ON THE COMMUTATIVE FAMILY OF SUBNORMAL OPERATORS

By

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Introduction. HALMOS has given in [3] the definition of a subnormal operator and the characteristic property of it. A bounded operator A defined on a HILBERT space $\mathfrak{H}$ is said to be *subnormal* if there exist a HILBERT space $\mathfrak{K}$ containing $\mathfrak{H}$ and a bounded normal operator N on $\mathfrak{K}$ such that $Ax=Nx$ for every x in $\mathfrak{H}$. Recently in [1] BRAM has made HALMOS' characterization simpler ([1], Theorem 1) and given another characteristic property ([1], Theorem 2) and some results about subnormal operators (for example, [1], Theorems 4, 7, 8, 9).

In this paper first we shall study the problem under what conditions it is possible to extend the commutative family of subnormal operators acting on a HILBERT space $\mathfrak{H}$ to the commutative family of normal operators on a HILBERT space $\mathfrak{K}$ containing $\mathfrak{H}$. Theorem 1 answers to this question. Then we shall give a generalization of BRAM's theorems (for example Theorem 6 and Theorem 7) and another simpler proof of BRAM's theorm about the spectrum of subnormal operators (Theorem 8). Theorem 3 is a generalization of COOPER's result in [2] (cf. [9], p. 393). Theorem 5 gives a new characterization of subnormal operators.

1. An abelian semi-group of subnormal operators. Throughout the paper, a HILBERT space is a vector space over the complex numbers, an operator is a bounded linear transformation unless denoted explicitly. For an operator A we denote by A^* an adjoint operator of A.

Lemma 1. *Let $A_l\,(l=1,2,\cdots,n)$ be n commutative operators on a HILBERT space $\mathfrak{H}$. If for every non-negative integer M and element $x_{i_1,i_2,\cdots,i_n}$ in $\mathfrak{H}$ $(0\leqq i_l\leqq M,\ l=1,2,\cdots,n)$*

$$(1.1)\qquad \sum_{\substack{i_l,j_l\geqq 0\\ l=1,2,\cdots,n}}^{M} (A_1^{i_1}A_2^{i_2}\cdots A_n^{i_n}x_{j_1,j_2,\cdots,j_n},\ A_1^{j_1}A_2^{j_2}\cdots A_n^{j_n}x_{i_1,i_2,\cdots,i_n})\geqq 0\ ,$$

then we have the inequality such that for every M, $x_{i_1,i_2,\cdots i_n}$ in $\mathfrak{H}$ $(0\leqq i_l\leqq M,$ $l=1,2,\cdots,n)$ and non-negative integer $\nu_l\,(l=1,2,\cdots,n)$

$$(1.2)\qquad \begin{aligned}&\sum_{\substack{i_l, j_l \geqq 0\\ l=1,2,\cdots,n}}^{M} (A_1^{i_1+\nu_1}A_2^{i_2+\nu_2}\cdots A_n^{i_n+\nu_n}x_{j_1,j_2,\cdots,j_n},\ A_1^{j_1+\nu_1}A_2^{j_2+\nu_2}\cdots A_n^{j_n+\nu_n}x_{i_1,i_2,\cdots,i_n})\\ &\leqq \|A_1\|^{2\nu_1}\|A_2\|^{2\nu_2}\cdots\|A_n\|^{2\nu_n}\sum_{\substack{i_l, j_l \geqq 0\\ l=1,2,\cdots,n}}^{M}(A_1^{i_1}\cdots A_n^{i_n}x_{j_1,\cdots,j_n},\ A_1^{j_1}\cdots A_n^{j_n}x_{i_1,\cdots,i_n}).\end{aligned}$$

Proof. Essentially the proof is the same as that of [1] Theorem 1. HEINZ's theorem ([5]) is essential.

