ON SOME TYPE OF THE MODULARED LINEAR SPACE

By

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§ 1. **Preliminary.** Modulared linear spaces with order structure in which there are functionals called modulars were discussed by H. NAKANO in his book [3]. H. NAKANO studied these spaces through the properties of these functionals. Above all he defined the uniform properties such as "uniformly simple", uniformly monotone", "uniformly increasing" and "uniformly finite", in his paper [2].

In this paper we investigate some new uniform properties of modulars, and give some examples of the new spaces with these properties.

In the following, we denote a modulared semi-ordered linear space by R, and an additive modular on R by m, and it will be supposed that R is semi-regular, i. e. there exist sufficiently many order-continuous linear functionals.

The terminologies and notations will be the same as [1] and [3].

§ 2. **Uniformly ascending modulars.** A modular m is said to be *ascending* if for any $a(\neq 0)\in R$ we have $\inf\limits_{\xi>0}\frac{m(\xi a)}{\xi}>0$. Uniformizing these properties, we have the following definition.

Definition 1.[1] *We call the modular m uniformly ascending if*

$$\inf_{R\ni a\neq 0}\frac{m(a)}{|||a|||}\geqq\delta>0\,,$$

where $|||a|||$ *is the second norm of m.* (cf. [1] p. 180).

In the above definition we may change the second norm by the first norm of m.

The conjugate property of an ascending modular is *semi-singular*: that is, for any $a>0$ there exists an element b such that $m(b)=0$ and

1) These definitions are also found in the paper [1] p. 185, which concerns with the another problem; they are sufficient conditions in order that a semi-additive modular is essentially additive.

$0 < b \leqq a$.

Definition 2.[1] *We call m uniformly semi-singular if for some number $\varepsilon > 0$, $|||x||| \leqq \varepsilon$, $x \in R$ implies $m(x) = 0$.*

Theorem 1. *Let m be an ascending modular. Then $m'(a) = \inf_{\xi > 0} \frac{m(\xi a)}{\xi}$ is linear modular and m' is also considered a modular norm of m' itself. m' is equivalent to the modular norm of m if and only if m is uniformly ascending.*

Proof. Suppose that m is uniformly ascending. Since we have $m(a) \geqq \delta |||a|||$ for some $\delta > 0$, it follows that $m(\xi a) \geqq \xi\delta |||a|||$ for $\xi \geqq 0$ i.e.

$$\inf_{\xi > 0} \frac{m(\xi a)}{\xi} \geqq \delta |||a||| \qquad \text{for every} \quad a \in R\,. \tag{1}$$

From the above inequality (1), it is easy to see that m' is a linear modular, because $m'(\xi a) = \xi m'(a)$ for any $\xi \geqq 0$ and $a \in R$. On the other hand, for any $a \in R$ and $1 \geqq \xi > 0$, we have

$$m(a) \geqq \frac{m(\xi a)}{\xi} \geqq m'(a)\,.$$

Since $1 \geqq m(a)$ implies $m(a) \leqq |||a|||$, we have $|||a||| \geqq m'(a)$. Hence m' and $|||\ |||$ are equivalent to each others. Conversely if $m'(a)$ is equivalent to the modular norm of m, then there exists a number $\delta > 0$ such that $m'(a) \geqq \delta |||a|||$.

Therefore

$$m(a) \geqq m'(a) \geqq \delta |||a|||\,,$$

which implies $\inf_{a \neq 0} \frac{m(a)}{|||a|||} \geqq \delta > 0$.

Theorem 2. *Let m be a semi-singular modular on R. Then $\|a\|_0 = \inf_{m(\xi a) = 0} \frac{1}{|\xi|}$ $(a \in R)$ is a norm on R. m is uniformly semi-singular if and only if this norm is equivalent to the modular norm.*

The proof of this theorem is followed easily from **Definition 2.** We have the dual relation between the uniformly ascending modulars and the uniformly semi-singular ones as in [1] p. 185.

