ON SEMI-LOWER BOUNDED MODULARS

By

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W. Orlicz and Z. Birnbaum proved in [7] that an Orlicz space $L_{\Phi}(G)$ is finite if and only if the function Φ satisfies the following condition: for some $\gamma>0$ and $t_0>0$, $\Phi(2t)\leqq\gamma\Phi(t)$ for every $t\geqq t_0$. (In case of mes $(G)=+\infty$, $\Phi(2t)\leqq\gamma\Phi(t)$ for all $t\geqq 0$.)

This fact was generalized for arbitrary monotone complete modulars on non-atomic space by I. Amemiya in [1], that is, suppose that R is a universally continuous semi-ordered linear space and has no atomic element, then every monotone complete finite modular on R is semi-upper bounded. T. Shimogaki showed in [8] a new simple proof of this Amemiya's Theorem. In this paper we investigate the properties of the conjugate modular of a semi-upper bounded modular, i.e. the semi-lower bounded modular. Throughout this paper we use the terminologies and notations used in [5].

In §1 we give corollaries of Amemiya's Theorem and a theorem relate to Amemiya's Theorem. In §2 we investigate the relations between a modular or the modular norms and semi-lower bounded modular. In §3 we express the properties of a semi-upper and semi-lower bounded modular.

§1. Let R be a universally continuous semi-ordered linear space and m be a modular on R[1]. A modular m is said to be "*finite*", if $m(x)<+\infty$ for every $x\in R$. A modular m is said to be "*monotone complete*", if for $0\leqq a_{\lambda}\uparrow_{\lambda\in\Lambda}$, $\sup_{\lambda\in\Lambda} m(a_{\lambda})<+\infty$ there exists $a\in R$ for which $a_{\lambda}\uparrow_{\lambda\in\Lambda}a$. And a modular m is said to be "*semi-upper bounded*", if for every $\varepsilon>0$ there exists $\gamma=\gamma(\varepsilon)>0$ such that $m(x)\geqq\varepsilon$ implies $m(2x)\leqq\gamma m(x)$.

In [1] I. Amemiya proved:

Theorem 1.1. *Suppose that R has no atomic element, then every monotone complete, finite modular on R is semi-upper bounded.*

We say a modular m on R to be "*domestic*", if for any $a\in\{a: m(a)<+\infty, a\in R\}$ there exists $\xi=\xi(a)>1$ such that $m(\xi a)<+\infty$. On R, we define the two functionals $\|a\|$, $|||a|||$ $(a\in R)$ as follows:

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1) For the definition of the modular see H. Nakano [5].

$$\|a\|=\inf_{\xi>0}\frac{1+m(\xi a)}{\xi},\quad |||a|||=\inf_{m(\xi a)\leqq 1}\frac{1}{|\xi|}.$$

Then it is easily seen that both $\|a\|$ and $|||a|||$ are norms on R and satisfy always $|||a|||\leqq\|a\|\leqq 2|||a|||$ for all $a\in R$ (cf. [6]). The norms $\|a\|$ and $|||a|||$ are called the *first norm* and the *second* (or *modular*) *norm* by m respectively.

Remark 1.1. (i) *If a modular m on R is finite, then m is domestic*; (ii) *if m is domestic, then* $\inf_{0\neq x\in R} m\left(\frac{x}{|||x|||}\right)=1$; (iii) $\inf_{0\neq x\in R} m\left(\frac{x}{|||x|||}\right)>0$ *implies $|||\cdot|||$ is continuous*; (iv) *if $|||\cdot|||$ is continuons, then m is finite, when R has no atomic element.*

Because, (i) is trivial. (iii) and (iv) is well known[2]. Therefore we have only to prove (ii). If $m\left(\frac{x}{|||x|||}\right)<1$ for some $x\in R$, there exists $\varepsilon>0$ by domesticness such that

$$1<m\left((1+\varepsilon)\frac{x}{|||x|||}\right)<+\infty.$$

Thus there exists $\gamma<1$, for which $m\left(\gamma(1+\varepsilon)\frac{x}{|||x|||}\right)=1$. Therefore we obtain $\gamma(1+\varepsilon)=\left|\left|\left|\gamma(1+\varepsilon)\frac{x}{|||x|||}\right|\right|\right|=1$, and hence $m\left(\frac{x}{|||x|||}\right)=1$, contradicting $m\left(\frac{a}{|||x|||}\right)<1$.