Let $\mathfrak{H}_{i_1,i_2,\cdots i_n}$ $(i_l=0,1,2,\cdots;\ l=1,2,\cdots,n)$ be spaces isomorphic to $\mathfrak{H}$, $\mathfrak{R}$ be the direct sum of $\mathfrak{H}_{i_1,i_2,\cdots,i_n}$, that is, $\mathfrak{R}=\sum_{\substack{i_l\geqq 0\\ l=1,2,\cdots,n}}\oplus\mathfrak{H}_{i_1,i_2,\cdots,i_n}$. We denote the element of $\mathfrak{R}$ by $\bar{x}=\{x_{i_1,i_2,\cdots,i_n}\}$, where $x_{i_1,i_2,\cdots,i_n}$ is $i_1,i_2,\cdots,i_n$-component of $\bar{x}$. For a positive number $\varepsilon>0$ we put $B_l=(\|A_l\|+\varepsilon)^{-1}A_l (l=1,2,\cdots,n)$, then $\|B_l\|<1$ $(l=1,2,\cdots,n)$.

We can define a linear transformation S on $\mathfrak{R}$ such that

$$(1.3)\qquad \begin{cases} S\bar{x}=\bar{y}=\{y_{i_1,i_2,\cdots,i_n}\},\quad \bar{x}=\{x_{i_1,i_2,\cdots,i_n}\},\\ y_{i_1,i_2,\cdots,i_n}=\sum_{\substack{j_l\geqq 0\\ l=1,2,\cdots,n}} B_n^{*j_n}\cdots B_2^{*j_2}B_1^{*j_1}B_1^{i_1}B_2^{i_2}\cdots B_n^{i_n}x_{j_1,j_2,\cdots j_n}.\end{cases}$$

As $\|B_l\|<1$ $(l=1,2,\cdots,n)$, the right hand of (1.3) is convergent and S is a bounded operator. Because

$$\begin{aligned}\|S\bar{x}\|^2&=\sum_{\substack{i_l\geqq 0\\ l=1,2,\cdots n}}\Big\|\sum_{\substack{j_l\geqq 0\\ l=1,2,\cdots,n}}B_n^{*j_n}\cdots B_1^{*j_1}B_1^{i_1}\cdots B_n^{i_n}x_{j_1,j_2,\cdots,j_n}\Big\|^2\\ &\leqq\sum_{\substack{i_l\geqq 0\\ l=1,2,\cdots,n}}\Big\{\sum_{\substack{j_l\geqq 0\\ l=1,2,\cdots,n}}\|B_n^{*j_n}\cdots B_1^{*j_1}B_1^{i_1}\cdots B_n^{i_n}\|\,\|x_{j_1,j_2,\cdots,j_n}\|\Big\}^2\\ &\leqq\sum_{\substack{i_l\geqq 0\\ l=1,2,\cdots,n}}\Big\{\sum_{\substack{j_l\geqq 0\\ l=1,2,\cdots,n}}\|B_n\|^{2j_n+2i_n}\cdots\|B_1\|^{2j_1+2i_1}\Big\}\Big\{\sum_{\substack{j_l\geqq 0\\ l=1,2,\cdots,n}}\|x_{j_1,j_2,\cdots j_n}\|^2\Big\}\\ &=(1-\|B_n\|^2)^{-2}\cdots(1-\|B_1\|^2)^{-2}\|x\|^2.\end{aligned}$$

So we have

$$(1.4)\qquad \|S\bar{x}\|\leqq(1-\|B_n\|^2)^{-1}\cdots(1-\|B_1\|^2)^{-1}\|\bar{x}\|\qquad(\bar{x}\in\mathfrak{R}).$$

On the other hand for $\bar{x}=\{x_{i_1,i_2,\cdots,i_n}\}$ whose components equal to zero except for finite number of $x_{i_1,i_2,\cdots,i_n}$ we have by assumption (1.1)

$$(1.5)\qquad\begin{aligned}(S\bar{x},\bar{x})&=\sum_{\substack{i_l,j_l\geqq 0\\ l=1,2,\cdots,n}}^{M}(B_1^{i_1}\cdots B_n^{i_n}x_{j_1,\cdots,j_n},\ B_1^{j_1}\cdots B_n^{j_n}x_{i_1,\cdots,i_n})\\ &=\sum_{\substack{i_l,j_l\geqq 0\\ l=1,2,\cdots,n}}^{M}(A_1^{i_1}\cdots A_n^{i_n}z_{j_1,j_2,\cdots,j_n},\ A_1^{j_1}\cdots A_n^{j_n}z_{i_1,\cdots,i_n})\geqq 0,\end{aligned}$$

where $\quad z_{i_1,i_2,\cdots,i_n}=(\|A_1\|+\varepsilon)^{-i_1}\cdots(\|A_n\|+\varepsilon)^{-i_n}x_{i_1,i_2,\cdots,i_n}.$

