gamma\delta}=-S_{\beta\alpha\gamma\delta}=-S_{\alpha\beta\delta\gamma}$, we get the relation

$$\eta^{\delta}_{;\lambda}(\delta^{\alpha}_{\delta}S^{\lambda}_{.\beta\gamma\varepsilon}+g^{\alpha\lambda}S_{\beta\delta\varepsilon\gamma})=2\eta^{\sigma}A^{\delta\alpha}_{..\sigma}S_{\delta\beta\gamma\varepsilon}\,. \tag{7.15}$$

In consequence of (7.12), (7.13) and (7.15), the integrability conditions (7.11) can be expressed as follows:

$$\begin{aligned}(7.16)\qquad &\eta^{\delta}\{-A^{\omega}_{\cdot\beta\delta}R^{\alpha}_{\cdot\omega\gamma\varepsilon}+A^{\alpha}_{\cdot\omega\delta}R^{\omega}_{\cdot\beta\gamma\varepsilon}+(R^{\alpha}_{\cdot\beta\gamma\varepsilon}-S^{\alpha}_{\cdot\beta\gamma\varepsilon})_{;\delta}\\ &\quad-2(A^{\alpha}_{\cdot\omega[\gamma}R^{\omega}_{\cdot|\beta|\varepsilon]\delta}+A^{\omega}_{\cdot\beta[\varepsilon}R^{\alpha}_{\cdot|\omega\delta|\gamma]}-A^{\alpha\omega}_{\cdot\cdot\delta}S_{\omega\beta\gamma\varepsilon})\}\\ &+\eta^{\delta}_{;\lambda}\{\delta^{\lambda}_{\beta}(R^{\alpha}_{\cdot\delta\gamma\varepsilon}-S^{\alpha}_{\cdot\delta\gamma\varepsilon})-\delta^{\alpha}_{\delta}(R^{\lambda}_{\cdot\beta\gamma\varepsilon}-S^{\lambda}_{\cdot\beta\gamma\varepsilon})+2A^{\alpha}_{\cdot\delta[\varepsilon}A^{\lambda}_{\cdot\gamma]\beta}\\ &\quad+2\delta^{\lambda}_{[\varepsilon}(R^{\alpha}_{\cdot|\beta|\gamma]\delta}-S^{\alpha}_{\cdot|\beta|\gamma]\delta})-2g^{\omega\lambda}A_{\beta\delta[\varepsilon}A^{\alpha}_{\cdot\gamma]\omega}\}=0\,.\end{aligned}$$

If we call our attention to the fact that in an n-dimensional Riemannian space I_n the curvature tensor has the form (3.7), we obtain the following theorem:

Theorem 7.2. *In an n-dimensional Riemannian space I_n, the system of linear partial differential equations for determination of groups of infinitesimal rotations on I_n is completely integrable.*

Proof. In the following we shall show that (7.16) holds good identically if the relation (3.7) is satisfied. By means of (3.7) and (7.7) it is easily seen that

$$\begin{aligned}&\eta^{\delta}_{;\lambda}\{\delta^{\lambda}_{\beta}(R^{\alpha}_{\cdot\delta\gamma\varepsilon}-S^{\alpha}_{\cdot\delta\gamma\varepsilon})-\delta^{\alpha}_{\delta}(R^{\lambda}_{\cdot\beta\gamma\varepsilon}-S^{\lambda}_{\cdot\beta\gamma\varepsilon})+2\delta^{\lambda}_{[\varepsilon}(R^{\alpha}_{\cdot|\beta|\gamma]\delta}-S^{\alpha}_{\cdot|\beta|\gamma]\delta})\}\\ &\quad=2\eta^{\delta}(\delta^{\alpha}_{\gamma}A_{\varepsilon\beta\delta}-\delta^{\alpha}_{\varepsilon}A_{\gamma\beta\delta})\,.\end{aligned}$$