A modular norm $|||x|||\,(x\in R)$ is said to be "*finitely monotone*" (cf. [9]), if for every $\varepsilon>0$, there exists an integer $n_0=n_0(\varepsilon)$ such that $x=\oplus\sum_{i=1}^{n}x_i$, $|||x|||\leqq 1$, $|||x_i|||\geqq\varepsilon$ $(i=1,2,\cdots,n)$ implies $n\leqq n_0$. A modular m is said to be "*uniformly finite*", if

$$\sup_{m(x)\leqq 1} m(\xi x)<+\infty \quad \text{for all} \quad \xi\geqq 0.$$

In [9, Theorems 1.1, 2.1 and 2.2], it is shown that if a norm on R is uniformly monotone[3], then it is finitely monotone; if a modular m is uniformly finite, then the modular norm by m is finitely monotone; if the modular norm by m is finitely monotone, then m is uniformly finite when R has no atomic element; if a norm is finitely monotone, then the every norms which is equivalent to it is also finitely monotone.

2) T. Andô obtained (iii). For (iv) see [1].

3) A norm on R is said to be uniformly monotone, if for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $a\frown b=0$, $\|a\|=1$, $\|b\|\geqq\varepsilon$ implies $\|a+b\|\geqq 1+\delta$ (cf. [4]).

$\bar{R}^m$ denotes the totality of all universally continuous linear functionals[4)] on R which are bounded under the modular norm $|||\cdot|||$ by m. On $\bar{R}^m$ the conjugate modular of $m(x)$ is defined as follows

$$\bar{m}(\bar{a}) = \sup_{x \in R} \{\bar{a}(x) - m(x)\} \qquad \text{for every } \bar{a} \in \bar{R}^m.$$

$\bar{m}(\bar{a})$ satisfies the modular conditions and is monotone complete (cf. [5, §38]).

It has been known that if R is semi-regular[5)], the first norm by the conjugate modular $\bar{m}$ is the conjugate norm of the second norm by m and the second norm by the conjugate modular $\bar{m}$ is the conjugate norm of the first norm by m.

Lemma 1 ([5, Theorem 39.4]). *If R is semi-regular, then R is isometric[6)] to a complete semi-normal manifold of the conjugate space $\bar{\bar{R}}^m$ of $\bar{R}^m$ by the correspondence*

$$R \ni a \to a^{\bar{R}^m} \in \bar{\bar{R}}^m, \quad a^{\bar{R}^m}(\bar{x}) = \bar{x}(a) \quad \text{for} \quad \bar{x} \in \bar{R}^m.$$

Corollary 1 of Theorem 1.1. *Suppose that R has no atomic element. If the modular norm $|||\cdot|||$ by m is finitely monotone, then m is semi-upper bounded.*

Proof. Since m is uniformly finite by assumption, $\bar{\bar{m}}$ is uniformly finite on $\bar{\bar{R}}^m$ ([5, Theorems 48.4, 48.5]). Since $\bar{\bar{m}}$ is monotone complete and $\bar{\bar{R}}^m$ has no atomic element, we obtain by Theorem 1.1 $\bar{\bar{m}}$ is semi-upper bounded on $\bar{\bar{R}}^m$. Therefore m is semi-upper bounded by Lemma 1. Q.E.D.

Remark 1.2. *If a modular m is semi-upper bounded and semi-simple, then m is uniformly finite.*

Because, if for some $\gamma > 1$ we have $m(2x) \leqq \gamma m(x)$ for every x such that $m(x) \geqq 1$, then we have obviously $m(2^\nu x) \leqq \gamma^\nu m(x)$ $(\nu = 1, 2, \cdots)$ for every x such that $m(x) \geqq 1$. Since m is finite by assumption, we obtain

$$\sup_{m(x) \leqq 1} m(2^\nu x) \leqq \sup_{1 \leqq m(x) \leqq 2} m(2^\nu x)$$
$$\leqq \sup_{1 \leqq m(x) \leqq 2} \gamma^\nu m(x) \leqq 2\gamma^\nu < +\infty \quad (\nu = 1, 2, \cdots).$$

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4) A linear functional L on R is said to be universally continuous, if for any $a_\lambda \downarrow_{\lambda \in \Lambda} 0$ we have $\inf_{\lambda \in \Lambda} |L(a_\lambda)| = 0$.