The whale of such $\bar{x}$ is dense in $\mathfrak{R}$ evidently. Therefore S is a bounded positive symmetric operator on $\mathfrak{R}$.

In the same way we define a linear transformation T on $\mathfrak{R}$ such that

$$(1.6)\qquad \begin{cases} T\bar{x}=\bar{z}=\{z_{i_1,i_2,\cdots,i_n}\}, \quad \bar{x}=\{x_{i_1,i_2,\cdots,i_n}\}, \\ z_{i_1,i_2,\cdots,i_n}=\sum\limits_{\substack{j_l\geqq 0\\ l=1,2,\cdots,n}} B_n^{*j_n+\nu_n}\cdots B_1^{*j_1+\nu_1}B_1^{i_1+\nu_1}\cdots B_n^{i_n+\nu_n}x_{j_1,j_2,\cdots j_n}. \end{cases}$$

It is proved like S that T is a bounded positive symmetric operator on $\mathfrak{R}$.

Next we see

$$(1.7)\qquad \|T\bar{x}\|\leqq\|S\bar{x}\| \qquad (\bar{x}\in\mathfrak{R}).$$

Because

$$\|T\bar{x}\|^2\leqq\sum_{\substack{i_l\geqq 0\\ l=1,2,\cdots,n}}\Big\|\sum_{\substack{j_l\geqq 0\\ l=1,2,\cdots,n}} B_n^{*j_n+\nu_n}\cdots B_1^{*j_1+\nu_1}B_1^{i_1+\nu_1}\cdots B_n^{i_n+\nu_n}x_{j_1,j_2,\cdots,j_n}\Big\|^2$$
$$\leqq\sum_{\substack{i_l\geqq 0\\ l=1,2,\cdots,n}}\Big\|\sum_{\substack{j_l\geqq 0\\ l=1,2,\cdots,n}} B_n^{*j_n+\nu_n}\cdots B_1^{*j_1+\nu_1}B_1^{i_1}\cdots B_n^{i_n}x_{j_1,j_2,\cdots,j_n}\Big\|^2,$$

as B_l $(l=1,2,\cdots,n)$ are commutative we have

$$\leqq\sum_{\substack{i_l\geqq 0\\ l=1,2,\cdots,n}}\|B_n^{*\nu_n}\cdots B_1^{*\nu_1}\|^2\Big\|\sum_{\substack{j_l\geqq 0\\ l=1,2,\cdots,n}} B_n^{*j_n}\cdots B_1^{*j_1}B_1^{i_1}\cdots B_n^{i_n}x_{j_1,j_2,\cdots,j_n}\Big\|^2$$
$$\leqq\|B_1\|^{2\nu_1}\cdots\|B_n\|^{2\nu_n}\|S\bar{x}\|^2\leqq\|S\bar{x}\|^2.$$

Owing to Heinz's theorem (cf. [5] or Kato [7]) we obtain from (1.7)

$$(1.8)\qquad (T\bar{x},\bar{x})\leqq(S\bar{x},\bar{x}) \qquad (\bar{x}\in\mathfrak{R}).$$

Hence

$$\sum_{\substack{i_l,j_l\geqq 0\\ l=1,2,\cdots,n}}^{M}(B_1^{i_1+\nu_1}\cdots B_n^{i_n+\nu_n}x_{j_1,\cdots,j_n},\ B^{j_1+\nu_1}\cdots B_n^{j_n+\nu_n}x_{i_1,\cdots,i_n})$$
$$\leqq\sum_{\substack{i_l,j_l\geqq 0\\ l=1,2,\cdots,n}}^{M}(B_1^{i_1}\cdots B_n^{i_n}x_{j_1,\cdots,j_n},\ B_1^{j_1}\cdots B_n^{j_n}x_{i_1,\cdots,i_n}).$$