Also, from (3.7) we must have $(R^{\alpha}_{\cdot\beta\gamma\varepsilon}-S^{\alpha}_{\cdot\beta\gamma\varepsilon})_{;\delta}=0$. Accordingly (7.16) can be rewritten as follows:

$$\begin{aligned}(7.16')\qquad &\eta^{\delta}\{-A^{\omega}_{\cdot\beta\delta}R^{\alpha}_{\cdot\omega\gamma\varepsilon}+A^{\alpha}_{\cdot\omega\delta}R^{\omega}_{\cdot\beta\gamma\varepsilon}-2(A^{\alpha}_{\cdot\omega[\gamma}R^{\omega}_{\cdot|\beta|\varepsilon]\delta}+A^{\omega}_{\cdot\beta[\varepsilon}R^{\alpha}_{\cdot|\omega\delta|\gamma]})\\ &\quad-2A^{\alpha\omega}_{\cdot\cdot\delta}S_{\omega\beta\gamma\varepsilon}+2(\delta^{\alpha}_{\gamma}A_{\varepsilon\beta\delta}-\delta^{\alpha}_{\varepsilon}A_{\gamma\beta\delta})\}\\ &+\eta^{\delta}_{;\lambda}(2A^{\alpha}_{\cdot\delta[\varepsilon}A^{\lambda}_{\cdot\gamma]\beta}-2g^{\omega\lambda}A_{\beta\delta[\varepsilon}A^{\alpha}_{\cdot\gamma]\omega})=0\,.\end{aligned}$$

On the other hand, making use of (5.3), (5.4) and (5.7), after some complicated calculations we get

$$\begin{aligned}&\mathfrak{L}R^{\alpha}_{\cdot\beta\varepsilon\gamma}=\eta^{\delta}\{-A^{\omega}_{\cdot\beta\delta}R^{\alpha}_{\cdot\omega\gamma\varepsilon}+A^{\alpha}_{\cdot\omega\delta}R^{\omega}_{\cdot\beta\gamma\varepsilon}-2(A^{\alpha}_{\cdot\omega[\gamma}R^{\omega}_{\cdot|\beta|\varepsilon]\delta}+A^{\omega}_{\cdot\beta[\varepsilon}R^{\alpha}_{\cdot|\omega\delta|\gamma]})\}\\ (7.17)\qquad &\quad+\mathfrak{L}S^{\alpha}_{\cdot\beta\varepsilon\gamma}+2\eta^{\delta}A^{\alpha}_{\cdot\delta\omega}S^{\omega}_{\cdot\beta\varepsilon\gamma}\\ &\quad+\eta^{\delta}_{;\gamma}(2A^{\alpha}_{\cdot\delta[\varepsilon}A^{\lambda}_{\cdot\gamma]\beta}-2g^{\omega\gamma}A_{\beta\delta[\varepsilon}A^{\alpha}_{\cdot\gamma]\omega})\,.\end{aligned}$$

Comparing (7.16′) and (7.17), the integrability conditions may be reduced in the form

$$\mathfrak{L}R^{\alpha}_{\cdot\beta\varepsilon\gamma}-\mathfrak{L}S^{\alpha}_{\cdot\beta\varepsilon\gamma}+\eta^{\delta}(2\delta^{\alpha}_{\gamma}A_{\varepsilon\beta\delta}-2\delta^{\alpha}_{\varepsilon}A_{\gamma\beta\delta})=0\,.$$

However, since $\mathfrak{L}g_{\alpha\beta}=2\eta^{\gamma}A_{\alpha\beta\gamma}$, it is readily seen that

$$\eta^{\delta}(2\delta^{\alpha}_{\gamma}A_{\varepsilon\beta\delta}-2\delta^{\alpha}_{\varepsilon}A_{\gamma\beta\delta})=\mathfrak{L}(\delta^{\alpha}_{\gamma}g_{\beta\varepsilon}-\delta^{\alpha}_{\varepsilon}g_{\beta\gamma})\,.$$

Therefore, finally we have the following form of the integrability conditions:

$$\mathfrak{L}\{R^{\alpha}_{.\beta\varepsilon\gamma}-S^{\alpha}_{.\beta\varepsilon\gamma}-(\delta^{\alpha}_{\varepsilon}g_{\beta\gamma}-\delta^{\alpha}_{\gamma}g_{\beta\varepsilon})\}=0.$$

The above relation gives us the result of Theorem 7.2.