5) R is said to be semi-regular, if $\bar{a}[p] = 0$ for all $\bar{a} \in \bar{R}^m$ implies $p = 0$. For $p \in R$, $[p]$ denotes the projection operator defined by $[p]x = \bigcup_{\nu=1}^{\infty} (x \frown \nu |p|)$ for all $x \geqq 0$.

6) A modulared space R with a modular m is said to be isometric to a modulared space $\hat{R}$ with a modular $\hat{m}$ by a correspondence $R \ni a \to a^{\hat{R}} \in \hat{R}$, if R is isomorphic to $\hat{R}$ by this correspondence and $m(a) = \hat{m}(a^{\hat{R}})$ for all $a \in R$.

Thus, m is uniformly finite.

A norm on R is said to be "*monotone*", if $0 \leqq a < b$ implies $\|a\| < \|b\|$. A norm on R is said to be "*universally monotone complete*", if for $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda}$, $\sup_{\lambda \in \Lambda} \|a_\lambda\| < +\infty$ there exists $a \in R$ such that $a_\lambda \uparrow_{\lambda \in \Lambda} a$; if $\Lambda = \{1, 2, \cdots\}$ we say to be "*monotone complete*".

Corollary 2 of Theorem 1.1. *If the modular norm $|||\cdot|||$ by m is monotone and monotone complete, then m is uniformly simple*[7]*, and m is semi-upper bounded when R has no atomic element.*

Proof. (i) If the modular norm $|||\cdot|||$ by m is monotone, than $|||\cdot|||$ is continuous.

Because, if $\inf_{0 \neq x \in R} m\left(\frac{x}{|||x|||}\right) < 1$, there exists $a \in R$ such that $|||a||| = 1$ and $m(a) < 1$, therefore we can suppose $[a] < 1$ without difficulty, and hence there exists $0 < b \in R$ such that $a \frown b = 0$, $m(a+b) \leqq 1$. Thus we obtain obviously $|||a+b||| = |||a||| = 1$, which is contradicting $|||\cdot|||$ is monotone. Consequently we obtain $\inf_{0 \neq x \in R} \left(\frac{x}{|||x|||}\right) = 1$, and hence $|||\cdot|||$ is continuous by Remark 1.1.

(ii) If the modular norm $|||\cdot|||$ by m is monotone, then m is simple[8].

Because, if m is not simple there exists $a \in R$ such that $0 < a$ and $m(a) = 0$, then $m(a+b) = m(b) \leqq 1$ for any $0 < b$, $a \frown b = 0$ and $|||b||| = 1$. Thus we have $|||a+b||| = |||b||| = 1$, contradicting assumption that $|||\cdot|||$ is monotone. Thus m is simple.

If the modular norm $|||\cdot|||$ by m is continuous and monotone complete, then m is monotone complete (cf. [5, Theorems 30.20, 40.7]). Thus we obtain m is monotone complete, simple and $|||\cdot|||$ is continuous by (i) and (ii). Therefore m is uniformly simple (cf. [11, Theorem 2.1]).

If R has no atomic element, then uniformly simple modular m is uniformly finite ([10, Theorem 1.2]), and hence we obtain m is semi-upper bounded by Corollary 1 of Theorem 1.1. Q.E.D.

Theorem 1.2. *Suppose that R has no atomic element. Each of the following conditions implies that m is semi-upper bounded*

(1): $$\inf_{0 \neq x \in R} m\left(\frac{\alpha}{|||x|||} x\right) > 0 \qquad \textit{for some } 0 < \alpha < 1,$$

7) A modular m is said to be uniformly simple, if $\inf_{m(x) \geqq 1} m(\xi x) > 0$ for all $\xi > 0$, that is, $\lim_{\nu \to \infty} m(a_\nu) = 0$ implies $\lim_{\nu \to \infty} |||a_\nu||| = 0$.

8) A modular m on R is said to be simple, if $m(a) = 0$ implies $a = 0$.