Theorem 3. *Let m be uniformly ascending, then $\bar{m}$ is uniformly semi-singular modular on $\bar{R}^m$.*

Theorem 4. *Let m be uniformly semi-singular, then $\bar{m}$ is uniformly ascending modular on $\bar{R}^m$.*

The proofs of the above two theorems are found in [1] and so omitted. (cf. [1] p. 185).

If m is uniformly semi-singular, we will write

$$\|a\|_0 = \inf_{m(\xi a)=0} \frac{1}{|\xi|} \qquad \text{for} \quad a \in R ,$$

and if m is uniformly ascending, we will make use of the notation:

$$\|a\|' = m'(a) = \inf_{\xi>0} \frac{m(\xi a)}{\xi} \qquad \text{for} \quad a \in R .$$

Theorem 5. *Let m be uniformly semi-singular. Then the norm $\|\bar{a}\|' = \bar{m}'(\bar{a}) = \inf_{\xi>0} \frac{\bar{m}(\xi\bar{a})}{\xi}$, $\bar{a} \in \bar{R}^m$ is exactly the dual norm of $\|a\|_0$, $a \in R$. Conversely if m is uniformly ascending, the norm $\|\bar{a}\|_0$, $\bar{a} \in \bar{R}^m$ is exactly the dual norm of $\|a\|'$, $a \in R$.*

Proof. We need only prove the first part of the theorem. Let m be uniformly semi-singular. For any $\bar{a} \in \bar{R}^m$, and $\xi > 0$ we have

$$\frac{\bar{m}(\xi\bar{a})}{\xi} = \sup_{a \in R} \left\{ \bar{a}(a) - \frac{m(a)}{\xi} \right\} \geqq \sup_{a \in A} \left\{ \bar{a}(a) - \frac{m(a)}{\xi} \right\}$$

where $A = \{a \mid m(a) = 0\}$. Hence we have

$$\frac{\bar{m}(\xi\bar{a})}{\xi} \geqq \sup_{a \in A} |\bar{a}(a)| = \sup_{\|a_0\| \leqq 1} |\bar{a}(a)| ,$$

therefore, $\|\bar{a}\|' \geqq \sup_{\|a_0\| \leqq 1} |\bar{a}(a)|$.

On the other hand, we have

$$m(a) = \sup_{\bar{a} \in \bar{R}^m} \{\bar{a}(a) - \bar{m}(\bar{a})\} \leqq \sup_{\bar{a} \in \bar{R}^m} \{\bar{a}(a) - \bar{m}'(\bar{a})\} = \sup_{\bar{a} \in \bar{R}^m} \{\bar{a}(a) - \|\bar{a}\|'\} .$$

Hence $\quad \sup_{\bar{m}'(\bar{a}) \leqq 1} |\bar{a}(a)| = \sup_{\|\bar{a}\|' \leqq 1} |\bar{a}(a)| \leqq 1$ implies $m(a) = 0$.

Putting $B = \{a \mid \sup_{\|\bar{a}\|' \leqq 1} |\bar{a}(a)| \leqq 1\}$, we have $B \subset A$. By the reflexivity of the norm, (cf. [4]) we have for any $\bar{a} \in \bar{R}^m$

$$\|\bar{a}\|' = \sup_{a \in B} |\bar{a}(a)| \leqq \sup_{a \in A} |\bar{a}(a)| .$$

Hence we have

$$\|\bar{a}\|' = \sup_{a \in A} |\bar{a}(a)| = \sup_{\|a\|_0 \leqq 1} |\bar{a}(a)| .$$

Another theorem for uniformly ascending modulars is

Theorem 6. *Suppose that m is uniformly ascending. Then m is also* *ascending and uniformly simple. Hence if R has no atomic element, then m is uniformly finite.*

The first of this theorem is followed by the definition. If R has no atomic element and m is uniformly simple, then m is uniformly finite by the theorem of SHIMOGAKI [5] p. 205 or Theorem 5.1 in [1].

A dual form of this theorem is

Theorem 7. *Suppose that m is uniformly semi-singular. Then m is semi-singular and uniformly monotone. Hence, if R has no atomic element, m is uniformly increasing.*

§ 3. Uniformizations of infinitely linear modulars.