Therefore we have

$$(1.9)\qquad \begin{aligned}&\sum_{\substack{i_l,j_l\geqq 0\\ l=1,2,\cdots,n}}^{M}(A_1^{i_1+\nu_1}\cdots A_n^{i_n+\nu_n}x_{j_1,\cdots,j_n},\ A_1^{j_1+\nu_1}\cdots A_n^{j_n+\nu_n}x_{i_1,\cdots,i_n})\\ &\leqq(\|A_1\|+\varepsilon)^{2\nu_1}\cdots(\|A_n\|+\varepsilon)^{2\nu_n}\sum_{\substack{i_l,j_l\geqq 0\\ l=1,2,\cdots,n}}^{M}(A_1^{i_1}\cdots A_n^{i_n}x_{j_1,\cdots,j_n},\ A_1^{j_1}\cdots A_n^{j_n}x_{i_1,\cdots,i_n}).\end{aligned}$$

Remenbering ε was an arbitrary positive number we obtain the inequality (1.2) from (1.9).

Let Γ be an abelian semi-group having at least one zero element 0. The function A_γ $(\gamma \in \Gamma)$ from Γ into the algebra of bounded operators on a HILBERT space $\mathfrak{H}$ is called an operator representation of Γ if

$$(1.10) \qquad \begin{cases} A_{\gamma_1} A_{\gamma_2} = A_{\gamma_1+\gamma_2} \ (\gamma_1, \gamma_2 \in \Gamma) \text{ and} \\ A_0 \quad = I \qquad \text{(an identity operator on } \mathfrak{H}) . \end{cases}$$

Through the paper such an operator representation of Γ will be denoted by A_γ $(\gamma \in \Gamma, \mathfrak{H})$.

Definition 1. *An operator representation* A_γ $(\gamma \in \Gamma, \mathfrak{H})$ *will be called positive definite if*

$$(1.11) \qquad \sum_{i,j} (A_{\gamma_i} x_j,\ A_{\gamma_j} x_i) \geqq 0$$

for every finite number of x_i *in* $\mathfrak{H}$ *and* γ_i *in* Γ.

From Lemma 1 following lemma is proved.

Lemma 2. *Let an operator representation* A_γ $(\gamma \in \Gamma, \mathfrak{H})$ *be positive definite. Then we have for every finite number of* x_i *in* $\mathfrak{H}$, γ_i *in* Γ *and an arbitrary* ρ *in* Γ

$$(1.12) \qquad \sum_{i,j} (A_{\gamma_i+\rho} x_j,\ A_{\gamma_j+\rho} x_i) \leqq \|A_\rho\|^2 \sum_{i,j} (A_{\gamma_i} x_j, A_{\gamma_j} x_i).$$

Proof. Assuming that i and j run from 1 to $n-1$ we put $A_i = A_{\gamma_i}$ $(1 \leqq i \leqq n-1)$ and $A_n = A_\rho$. By the fact that $A_\gamma (\gamma \in \Gamma, \mathfrak{H})$ is positive definite we can see easily A_l $(l = 1, 2, \cdots, n)$ satisfy the assumption of Lemma 1 namely the inequality (1.1). Therefore putting in (1.2)

$$\begin{cases} x_{i_1, i_2, \cdots, i_n} = x_l \text{ if } i_l = 1,\ i_m = 0\,(m \neq l),\ l = 1, 2, \cdots, n-1, \\ x_{i_1, i_2, \cdots, i_n} = 0 \quad \text{besides}, \\ \nu_l = 0 \qquad (l = 1, 2, \cdots, n-1),\ \nu_n = 1, \end{cases}$$

we have (1.12).