(2): $$\sup_{0\neq x\in R} m\left(\frac{\alpha}{|||x|||}x\right)>0 \qquad \textit{for some } \alpha\geqq 1.$$

Proof. (1): We prove first that the condition:

$$\inf_{0\neq x\in R} m\left(\frac{1-\varepsilon}{|||x|||}x\right)=\xi>0 \qquad \text{for some } 1>\varepsilon>0$$

implies the condition:

$$\inf_{0\neq \bar{\bar{x}}\in \overline{\overline{R}}^m} \overline{\overline{m}}\left(\frac{1-\varepsilon'}{|||\bar{\bar{x}}|||}\bar{\bar{x}}\right)>0 \qquad \text{for some } \varepsilon>\varepsilon'>0.$$

For $\bar{\bar{x}}\in\overline{\overline{R}}^m$ with $|||\bar{\bar{x}}|||=1$ there exists $x_\lambda\in R$ $(\lambda\in\Lambda)$ such that $x_\lambda\uparrow_{\lambda\in\Lambda}\bar{\bar{x}}$ (cf. [5, Theorem 5.34]), because R is a complete semi-normal manifold of $\overline{\overline{R}}^m$ by Lemma 1. Since the modular norm is semi-continuous and reflexive (cf. [3]), we obtain $|||x_\lambda|||\uparrow_{\lambda\in\Lambda}|||\bar{\bar{x}}|||$, and hence we have

$$\left(1-\frac{\varepsilon}{2}\right)|||x_\lambda|||\uparrow_{\lambda\in\Lambda}\left(1-\frac{\varepsilon}{2}\right).$$

Consequently there exists λ_0 such that $\left(1-\frac{\varepsilon}{2}\right)|||x_\lambda|||\geqq 1-\varepsilon$ for $\lambda\geqq\lambda_0$. If $\inf_{0\neq x\in R} m\left(\frac{1-\varepsilon}{|||x|||}x\right)=\xi>0$, we obtain easily $m(x)\geqq\xi$ for every x such that $|||x|||\geqq 1-\varepsilon$, thus we have obviously $m\left(\left(1-\frac{\varepsilon}{2}\right)x_\lambda\right)\geqq\xi$ for $\lambda\geqq\lambda_0$.

Therefore we have $$\inf_{0\neq \bar{\bar{x}}\in \overline{\overline{R}}^m} \overline{\overline{m}}\left(\frac{1-\frac{\varepsilon}{2}}{|||\bar{\bar{x}}|||}\bar{\bar{x}}\right)>0.$$

Therefore, we obtain $|||\bar{\bar{a}}|||$ $(\bar{\bar{a}}\in\overline{\overline{R}}^m)$ is continuous by Remark 1.1, and, since $\overline{\overline{R}}^m$ is non-atomic, $\overline{\overline{m}}$ is finite on $\overline{\overline{R}}^m$ by Remark 1.1. As $\overline{\overline{m}}$ is monotone complete, we obtain $\overline{\overline{m}}$ is semi-upper bounded by Theorem 1.1, and hence we obtain finally that m is semi-upper bounded by Lemma 1.

The proof for the condition (2) is similar. Q.E.D.

§2. Let R be a modulared semi-ordered linear space with a modular m and be semi-regular. In this section, our aim is to consider the relations between properties of a modular or the modular norms and its semi-lower boundedness.

A modular m on R is said to be "*semi-lower bounded*" if for every $\varepsilon>0$, there exist $1<\alpha=\alpha(\varepsilon)<\gamma(\varepsilon)=\gamma$ such that $m(x)\geqq\varepsilon$ implies $m(\alpha x)\geqq\gamma m(x)$.