For any $a \in R$, we consider the functional defined by

$$\sup_{\xi > 0} \frac{m(\xi a)}{\xi} = \|a\|_\infty .$$

The set of all elements $\|a\|_\infty < \infty$, $a \in R$ is a semi-normal manifold of R. If this set is complete in R, then we call m *infinitely linear.*

Uniformizing this property, we have the following definition:

Definition 3. *We call m uniformly infinitely linear if there exists a positive number $\varepsilon > 0$ such that for some positive number n,*

$$|||x||| \geqq \varepsilon \quad \textit{implies} \quad m(x) \leqq n|||x||| .$$

Here the second norm may be changed by the first norm. We call an element $a \in R$ a *finite element by m* if

$$m(\xi a) < +\infty \qquad \text{for every} \quad \xi \geqq 0 .$$

If there is no finite element except 0, then we call m *infinite.* This property is dual to an infinitely linear modular. Hence we have the dual definition:

Definition 4. *A modular m is said to be uniformly infinite if there exists a positive number n such that*

$$|||x||| \geqq n \quad \textit{implies} \quad m(x) = +\infty .$$

By the above definitions we have the following:

Theorem 8. *If m is uniformly infinitely linear, then $\overline{m}$ is uniformly infinite.*

Proof. By assumption, there exist positive numbers n, $\varepsilon > 0$ such that $|||x||| \geqq \varepsilon$ implies $m(x) \leqq n|||x|||$.

Hence we have

$$\bar{m}(\bar{x}) = \sup_{x\in R} \left\{\bar{x}(x) - m(x)\right\}$$
$$= \sup_{0\neq x\in R} \left\{|||x|||\left(\bar{x}\left(\frac{x}{|||x|||}\right) - \frac{m(x)}{|||x|||}\right)\right\} \geqq \sup_{\substack{x\in R\\ |||x|||\geqq\varepsilon}} |||x|||\left\{\bar{x}\left(\frac{x}{|||x|||}\right) - n\right\}.$$

Since $\|\bar{x}\| = \sup_{|||x|||\leqq 1} |\bar{x}(x)| = \sup_{|||x|||=1} |\bar{x}(x)| = \sup_{|||x|||=1} \{\bar{x}(x)\}$, we can see that

$$\|\bar{x}\| \geqq n+1 \quad \text{implies} \quad \sup_{|||x|||\leqq 1} |\bar{x}(x)| = \sup_{0\neq x\in R} \left|\bar{x}\left(\frac{x}{|||x|||}\right)\right| \geqq n+1\,;$$

$$\text{i. e.} \qquad \sup_{0\neq x\in R} \left\{\bar{x}\left(\frac{x}{|||x|||}\right) - n\right\} > 0\,.$$

Therefore

$$\|\bar{x}\| \geqq n+1 \quad \text{implies} \quad \bar{m}(\bar{x}) = +\infty\,.$$

This shows the assertion.

A dual form of Theorem 8 is

Theorem 9. *If m is uniformly infinite, then $\bar{m}$ is uniformly infinitely linear.*

Proof. Since there exists a positive number ε such that $\|x\| \geqq \varepsilon$ implies $m(x) = +\infty$, for any $\bar{x}\in \bar{R}^m$, we have

$$\bar{m}(\bar{x}) = \sup_{x\in R} \left\{\bar{x}(x) - m(x)\right\}$$
$$= \sup_{x\in R, \|x\|\leqq\varepsilon} \left\{\bar{x}(x) - m(x)\right\} \leqq \sup_{\|x\|\leqq\varepsilon} \left\{\bar{x}(x)\right\} = \varepsilon\,|||\bar{x}|||\,.$$

Theorem 10. *Let m be uniformly infinitely linear. Then $\lim_{\xi\to\infty} \frac{m(\xi x)}{\xi} = \|x\|_\infty < +\infty$ for every $x\in R$. Conversely, if m is monotone complete*[2]*, and $\|x\|_\infty < +\infty$ for every $x\in R$, then m is uniformly infinitely linear.*