Definition 2. *For two operator representations of* Γ A_γ $(\gamma \in \Gamma, \mathfrak{H})$ *and* B_γ $(\gamma \in \Gamma, \mathfrak{K})$ *it is defined that* B_γ $(\gamma \in \Gamma, \mathfrak{K})$ *is an extension of* A_γ $(\gamma \in \Gamma, \mathfrak{H})$ *if following conditions are satisfied*

$$(1.13) \qquad \mathfrak{K} \supset \mathfrak{H} \text{ and } B_\gamma x = A_\gamma x \ (x \in \mathfrak{H}) \text{ for all } \gamma \in \Gamma.$$

If all B_γ *are normal operators on* $\mathfrak{K}$ *we call* B_γ $(\gamma \in \Gamma, \mathfrak{H})$ *the normal extension of* A_γ $(\gamma \in \Gamma, \mathfrak{H})$.

We obtain the following theorem which is a generalization of

HALMOS' theorem ([3], Theorem 3).

Theorem 1. *An operator representation* A_γ $(\gamma\in\Gamma,\ \mathfrak{H})$ *of an abelian semi-group* Γ *has a normal extension* N_γ $(\gamma\in\Gamma,\ \mathfrak{K})$ *if and only if* A_γ $(\gamma\in\Gamma,\ \mathfrak{H})$ *is positive definite.*

Proof. Necessity. For every finite number of x_i in $\mathfrak{H}$ and γ_i in Γ we have

$$\sum_{i,j}(A_{\gamma_i}x_j,\ A_{\gamma_j}x_i)=\sum_{i,j}(N_{\gamma_i}x_j,\ N_{\gamma_j}x_i)=\sum_{i,j}(N^*_{\gamma_j}x_j,\ N^*_{\gamma_i}x_i)$$
$$=\|\sum_i N^*_{\gamma_i}x_i\|^2\geqq 0\,.$$

Sufficiency. The construction of $\mathfrak{K}$ and N_γ $(\gamma\in\Gamma)$ are obtained by generalizing HALMOS' method ([3]) to the case of semi-groups.

Putting $\mathfrak{F}$ the Cartesian product of $\mathfrak{H}_\gamma$ $(\gamma\in\Gamma)$, namely $\mathfrak{F}=\prod_{\gamma\in\Gamma}\mathfrak{H}_\gamma$, here every $\mathfrak{H}_\gamma$ is isomorphic to $\mathfrak{H}$. We shall denote the element of $\mathfrak{F}$ by $\bar{x}=\{x_\gamma\}$ whose γ-component is x_γ. Let $\mathfrak{D}=\{\bar{x};\ \bar{x}=\{x_\gamma\},\ x_\gamma\neq 0$ at most finite number of $\gamma\}$, then $\mathfrak{D}$ is a linear manifold in $\mathfrak{F}$. We shall introduce onto $\mathfrak{D}$ a bilinear functional such that

$$\langle\bar{x},\bar{y}\rangle=\sum_{\gamma,\gamma'\in\Gamma}(A_\gamma x_{\gamma'},\ A_{\gamma'}y_\gamma)\qquad(\bar{x},\bar{y}\in\mathfrak{D}),\tag{1.15}$$

for brevity we identify all $\mathfrak{H}_\gamma$ with $\mathfrak{H}$. Since A_γ $(\gamma\in\Gamma,\ \mathfrak{H})$ is positive definite, $\langle\bar{x},\bar{y}\rangle$ is a positive symmetric bilinear functional. Putting $\mathfrak{Z}=\{\bar{x};\ \langle\bar{x},\bar{x}\rangle=0\}$, then naturally the quotient space $\mathfrak{D}/\mathfrak{Z}$ is an inner product space. The completion $\mathfrak{K}$ of $\mathfrak{D}/\mathfrak{Z}$ by this inner product is a HILBERT space. Evidently the correspondence $\mathfrak{H}\ni x\longrightarrow\bar{x}=\{x_\gamma\}$, where $x_0=x$ and $x_\gamma=0\ (\gamma\neq 0)$, is an isomorphism from $\mathfrak{H}$ into $\mathfrak{D}$. Thus $\mathfrak{H}$ is imbeded into $\mathfrak{K}$.