In consequence of Theorem 7.2 and the theory of linear partial differential equations, we can see that the general solution involves at most $\frac{1}{2}n(n+1)$ parameters, because the system of partial differential equations (7.5) and (7.6) with respect to $n+n^2$ unknown function η^{α} and $\eta^{\alpha}_{;\beta}$, is completely integrable under $\frac{1}{2}n(n+1)$ conditions (7.7). Then we have

Theorem 7.3. *The maximum order of groups of infinitesimal rotations on I_n is equal to $\frac{1}{2}n(n+1)$.*

§ 8. Properties of an infinitesimal rotation on I_n. In the following, we shall consider properties of an infinitesimal transformation $\bar{u}^{\alpha}=u^{\alpha}+\eta^{\alpha}(u)\delta t$, where η^{α} satisfies the relation

$$\eta_{\alpha;\beta}+\eta_{\beta;\alpha}=2\eta^{\gamma}A_{\alpha\beta\gamma} \quad (\eta^{\alpha}=g_{\alpha\beta}\eta^{\beta})\,. \tag{8.1}$$

As a special case, if the vector A^{α} generates an infinitesimal rotation on I_n, it should be satisfied that

$$A_{\alpha;\beta}+A_{\beta;\alpha}=2A^{\gamma}A_{\alpha\beta\gamma}\,. \tag{8.2}$$

Multiplying (8.2) by $g^{\alpha\beta}$ and summing for α and β, it follows that $A^{\gamma}_{;\gamma}=A^{\gamma}A_{\gamma}$. Since I_n is a compact Riemannian space, if I_n is an orientable manifold, by means of the theorem of Green, we find that

$$\int_{I_n} A^{\gamma}_{;\gamma}d\sigma=\int_{I_n} A^{\gamma}A_{\gamma}d\sigma=0\,. \tag{8.3}$$

In consequence of $A^{\gamma}A_{\gamma}=g_{\gamma\delta}A^{\gamma}A^{\delta}\geq 0$, (8.3) implies $A^{\gamma}A_{\gamma}=0$. Therefore $A_{\gamma}=0$ should be held at every point on I_n. This means that the considered Minkowski space M_{n+1} is essentially a Euclidean space [9]. Thus we have

Theorem 8.1. *In I_n the generating vector η^{α} of an infinitesimal rotation does not coincides with the vector A^{α}, otherwise M_{n+1} is essentially a Euclidean space and the infinitesimal transformation becomes the identity transformation.*

Let us suppose that a vector η^{α} generates an infinitesimal rotation on I_n. In order that a vector $\rho\eta^{\alpha}$, where ρ is a scalar function, generates also an infinitesimal rotation on I_n, it is necessary and sufficient that we have

$$(\rho\eta_{\alpha})_{;\beta}+(\rho\eta_{\beta})_{;\alpha}=2\rho\eta^{\gamma}A_{\alpha\beta\gamma}\,. \tag{8.4}$$

However, since the vector η_α satisfies the relation (8.1), we have from (8.4)

$$\rho_{,\beta}\eta_\alpha+\rho_{,\alpha}\eta_\beta=0\,. \quad \left(\rho_{,\alpha}=\frac{\partial\rho}{\partial u^\alpha}\right) \tag{8.5}$$

Replacing β by γ in (8.5), we have a similar relation

$$\rho_{,\gamma}\eta_\alpha+\rho_{,\alpha}\eta_\gamma=0\,. \tag{8.6}$$

Then, eliminating η_α from (8.5) and (8.6), we get

$$\rho_{,\alpha}(\rho_{,\gamma}\rho_\beta-\rho_{,\beta}\eta_\gamma)=0\,. \tag{8.7}$$

The relation (8.7) shows us that if we suppose that $\rho_{,\alpha}\neq 0$, then $\rho_{,\gamma}\eta_\beta-\rho_{,\beta}\eta_\gamma=0$ must be satisfied. However, in such a case, since we have $\rho_{,\gamma}\eta_\beta+\rho_{,\beta}\eta_\gamma=0$, it follows that $\rho_{,\gamma}\eta_\beta=0$. Therefore, from our assumption we should have $\eta_\beta=0$. Accordingly, we must have $\rho_{,\alpha}=0$. Then,

Theorem 8.2. *Two infinitesimal rotations on I_n cannot have the same trajectory.*