Proof. The first part of this theorem follows from **Definition 3.** If m is monotone complete and $\|x\|_\infty$ is finite for every $x\in R$, then $\|x\|_\infty$ can be considered a norm on R. Furthermore we have

$$|||x||| \leqq \|x\|_\infty \qquad \text{for every} \quad x\in R\,.$$

Since the second norm $|||\ |||$ is complete by virtue of the monotone completeness of m, the second norm $|||\ |||$ and $\|\ \|_\infty$ are equivalent to each others,

$$\text{i. e.} \quad \|x\|_\infty \leqq n|||x||| \qquad \text{for some number} \quad n>0.$$

2) For the definition of the monotone complete modular, see, [2] p. 129.

This shows that m is uniformly infinitely linear.

Theorem 11. *Suppose that m is uniformly infinitely linear. Then m is uniformly finite. If furthermore m is simple, then m is uniformly simple.*

The proof of the first part of this theorem is deduced from the definition. If m is uniformly finite and simple, then we can suppose that m is furthermore monotone complete. Hence, by the theorem of YAMAMURO (cf. [1] p. 190) we can deduce the second assertion.

A dual form of the above theorem is

Theorem 12. *Let m be uniformly infinite. Then m is uniformly increasing. Hence, if m is monotone, then m is uniformly monotone.*

If m is uniformly infinite, then a norm is defined by the set $F=\{x:\ m(x)<+\infty\}$, such that

$$\|x\|_f = \inf_{\xi x \in F} \frac{1}{|\xi|}\,.$$

Theorem 13. *Let m be uniformly infinitely linear. Then $\|\bar{x}\|_\infty (\bar{x}\in\bar{R}^m)$ is the dual norm of $\|x\|_f$, $(x\in R)$.*

Proof. For any $\bar{x}\in\bar{R}^m$, we will show that

$$\sup_{x\in F} |\,\bar{x}(x)\,| = \sup_{\xi>0} \frac{\bar{m}(\xi\bar{x})}{\xi} = \|\bar{x}\|_\infty$$

Since

$$\frac{\bar{m}(\xi\bar{x})}{\xi} = \sup_{x\in F}\left\{\bar{x}(x) - \frac{m(x)}{\xi}\right\} \leqq \sup_{x\in F} |\,\bar{x}(x)\,| \qquad \text{for any } \xi>0,$$

we have $$\|\bar{x}\|_\infty \leqq \sup_{x\in F} |\,\bar{x}(x)\,|\,.$$

On the other hand, for every $x\in F$, and for any $\xi>0$, we have

$$\sup_{\xi>0} \frac{\bar{m}(\xi\bar{x})}{\xi} \geqq \bar{x}(x) - \frac{m(x)}{\xi}\,,$$

that is, $$\sup_{\xi>0} \frac{\bar{m}(\xi\bar{x})}{\xi} \geqq \bar{x}(x)\,.$$

Hence, we have $\|\bar{x}\|_\infty = \sup_{\xi>0} \frac{\bar{m}(\xi\bar{x})}{\xi} \geqq \sup_{x\in F} |\bar{x}(x)|$, therefore we have the proof.

Concerning the relation between the uniformly infinitely linear modulars and the uniformly ascending ones, we have the following:

Theorem 14. *Let R have no atomic element. If m is uniformly* *ascending, then m is uniformly infinitely linear.*

Proof. We will prove that for some $n>0$,

$$m(x)\geqq 1 \quad \text{implies} \quad n\xi m(x)\geqq m(\xi x) \quad (\text{for } \xi>1).$$

Suppose that for any integer $n>0$, there exist a real number $\xi>1$ and $x>0$, $x\in R$ such that

$$n\xi m(x)<m(\xi x)\ .$$

Decomposing x orthogonally, we find the projection operators $[p_i]$, $i=1,\cdots,n_0$ such that

$$m(\xi[p_i]x)\leqq 1,\quad \xi[p_i]x\neq 0\ ;\quad i=1,\cdots,n_0,$$

since R has no atomic element.