Next we shall define linear transformations N_ρ $(\rho\in\Gamma)$ on $\mathfrak{D}$ such that

$$\begin{cases} N_\rho\bar{x}=\bar{y}=\{y_\gamma\}, & \bar{x}=\{x_\gamma\}\,,\\ y_\gamma=A_\rho x_\gamma & (x\in\Gamma)\,.\end{cases}\tag{1.16}$$

Then we have from Lemma 2

$$\langle N_\rho\bar{x},N_\rho\bar{x}\rangle=\sum_{\gamma,\gamma'\in\Gamma}(A_{\gamma+\rho}x_{\gamma'},\ A_{\gamma'+\rho}x_\gamma)$$
$$\leqq\|A_\rho\|^2\sum_{\gamma,\gamma'\in\Gamma}(A_\gamma x_{\gamma'},\ A_{\gamma'}x_\gamma)=\|A_\rho\|^2\langle\bar{x},\bar{x}\rangle\,.$$

Therefore N_ρ is regarded as a bounded operator on $\mathfrak{K}$. We shall denote this operator on $\mathfrak{K}$ by the same notation N_ρ.

We shall show N_ρ is a normal operator on $\mathfrak{K}$. For every $\rho,\gamma\in\Gamma$, putting $\Gamma_{\gamma-\rho}=\{\delta;\ \delta+\rho=\gamma\}$, and we introduce linear transformations

$L_\rho\,(\rho\in\Gamma)$ on $\mathfrak{D}$ such that

$$(1.18)\qquad \begin{cases} L_\rho\bar{x}=\bar{z}=\{z_\gamma\}, \quad \bar{x}=\{x_\gamma\}, \\ z_\gamma = \sum_{\delta\in\Gamma_{\gamma-\rho}} x_\delta \text{ or } =0 \text{ if } \Gamma_{\gamma-\rho}=\phi, \end{cases}$$

generally $\Gamma_{\gamma-\rho}$ is an infinite set, but for $\bar{x}$ $x_\delta=0$ except for finite number of δ, so $\sum_{\delta\in\Gamma_{\gamma-\rho}} x_\delta$ has a meaning. Thus

$$(1.19)\qquad \begin{aligned}\langle L_\rho\bar{x}, L_\rho\bar{x}\rangle &= \sum_{\gamma,\gamma'\in\Gamma}(A_\gamma \sum_{\delta\ni\Gamma_{\gamma'-\rho}} x_\delta, A_{\gamma'} \sum_{\delta\in\Gamma_{\gamma-\rho}} x_\delta) \\ &= \sum_{\delta,\delta'\in\Gamma}(A_{\delta+\rho}x_{\delta'}, A_{\delta'+\rho}x_\delta)=\langle N_\rho\bar{x}, N_\rho\bar{x}\rangle \leqq \|A_\rho\|^2\langle\bar{x},\bar{x}\rangle.\end{aligned}$$

Therefore L_ρ defines a bounded operator on $\mathfrak{K}$, we shall denote that operator by the same notation L_ρ. Then likeweise we have

$$(1.20)\qquad \langle L_\rho\bar{x}, \bar{y}\rangle=\langle\bar{x}, N_\rho\bar{y}\rangle \qquad (\bar{x},\bar{y}\in\mathfrak{D}),$$

therefore $L_\rho^*=N_\rho\ (\rho\in\Gamma)$ on $\mathfrak{K}$. From (1.19) and (1.20) $N_\rho\ (\rho\in\Gamma)$ are normal operators on $\mathfrak{K}$. And evidently $N_\rho=A_\rho$ on $\mathfrak{H}$. Furthermore by (1.16) $N_0=I$ and $N_{\gamma_1}N_{\gamma_2}=N_{\gamma_1+\gamma_2}\,(\gamma_1,\gamma_2\in\Gamma)$. The proof is complete.