Next, let us consider the case when the trajectory for an infinitesimal rotation on I_n be geodesic of the Riemannian space I_n. Then we must have (8.1) and

$$\eta_{\alpha;\beta}\eta^\beta=\rho\eta_\alpha\,, \tag{8.8}$$

where ρ is a scalar function. Multiplying (8.1) by $\eta^\alpha\eta^\beta$ and summing for α and β, by means of (8.8) we get

$$\rho\eta_\delta\eta^\delta=A_{\alpha\beta\gamma}\eta^\alpha\eta^\beta\eta^\gamma\,. \tag{8.9}$$

On the other hand, if we contract η^α to (8.1) and making use of (8.8) it follows that

$$\frac{1}{2}(\eta_\alpha\eta^\delta)_{;\beta}\eta^\beta+\rho\eta_\beta=2A_{\alpha\beta\gamma}\eta^\alpha\eta^\gamma\,.$$

Moreover, making contraction by η^β, the above relation becomes

$$\frac{1}{2}(\eta_\delta\eta^\delta)_{;\beta}+\rho\eta_\beta\eta^\beta=2A_{\alpha\beta\gamma}\eta^\alpha\eta^\beta\eta^\gamma\,. \tag{8.10}$$

Comparing (8.9) and (8.10), we must have $(\eta_\delta\eta^\delta)_{;\beta}\eta^\beta=2\rho(\eta_\delta\eta^\delta)$, namely $\mathfrak{L}(g_{\lambda\nu}\eta^\lambda\eta^\nu)=2\rho(g_{\lambda\nu}\eta^\lambda\eta^\nu)$. Consequently it follows that $\mathfrak{L}g_{\lambda\nu}=2\rho g_{\lambda\nu}$. This implies that the infinitesimal rotation on I_n is an infinitesimal conformal transformation.

Theorem 8.3. *If the trajectory of an infinitesimal rotation on I_n is a geodesic of the Riemannian space I_n, the transformation must be an infinitesimal conformal one.*

Let $\eta^{\alpha}_{(a)}$ $(a=1, 2,\cdots, r)$ be vectors of r one-parameter groups of infinitesimal rotations on I_n. Then we have $\mathfrak{L}_a g_{\alpha\beta}=2\eta^{\gamma}_{(a)}A_{\alpha\beta\gamma}$ and from which it follows that

$$c^a\mathfrak{L}_a g_{\alpha\beta}=c^a(\eta_{(a)\alpha;\beta}+\eta_{(a)\beta;\alpha})=(c^a\eta_{(a)\alpha})_{;\beta}+(c^a\eta_{(a)\beta})_{;\alpha}=2(c^a\eta^{\gamma}_{(a)})A_{\alpha\beta\gamma}, \tag{8.11}$$

where c^a are arbitrary constants.

If we introduce the notation $X_a f=\eta^{\alpha}_{(a)}\dfrac{\partial f}{\partial u^{\alpha}}$, it is a symbol of the infinitesimal transformation $\overline{u}^{\alpha}=u^{\alpha}+\eta^{\alpha}_{(a)}(u)\delta t$. In consequence of (8.11), we have the following

Theorem 8.4. *If $X_a f$ are symbols of r one-parameter groups of infinitesimal rotations on I_n, then $c^a X_a f$ is also a symbol of a one-parameter group of infinitesimal rotations on I_n, where c^a are arbitrary constants.*

If $X_a f$ are the generators of an r-parameter group, then the transformations of this group consist of the transformations of one-parameter groups generated by the infinitesimal transformation $c^a X_a f$ and of the products of such transformations [8]. Thus in consequence of Theorem 8.4, we obtain

Theorem 8.5. *When each of r generators of an r-parameter group G_r of transformations is a generator of a one-parameter group of rotations on I_n, every transformation of the group G_r is a rotation on I_n.*

By virtue of the second fundamental theorem of Lie, if r independent linear operator $X_a f$ constitute a complete system of order r, then the transformations of the group, generated by the infinitesimal transformations $c^a X_a f$ or products of such transformations, form an r-parameter group G_r of transformations. Then making use of the result of Theorem 8.5 we obtain

Theorem 8.6. *If $X_a f$ are r generators of a complete set of one-parameter groups of infinitesimal rotations on I_n, then they are generators of an r-parameter group of infinitesimal rotations on I_n.*

The properties of an infinitesimal rotation on I_n, when the transformation coincides with a motion or transformations of other special classes and also the structures of r-parameter groups of infinitesimal rotations on I_n will be discussed more precisely in the next paper.

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