Hence we have for some $1\leqq i_0\leqq n_0$,

$$n\xi m([p_{i_0}]x)\leqq m(\xi[p_{i_0}]x)\ .$$

Putting $\xi[p_{i_0}]x=y\neq 0$, we have

$$n\xi m\left(\frac{y}{\xi}\right)\leqq m(y)\leqq |||y|||\ .$$

Hence we have

$$\|y\|'\leqq \frac{m\left(\frac{1}{\xi}y\right)}{\frac{1}{\xi}}\leqq \frac{1}{n}|||y|||,\quad y\neq 0\ .$$

This shows that $\|\ \|'$ is not equivalent to $|||\ |||$. By virtue of Theorem 1, this contradicts the uniformly ascending property of m.

Hence, for any $x\in R$ such that $m(x)=1$ and $\xi>1$, we have

$$n\xi m(x)\geqq m(\xi x)\ .$$

That is

$$|||x|||\geqq 1 \quad \text{implies} \quad n|||x|||\geqq \frac{m(\xi x)}{\xi}\ ,$$

and this shows that $n|||x|||\geqq \|x\|_\infty$; therefore we have the proof of Theorem 14.

A dual form of this theorem is

Theorem 15. *Let R have no atomic element. If m is uniformly semi-singular, then m is uniformly infinite.*

Remark. If R is discrete, there is no relation between uniformly ascending and uniformly infinitely linear properties as it will be seen later. (see proposition 1, 3 in §5.)

§ 4. Uniformizations of asymptotically linear modulars.

We say an element $x \in R$ to be asymptotically linear if

$$\sup_{\xi > 0} \{\xi \gamma(x) - m(\xi x)\} < +\infty \quad \text{where} \quad \gamma(x) = \sup_{\xi > 0} \frac{m(\xi x)}{\xi}.$$

The totality of the asymptotically linear elements constitutes a semi-normal manifold. (cf. [3] p. 202). If this manifold is complete in R, then we say m to be *asymptotically linear*.

Definition 5. *Furthermore if we suppose*

$$\sup_{x \in A} \sup_{\xi > 0} \{\xi \gamma(x) - m(\xi x)\} < +\infty$$

where $\gamma(x) = \sup_{\xi > 0} \frac{m(\xi x)}{\xi}$ *and A is the set of all asymptotically linear elements, we call m uniformly asymptotically linear.*

If m is asymptotically linear, then $\overline{m}$ is totally discontinuous. (cf. [3] p. 203: i. e. discontinuous units are complete in R). Uniformizing this property, we have

Definition 6. *We call m uniformly discontinuous, if for some $n > 0$, $m(x) < +\infty$ implies $m(x) \leqq n$.*

Theorem 16. *If m is uniformly asymptotically linear, then $\overline{m}$ is uniformly discontinuous.*

Proof. Let $x \in \overline{R}^m$ be such that $\overline{m}(\overline{x}) < +\infty$. Then we have,

$$\overline{m}(\overline{x}) = \sup_{x \in R} \{\overline{x}(x) - m(x)\} = \sup_{x \in A} \{\overline{x}(x) - m(x)\}$$
$$= \sup_{0 \leqq \xi, x \in A} \{\xi \overline{x}(x) - m(\xi x)\}$$

because A is a complete semi-normal manifold of R.

If $\overline{x}(x) > \gamma(x)$ for some $x \in A$, then $\overline{m}(\overline{x}) = \infty$, but this is a contradiction. Hence we have $\overline{x}(x) \leqq \gamma(x)$, therefore by assumption

$$\overline{m}(\overline{x}) \leqq \sup_{0 \leqq \xi, x \in A} \{\xi \overline{x}(x) - m(\xi x)\} = \sup_{0 \leqq \xi, x \in A} \{\xi \gamma(x) - m(\xi x)\} \leqq n$$

for some number $n > 0$.

This fact shows that $\overline{m}$ is uniformly discontinuous.