Definition 3. *Let $N_\gamma\,(\gamma\in\Gamma,\mathfrak{K})$ be a normal extension of $A_\gamma\,(\gamma\in\Gamma,\mathfrak{H})$. If for any subspace $\mathfrak{K}_0$ such that $\mathfrak{K}\supset\mathfrak{K}_0\supset\mathfrak{H}$ and every N_γ is reduced by $\mathfrak{K}_0$ we have $\mathfrak{K}_0=\mathfrak{K}$, then $N_\gamma\,(\gamma\in\Gamma,\mathfrak{K})$ is called a minimal normal extension of $A_\gamma\,(\gamma\in\Gamma,\mathfrak{H})$.*

Putting $\mathfrak{L}=\{\sum_i N_{\gamma_i}^* x_i;$ for every finite number of x_i in $\mathfrak{H}$ and γ_i in $\Gamma\}$, then evidently the closure of $\mathfrak{L}$ in $\mathfrak{K}$ is a subspace containing $\mathfrak{H}$ and invariant under every N_γ and N_γ^*. Therefore the necessary and sufficient condition that $N_\gamma\ (\gamma\in\Gamma,\mathfrak{K})$ be a minimal normal extension of $A_\gamma\,(\gamma\in\Gamma,\mathfrak{H})$ is that a linear manifold $\mathfrak{L}$ be dense in $\mathfrak{K}$. It is noted that the normal extension $N_\gamma\,(\gamma\in\Gamma,\mathfrak{K})$ which was obtained in Theorem 1 is a minimal normal extension of $A_\gamma\,(\gamma\in\Gamma,\mathfrak{H})$.

Theorem 2. *A minimal normal extension $N_\gamma\,(\gamma\in\Gamma,\mathfrak{K})$ of $A_\gamma\,(\gamma\in\Gamma,\mathfrak{H})$ is unique except for unitary isomorphism and $\|N_\gamma\|_{\mathfrak{K}}=\|A_\gamma\|_{\mathfrak{H}}\,(\gamma\in\Gamma)$, where $\|N_\gamma\|_{\mathfrak{K}}$ and $\|A_\gamma\|_{\mathfrak{H}}$ are respectively the operator norms on $\mathfrak{K}$ and $\mathfrak{H}$.*

Proof. Let $N_\gamma\,(\gamma\in\Gamma,\mathfrak{K}_1)$ and $M_\gamma\,(\gamma\in\Gamma,\mathfrak{K}_2)$ be two minimal normal extensions of $A_\gamma\,(\gamma\in\Gamma,\mathfrak{H})$ and $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be respectively linear manifolds defined above (cf. after Definition 3). Then we have

$$\begin{aligned}\|\sum_i N_{\gamma_i}^* x_i\|_1^2 &= \sum_{i,j}(N_{\gamma_j}x_i, N_{\gamma_i}x_j)=\sum_{i,j}(A_{\gamma_j}x_i, A_{\gamma_i}x_j) \\ &= \sum_{i,j}(M_{\gamma_j}x_i, M_{\gamma_i}x_j)=\|\sum_i M_{\gamma_i}^* x_i\|_2^2.\end{aligned}$$

Therefore if we make correspond $\sum_i N^*_{\gamma_i} x_i$ to $\sum_i M^*_{\gamma_i} x_i$, then we have an isometric transformation from $\mathfrak{L}_1$ onto $\mathfrak{L}_2$. Since $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are dense respectively in $\mathfrak{K}_1$ and $\mathfrak{K}_2$. Consequently $N_\gamma\,(\gamma\in\Gamma,\ \mathfrak{K}_1)$ and $M_\gamma\,(\gamma\in\Gamma,\ \mathfrak{K}_3)$ are unitary equivalent.

From (1.17) we have $\|N_\rho\|_{\mathfrak{K}}\leqq\|A_\rho\|_{\mathfrak{H}}$, on the other hand from $\mathfrak{H}\subset\mathfrak{K}$ we have $\|A_\rho\|_{\mathfrak{H}}\leqq\|N_\rho\|_{\mathfrak{K}}$, therefore we obtain $\|N_\rho\|_{\mathfrak{K}}=\|A_\rho\|_{\mathfrak{H}}$ $(\rho\in\Gamma)$.

Remark. The fact $\|N_\gamma\|_{\mathfrak{K}}=\|A_\gamma\|_{\mathfrak{H}}$ $(\gamma\in\Gamma)$ is a generalization of BRAM [1] Lemma 2, but remarked that HALMOS' theorem about the spectrum of subnormal operators is not necessary.