Theorem 2.1. *If a modular m is semi-upper bounded and semi-simple, then the conjugate modular $\overline{m}$ of m is semi-lower bounded.*

Proof. Since the case $\overline{m}(\overline{a})=+\infty$ is trivial, we can assume that $\overline{m}(\overline{a})<+\infty$. For every $\varepsilon>0$ there exists $\gamma=\gamma(\varepsilon)>0$ such that $m(x)\geqq\frac{\varepsilon}{3}$ implies $m(2x)\leqq\gamma m(x)$, by assumption. Then we have definition

$$\overline{m}\left(\frac{\gamma}{2}\overline{a}\right)=\sup_{x\in R}\left\{\frac{\gamma}{2}\overline{a}(2x)-m(2x)\right\}\geqq\sup_{m(x)\geqq\frac{\varepsilon}{3}}\left\{\frac{\gamma}{2}\overline{a}(2x)-m(2x)\right\}$$

$$\geqq\gamma\sup_{m(x)\geqq\frac{\varepsilon}{3}}\{\overline{a}(x)-m(x)\}\quad(\overline{a}\in\overline{R}^m).$$

For every $0\leqq\overline{a}\in\overline{R}^m$ such that $\varepsilon\leqq\overline{m}(\overline{a})<+\infty$, we have to consider the case

$$\overline{m}(\overline{a})=\sup_{m(x)<\frac{\varepsilon}{3}}\{\overline{a}(x)-m(x)\}.$$

For any $\delta>0$ there exists $x\in R$ such that $m(x)<\frac{\varepsilon}{3}$ and $\overline{a}(x)-m(x)\geqq\overline{m}(\overline{a})-\delta$. Since m is uniformly finite by Remark 1.2 there exists $\beta=\beta(a)>1$ such that

$$m(\beta x)=\frac{\varepsilon}{3}.$$

Therefore we obtain $\overline{a}(\beta x)-m(\beta x)\geqq\overline{a}(x)-m(x)-m(\beta x)\geqq\overline{m}(\overline{a})-\delta-\frac{\varepsilon}{3}$. Thus we have

$$\gamma\sup_{m(x)\geqq\frac{\varepsilon}{3}}\{\overline{a}(x)-m(x)\}\geqq\gamma\left(\overline{m}(\overline{a})-\frac{\varepsilon}{3}\right)\geqq\gamma\left(\overline{m}(\overline{a})-\frac{\overline{m}(\overline{a})}{3}\right)=\frac{2}{3}\gamma\overline{m}(\overline{a}),$$

and hence $\overline{m}\left(\frac{\gamma}{2}\overline{a}\right)\geqq\frac{2}{3}\gamma\overline{m}(\overline{a})$ for every $\overline{a}$ such that $\overline{m}(\overline{a})\geqq\varepsilon$. Q.E.D.

Theorem 2.2. *If a modular m is semi-lower bounded, then $\overline{m}$ is semi-upper bounded.*

Proof. If for every $\varepsilon>0$ there exist $\gamma>\alpha>1$ such that $m(x)\geqq\varepsilon$ implies $m(\alpha x)\geqq\gamma m(x)$, then we have by definition

$$\overline{m}\left(\frac{\gamma}{\alpha}\overline{a}\right)=\sup_{x\in R}\left\{\frac{\gamma}{\alpha}\overline{a}(\alpha x)-m(\alpha x)\right\}\leqq\gamma\sup_{m(x)\geqq\varepsilon}\{\overline{a}(x)-m(x)\}+\sup_{m(x)<\varepsilon}\{\gamma\overline{a}(x)-m(\alpha x)\}$$

$$\leqq\gamma\overline{m}(\overline{a})+\gamma\sup_{m(x)<\varepsilon}\{\overline{a}(x)\}\leqq\gamma\overline{m}(\overline{a})+\gamma\sup_{m(x)<\varepsilon}\{\overline{m}(\overline{a})+m(x)\}$$

$$\leqq\gamma\overline{m}(\overline{a})+\gamma(\overline{m}(\overline{a})+\varepsilon)=\gamma\{2\overline{m}(\overline{a})+\varepsilon\},$$

since by definition $|\overline{a}(x)|\leqq\overline{m}(\overline{a})+m(x)$ for $\overline{a}\in\overline{R}^m$, $x\in R$.
Thus we have $\overline{m}\left(\frac{\gamma}{\alpha}\overline{a}\right)\leqq3\gamma\overline{m}(\overline{a})$ for every $\overline{a}$ such that $\overline{m}(\overline{a})\geqq\varepsilon$. Q.E.D.