Theorem 17. *If m is uniformly discontinuous, then $\overline{m}$ is uniformly* *asymptotically linear.*

Proof. If m is uniformly discontinuous, then m is totally discontinuous. By Theorem 46.6 in [3], $\overline{m}$ is asymptotically linear. Let $\bar{x}\in\overline{R}$ be an asymptotically linear element by $\overline{m}$. Then we can find some number $n>0$ such that

$$+\infty>\overline{m}(\bar{x})=\sup_{x\in R}\left\{\bar{x}(x)-m(x)\right\}=\sup_{x\in F}\left\{\bar{x}(x)-m(x)\right\}$$
$$\geqq\sup_{x\in F}\left\{\bar{x}(x)\right\}-n$$

where $F=\{x\,|\,m(x)<+\infty\}$. For any $\xi>0$, we have thus

$$+\infty>\overline{m}(\xi\bar{x})\geqq\sup_{x\in F}\left\{\xi\bar{x}(x)\right\}-n\,.$$

Since $\dfrac{\overline{m}(\xi\bar{x})}{\xi}=\sup_{x\in R}\left\{\bar{x}(x)-\dfrac{m(x)}{\xi}\right\}=\sup_{x\in F}\left\{\bar{x}(x)-\dfrac{m(x)}{\xi}\right\}\leqq\sup_{x\in F}\{\bar{x}(x)\}$,

we have

$$\gamma(\bar{x})=\sup_{\xi>0}\frac{\overline{m}(\xi\bar{x})}{\xi}\leqq\sup_{x\in F}\left\{\bar{x}(x)\right\}, \quad \text{therefore}$$

$$\xi\gamma(\bar{x})-\overline{m}(\xi\bar{x})\leqq n \qquad \text{for} \quad \xi>0\,.$$

This shows that

$$\sup_{\bar{x}\in\bar{A}}\sup_{\xi>0}\left\{\xi\gamma(\bar{x})-\overline{m}(\xi\bar{x})\right\}\leqq n$$

where $\bar{A}$ is the set of all asymptotically linear elements by $\overline{m}$.

Concerning the relations to the previous sections we give the following theorems.

Theorem 18. *If m is uniformly asymptotically linear, then m is uniformly infinitely linear.*

A dual form of the above theorem is

Theorem 19. *If m is uniformly discontinuous, then m is uniformly infinite.*

Proof. We will only prove Theorem 19.

For any x, and a positive integer n,

$$m(x)\leqq n \text{ implies } m\left(\frac{x}{n}\right)\leqq 1,$$

therefore $|||x|||\geqq n+1$ implies $m(x)=+\infty$ if m is uniformly discontinuous. This shows the assertion of this theorem.

§ 5. Examples.

"Discrete cases". (*modulared sequence spaces*).

Let $\varphi_\lambda(\lambda\in\Lambda)$ be a system of positive convex functions of positive real variable. Then the set of sequence $\{\xi_\lambda\}_{\lambda\in\Lambda}$ such that $\sum_{\lambda\in\Lambda}\varphi_\lambda(|\xi\xi_\lambda|) <+\infty$ for some $\xi>0$, constitutes a modulared linear space with the usual order, defining a modular by $m(x)=\sum_\lambda\varphi_\lambda(\xi_\lambda)$ where $x=\{\xi_\lambda\}_{\lambda\in\Lambda}$. This space $l(\varphi_\lambda)_{\lambda\in\Lambda}$ is monotone complete by this modular. We will investigate the properties of these modulars on the line of the former conditions.