The "*conjugate*" of "*uniformly finite*" is "*uniformly increasing*", i.e.

$$\lim_{\xi\to\infty}\inf_{m(x)\geqq 1}\frac{m(\xi x)}{\xi}=+\infty \qquad \text{(cf. [5, §48])}.$$

Theorem 2.3. *If a modular m is semi-lower bounded, then m is uniformly increasing.*

Proof. By assumption there exist $1<\alpha<\gamma$ such that $m(x)\geqq 1$ implies $m(\alpha^\nu x)\geqq\gamma^\nu m(x)$ $(\nu=1,2,\cdots)$.
Therefore we obtain $\frac{1}{\alpha^\nu}m(\alpha^\nu x)\geqq\left(\frac{\gamma}{\alpha}\right)^\nu m(x)$ $(\nu=1,2,\cdots)$ for every x such that $m(x)\geqq 1$, and consequently m is uniformly increasing. Q.E.D.

Since the "*conjugate*" of "*finitely monotone*" is "*finitely flat*", i.e. for every $\gamma>0$ there exists $\varepsilon=\varepsilon(\gamma)$ such that

$$x=\oplus\sum_{i=1}^{n}x_i,\ \|x\|\geqq 1,\ \|x_i\|\leqq\varepsilon\ (i=1,2,\cdots,n)$$

implies $n\geqq\frac{\gamma}{\varepsilon}\|x\|$ (cf. [9, §1]), we have immediately by Corollary 1 of Theorem 1.1, Theorem 2.1 and Lemma 1 the following

Theorem 2.4. *Suppose that R has no atomic element. If the modular norm $|||\cdot|||$ by m is finitely flat, then m is semi-lower bounded.*

Remark 2.1. *If a modular m is uniformly increasing, then the modular norm is finitely flat. The converse of this is valid, if we suppose that R has no atomic element* (cf. [9]).

A norm $\|\cdot\|$ on R is said to be "*flat*", if for any $a\neq 0$, $a\frown b=0$ we have

$$\lim_{\xi\to 0}\frac{\|a+\xi b\|-\|a\|}{\xi}=0.$$

The "*conjugate*" of "*uniformly simple*" is "*uniformly monotone*", i.e.

$$\lim_{\xi\to 0}\frac{1}{\xi}\sup_{m(x)\leqq 1}m(\xi x)=0 \quad \text{(cf. [5, § 48])}.$$

Theorem 2.5. *If the first norm $\|\cdot\|$ by m is flat and the first norm $\|\bar{\cdot}\|$ by conjugate modular $\bar{m}$ of m is continuous, then m is uniformly monotone, and m is semi-lower bounded when R has no atomic element.*

Proof. Using Banach's theorem (cf. [6, §44]) and reflexivity of the norm $\|\cdot\|$, we can prove that flatness of $\|\cdot\|$ implies monotony of $|||\bar{\cdot}|||$. Thus we have $\bar{m}$ is simple by (ii) in proof of Corollary 2 of Theorem 1.1. Since $\|\bar{a}\|$ is continuous by assumption and $\bar{m}$ is monotone complete, we obtain $\bar{m}$ is uniformly simple ([11, Theorem 2.1]). Thus m is uniformly monotone.

On the other hand, if m is uniformly monotone then m is uniformly increasing when R has no atomic element ([10, Theorem 1.3]). By Theorem 2.4 and Remark 2.1 the proof is completed. Q.E.D.

A manifold K of R is said to be "*equi-continuous*", if for any $\bar{a}_\nu \downarrow_{\nu=1}^{\infty} 0$, $\bar{a}_\nu \in \bar{R}^m$ and $\varepsilon > 0$ there exists ν_0 for which we have $\bar{a}_{\nu_0}(x) \leqq \varepsilon$ for all $x \in K$.

Theorem 2.6. *If a modular m is semi-lower bounded, then a manifold $K=\{x: m(x) \leqq 1,\ x \in R\}$ is equi-continuous. The converse of this is true, if we suppose that R has no atomic element.*

Proof. If m is semi-lower bounded, m is uniformly increasing by Theorem 2.3. Then we have $\bar{m}$ is uniformly finite, and hence the conjugate norm of the modular norm by m is continuous by Remark 1.1. Therefore we obtain for any $\varepsilon > 0$ and $\bar{R}^m \ni \bar{a}_\nu \downarrow_{\nu=1}^{\infty} 0$ there exists ν_0 such that $\bar{a}_{\nu_0}(x) \leqq \varepsilon$ for all $x \in K$ ([5, Theorem 31.12]). That is, K is equi-continuous.