For a convention, we can suppose that the convex functions φ_λ $(\lambda\in\Lambda)$ is *normalized*, i.e. $|||e_\lambda|||=1(\lambda\in\Lambda)$, where e_λ is an element of $l(\varphi_\lambda)_{\lambda\in\Lambda}$ such that

$$e_\lambda=\{\xi_\rho\}_{\rho\in\Lambda},\quad \begin{cases}\xi_\rho=1 & \text{if } \rho=\lambda\\ \xi_\rho=0 & \text{if } \rho\neq\lambda.\end{cases}$$

Proposition 1. *$l(\varphi_\lambda)_{\lambda\in\Lambda}$ is uniformly ascending if and only if we can find a number $n>0$ such that*

$$\inf_{\xi>0}\frac{\varphi_\lambda(\xi)}{\xi}=\varphi'_\lambda(1)\geqq\frac{1}{n}\qquad(\lambda\in\Lambda)$$

Proof. We need only prove the sufficiency. If $x=\{\xi_\lambda\}_{\lambda\in\Lambda}$ and $m(x)=\sum_\lambda\varphi_\lambda(\xi_\lambda)\leqq1$, then we have

$$m'(x)\geqq\sum_\lambda|\xi_\lambda|\varphi'_\lambda(1)\geqq\frac{1}{n}\sum\varphi_\lambda(\xi_\lambda),$$

$$\text{i.e.}\qquad m'(x)\geqq\frac{1}{n}m(x),$$

this concludes the sufficiency of the proposition.

Proposition 2. *$l(\varphi_\lambda)_{\lambda\in\Lambda}$ is uniformly semi-singular if and only if we can find a number $n>0$ such that*

$$\varphi_\lambda(\xi)\leqq1\quad\textit{implies}\quad\varphi_\lambda(\xi/n)=0.$$

Proof of this proposition is similar to that of the former proposition, therefore it is omitted.

Proposition 3. *$l(\varphi_\lambda)_{\lambda\in\Lambda}$ is uniformly infinitely linear if and only if, the following conditions are satisfied:*

(1) *$\varphi_\lambda(\xi)<+\infty$ for every $\xi\geqq0$ and there is a definite number n_0 such that*

$$\varphi_\lambda(1) \geqq \frac{1}{n_0} \lim_{\xi \to \infty} \frac{\varphi_\lambda(\xi)}{\xi} = \frac{1}{n_0} \varphi_\lambda^\infty(1) .$$

(2) $$\sum_\lambda \varphi_\lambda(\xi_\lambda) < +\infty \quad \textit{implies} \quad \sum_\lambda |\xi_\lambda| < +\infty .$$

Proof. (1) is deduced from Definition 3. If we have

$$\sum_\lambda \varphi_\lambda(|\xi_\lambda|) < +\infty, \quad \text{then} \quad x = \{\xi_\lambda\} \in l(\varphi_\lambda) .$$

Hence $$\|x\|_\infty = \sum_\lambda |\xi_\lambda| \varphi_\lambda^\infty(1) < +\infty .$$

Because of the inequality $|\xi_\lambda| \leqq |\xi_\lambda| \varphi_\lambda^\infty(1)$, we have

$$\sum_\lambda |\xi_\lambda| \leqq \sum_\lambda |\xi_\lambda| \varphi_\lambda^\infty(1) .$$

This prove the necessity of the conditions (1), (2).

Let $\sum_\lambda \varphi_\lambda(\xi_\lambda) \leqq 1$ and $x = \{\xi_\lambda\}$.

Then (1) and (2) show that

$$\|x\|_\infty \leqq n_0 \sum_\lambda |\xi_\lambda| < +\infty$$

This shows the sufficiency by Theorem 10.

Proposition 4. *$l(\varphi_\lambda)_{\lambda \in \Lambda}$ is uniformly infinite if and only if the following conditions are satisfied:*

(1) *For some integer $n_0 > 0$, we have*

$$\varphi_\lambda(n_0) = +\infty , \qquad \lambda \in \Lambda .$$

(2) *For some number $\varepsilon > 0$, we have*

$$\sum_\lambda \varphi_\lambda(\varepsilon) < +\infty .$$

Proof. Suppose that

$$\sum_\lambda \varphi_\lambda\left(\frac{1}{2^m}\right) = +\infty \quad \text{for every integer } m > 0 .$$

Selecting λ_j, $j = 1, 2, \cdots, i_m$, from Λ, we have

$$\sum_{j=1}^{i_m} \varphi_{\lambda_j}\left(\frac{1}{2^m}\right) \geqq 1 .$$

Hence, putting $x_m = e_{\lambda_1} + \cdots + e_{\lambda_{i_m}}$

where, $$e_\lambda = \begin{cases} 1 & \text{at } \lambda \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$|||x_m||| \geqq 2^m \quad \text{and} \quad m(x_m) < \infty .$$

This is a contradiction. Therefore (2) is a necessary condition.

It is easy to see that (1) is also a necessary condition.

Suppose that (1), (2) are satisfied.

From (1), $\sum_{\lambda} \varphi_\lambda(|\xi_\lambda|) < +\infty$ implies $\sup_{\lambda \in \Lambda} |\xi_\lambda| < +\infty$.

Hence, if $x = \{\xi_\lambda\} \in l(\varphi_\lambda)$, then $\sup_{\lambda \in \Lambda} |\xi_\lambda| < +\infty$.

Let $x_1 = \{1\}$ {i.e. this element has a value 1 at any co-ordinate $\lambda \in \Lambda$}. By (2) we can find that

$$x_1 \in l(\varphi_\lambda).$$

Hence we can find a number ε_0 such that

$$|||\varepsilon_0 x_1||| = 1 .$$

If $x = \{\xi_\lambda\} \in l(\varphi_\lambda)$, $|||x||| \geqq 1$, then there exists at least one $\lambda_0 \in \Lambda$ such that

$$|\xi_{\lambda_0}| \geqq \varepsilon_0 .$$

By virtue of (1) $|||x||| \geqq \dfrac{n}{\varepsilon_0}$ implies $m(x) = +\infty$.

This prove the sufficiency of the conditions (1), (2).

Proposition 5. *$l(\varphi_\lambda)$ is uniformly discontinuous if and only if, there exist $n > 0$ and a system of $\xi_\lambda > 0$, $\lambda \in \Lambda$ such that*

(1) $\varphi_\lambda(\xi_\lambda) < +\infty$ *and* $\varphi_\lambda(\xi_\lambda + \varepsilon) = +\infty$ *for every* $\varepsilon > 0$,

(2) $\sum_{\lambda} \varphi_\lambda(\xi_\lambda) < +\infty$.

Proposition 6. *$l(\varphi_\lambda)$ is uniformly asymptotically linear if and only if*

$$\sum_{\lambda} \gamma_\lambda(1) < +\infty$$

where

$$\gamma_\lambda(1) = \sup_{\xi > 0} \left\{ \varphi_\lambda^\infty(1) - \varphi_\lambda(\xi) \right\} .$$

Proofs of the above propositions are easy, therefore omitted.

"Non discrete cases"

1. *The converse of Theorem 6 is not true.*

An example of a modulared linear space which is monotone complete, uniformly finite, uniformly simple and ascending, but not uniformly ascending and uniformly infinitely linear is the following:

Let M_n be a real function such that

$$M_n(x) = \frac{1}{2} x^2 + \frac{1}{n} x \qquad (n=1,2,\cdots)$$

and $L(M_n)$ be a totality of measurable functions f defined on $[0,1]$ such that

$$\sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} M_n(\alpha |f(t)|) dt < +\infty \qquad \text{for some} \quad \alpha > 0 .$$

Then $L(M_n)$ is such an example.

2. *The converse of Theorem* 13 *is not true.*

An example of a modulared linear space which is uniformly infinitely linear, but not uniformly ascending is the following:

Let φ_n be a real function such that

$$\varphi_n(x) = \frac{1}{2} x^2 + \frac{1}{n^4} x \qquad \text{if} \quad x \leqq 1$$

$$\varphi_n(x) = \left(1 + \frac{1}{n^4}\right) x - \frac{1}{2} \qquad \text{if} \quad x > 1$$

and $L(\varphi_n)$ be a totality of measurable functions f defined on $[0,1]$ such that

$$\sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi_n(\alpha |f(t)|) dt < +\infty \qquad \text{for some} \quad \alpha > 0 .$$

Then $L(\varphi_n)$ is such an example.

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