Conversely we suppose that R has no atomic element and the manifold $K=\{x: m(x) \leqq 1\}$ is equi-continuous. Since we have obviously by definition $\{x: |||x||| \leqq 1\} = \{x: m(x) \leqq 1\}$, the first norm by $\bar{m}$ is continuous ([5, Theorem 31.12]). Thus we obtain $\bar{m}$ is monotone complete and finite, because $\bar{R}^m$ is non-atomic by assumption. Thus we have $\bar{m}$ is semi-upper bounded by Theorem 1.1, therefore we obtain by Theorem 2.1 and Lemma 1 m is semi-lower bounded. Q.E.D.

A manifold K of R is said to be "*weakly bounded*", if

$$\sup_{x \in K} |\bar{a}(x)| < +\infty \quad \text{for all } \bar{a} \in \bar{R}^m.$$

Theorem 2.7. *If a modular m is semi-lower bounded, then every weakly bounded manifold is equi-continuous. The converse of this is truth, if we suppose that R has no atomic element.*

Proof. If m is semi-lower bounded, the conjugate norm of a norm by m is continuous. Consequently every manifold K for which $\sup_{x \in K} \|x\| < +\infty$ is equi-continuous ([5, Theorem 33.10]). Therefore we have $\sup_{x \in K} |\bar{a}(x)| \leqq \sup_{x \in K} |||\bar{a}||| \cdot \|x\|$ for all $\bar{a} \in \bar{R}^m$, and hence K is weakly bounded by definition.

Conversely we suppose that R has no atomic element. Since the norm $|||\cdot|||$ is reflexive (cf. [3]), if a manifold K is weakly bounded, then K is norm bounded, i.e. $\sup_{x \in K} |||x||| < +\infty$ ([5, Theorem 32.6]), and equi-continuous by assumption. Then the first norm by the conjugate modular $\bar{m}$ of m is continuous ([5, Theorem 33.10]. Thus we have obviously our conclusion by the method applied to Theorem 2.6. Q.E.D.

Theorem 2.7. *Suppose that R has no atomic element. Each of the following conditions implies m is semi-lower bounded*

(1) $$\inf_{0\neq x\in R}\frac{1}{\gamma}m\left(\frac{\gamma}{|||x|||}x\right)\geqq 1+\delta \qquad \textit{for some } \gamma, \delta>0,$$

(2) $$\sup_{0\neq x\in R} m\left(\frac{x}{\|x\|}\right)<1.$$

Proof. (1) For every $\bar{a}\in\bar{R}^m$ with $\|\bar{a}\|=1$, we have

$$(1+\delta)\bar{a}(\xi a)-m(\xi a)\leqq\xi(1+\delta)-\xi(1+\delta)=0$$

for every $a\in R$, $|||a|||=1$ and $\xi\geqq\gamma$.

Thus we have $\overline{m}((1+\delta)\bar{a})=\sup_{|||x|||\leqq\gamma}\{(1+\delta)\bar{a}(x)-m(x)\}\leqq\gamma(1+\delta)$.

Suppose that $\bar{R}^m\ni\bar{a}_\nu\downarrow_{\nu=1}^\infty 0$ and $\inf_{\nu\geqq 1}\|\bar{a}_\nu\|\doteq\alpha>0$,

then there exist $\varepsilon_0>0$, ν_0 such that

$$\left\|\frac{\bar{a}_\nu}{\alpha-\varepsilon_0}\right\|\leqq 1+\delta \quad \text{for every} \quad \nu\geqq\nu_0.$$

Since we have $$1+\overline{m}\left(\frac{\bar{a}_\nu}{\alpha-\varepsilon_0}\right)\geqq\left\|\frac{\bar{a}_\nu}{\alpha-\varepsilon_0}\right\|\geqq\frac{\alpha}{\alpha-\varepsilon_0} \quad \text{for every } \nu\geqq\nu_0.$$

we obtain $$1+\lim_{\nu\to\infty}\overline{m}\left(\frac{\bar{a}_\nu}{\alpha-\varepsilon_0}\right)\geqq\frac{\alpha}{\alpha-\varepsilon_0}>1.$$

Since $\lim_{\nu\to\infty}\overline{m}\left(\frac{\bar{a}_\nu}{\alpha-\varepsilon_0}\right)=0$, this is a contradiction.

Therefore $\|\bar{a}\|$ is continuous. Thus we have our conclusion by the method applied to Theorem 2.6.

The proof for the condition (2) is similar. Q.E.D.

§3. Let R be a modulared semi-ordered linear space with a semi-simple modular m. In this section, we express the properties of a semi-upper and semi-lower bounded modulars.

If a modular m is semi-upper and semi-lower bounded, then m is said to be "*semi-bounded*".

Lemma 3.1. *Suppose that R has no atomic element. If the norms by a modular m have the property:*

$\inf_{0\neq x\in R}\frac{\|x\|}{|||x|||}=\gamma$, *where* $\gamma>1$ *is a fixed constant, then m is semi-bounded.*

Proof. We have m is uniformly finite and uniformly increasing by the assumption (cf. [10, Theorem 1.1]). Therefore we obtain our conclusion by Corollary 1 of Theorem 1.1 and Theorem 2.4. Q.E.D.

Lemma 3.2. *If a modular m is semi-bounded, then the norms by m have the property*:

$$\inf_{0 \neq x \in R} \frac{\|x\|}{|||x|||} = \gamma \qquad \textit{for some } \gamma > 1.$$

Proof. Since m is uniformly finite and uniformly increasing by Remark 1.2 and Theorem 2.3, we have our conclusion (cf. [10, Theorem 1.4]). Q.E.D.

From these Lemmata, we obtain the following theorem

Theorem 3.1. *Suppose that R has no atomic element. A modular is semi-bounded, if and only if the norms by the modular have the property*:

$$\inf_{0 \neq x \in R} \frac{\|x\|}{|||x|||} = \gamma \qquad \textit{for some } \gamma > 1.$$

In the case when a modular m on R is of unique spectra (cf. [5, §54]), semi-boundedness of m implies boundedness[9] of m. In fact we have

Theorem 3.2. *If a modular m on R is of unique spectra*[10], *then semi-boundedness of m is equivalent to boundedness of m.*

Proof. If m is semi-bounded, then m is uniformly finite and uniformly increasing by Remark 1.2 and Theorem 2.3. Therefore m has the upper exponent[10] ρ_u and the lower exponent[10] ρ_l such that $1 \leqq \rho_l \leqq \rho_u < +\infty$ (cf. [5, Theorems 54.8, 54.10]). Thus m is bounded ([5, Theorems 54.4, 54.5]). Q.E.D.

A modular m of unique spectra is uniformly convex[10] (or uniformly even[10]) if and only if $1 < \rho_l \leqq \rho_u < +\infty$ for the upper exponent ρ_u and the lower exponent ρ_l (cf. [5, §50, §54]). Therefore we obtain also:

Theorem 3.3. *A modular m of unique spectra is uniformly convex (or uniformly even), if and only if m is semi-bounded.*

Theorem 3.4. *Suppose that R has no atomic element. If a modular m is uniformly convex (or uniformly even), then m is semi-bounded.*

Proof. Let m be uniformly convex. Then m is uniformly simple ([5, Theorem 50.1]). Since R is non-atomic by assumption, m and $\overline{m}$ are uniformly finite ([10, Theorem 1.2]), and hence m and $\overline{m}$ are semi-upper

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9) A modular m on R is said to be upper bounded, if there exist $\omega, \gamma > 1$, for which we have $m(\omega x) \leqq \gamma m(x)$ for all $x \in R$; and m is said to be lower bounded. if there exist $\gamma > \omega > 1$ such that $m(\omega x) \geqq \gamma m(x)$ for all $x \in R$; if a modular m is upper and lower bounded, then m is said to be bounded.

10) For the definitions see [5].

bounded by Corollary 1 of Theorem 1.1. Thus m is semi-bounded.

Let m be uniformly even. Then m is uniformly finite and uniformly monotone ([5, Theorems 51.1, 51.2]), and hence m is semi-bounded by Corollary 1 of Theorem 1.1 and Theorem 2.4. Q.E.D.

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