2. A commutative family of isometric operators. In this section the partially isometric operator V such that $V^*V=I$ will be called isometric simply. By the application of Theorem 1 we can show the following Theorem 3. This is a generalization of COOPER's result ([2] or cf. [9] p. 393) about the continuous one parameter semi-group $V_t(t\geqq0)$ consisting of isometric operators. In our proof any assumption about the parameter is not necessary.

Theorem 3. *Let V_γ $(\gamma\in\Gamma,\ \mathfrak{H})$ be an operator representation consisting of isometric operators. Then it can be extended to an unitary operator representation U_γ $(\gamma\in\Gamma,\mathfrak{K})$.*

Let $\mathfrak{V}=\{V\}$ be a commutative family of isometric operators on $\mathfrak{H}$. As the semi-group generated by $\mathfrak{V}$ consists of isometric operators, from Theorem 3 we can extend $\mathfrak{V}$ to a commutative family $\{\mathfrak{U}=U\}$ of unitary operators.

Before the proof we shall show the following Lemmas.

Lemma 3. *Let $A_\gamma\,(\gamma\in\Gamma,\mathfrak{H})$ be positive definite, $N_\gamma\,(\gamma\in\Gamma,\mathfrak{K})$ be a minimal extension of A_γ $(\gamma\in\Gamma,\mathfrak{H})$ and B be a bounded operator on $\mathfrak{H}$.*

a) The necessary and sufficient conditions that B can be extended to an operator L on $\mathfrak{K}$ being commutative with all N_γ $(\gamma\in\Gamma)$ is that

$$\text{(i)}\qquad BA_\gamma = A_\gamma B \qquad (\gamma\in\Gamma)$$

and some positive number $C>0$ exist such that

$$\text{(2. 1)}\quad\text{(ii)}\qquad \sum_{i,j}(A_{\gamma_i}Bx_j,\ A_{\gamma_j}Bx_i)\leqq C\sum_{i,j}(A_{\gamma_i}x_j,\ A_{\gamma_j}x_i)$$

for every finite number x_i in $\mathfrak{H}$ and γ_i in Γ. And such L is unique.

b) Let B_1 and B_2 be bounded operators on $\mathfrak{H}$ and satisfy conditions (i) *and* (ii) *in a) and L_1 and L_2 be the extensions on $\mathfrak{K}$ of B_1 and B_2 respectively. Then if B_1 and B_2 are commutative, L_1 and L_2 are commutative also.*

c) If adding to (i), (ii) *of a) B is a normal operator, then L is also a* *normal operator on* $\mathfrak{K}$.

Proof.

a). Necessity. (i) is evident. By observing that

$$(2.2)\qquad \sum_{i,j}(A_{\gamma_i}Bx_j,\ A_{\gamma_j}Bx_i)=\sum_{i,j}(N^*_{\gamma_i}Lx_i,\ N^*_{\gamma_j}Lx_j)=\|L(\sum_i N^*_{\gamma_i}x_i)\|^2$$
$$\leqq \|L\|^2\|\sum_i N^*_{\gamma_i}x_i\|^2=\|L\|^2\sum_{i,j}(A_{\gamma_i}x_j,\ A_{\gamma_j}x_i),$$

(ii) is obtained.

Sufficiency. First we define a linear transformation L for the element of $\mathfrak{L}$ such that $L(\sum_i N^*_{\gamma_i}x_i)=\sum_i N^*_{\gamma_i}Bx_i$. Then

$$\|L(\sum N^*_{\gamma_i}x_i)\|^2=\sum_{i,j}(N_{\gamma_i}Bx_j,\ N_{\gamma_j}Bx_i)$$
$$\leqq C\sum_{i,j}(A_{\gamma_i}Bx_j,\ A_{\gamma_j}Bx_i)=C\|\sum_i N^*_{\gamma_i}x_i\|^2.$$

Hence L is a bounded operator on $\mathfrak{L}$, and L can be extended onto $\mathfrak{K}$ uniquely. We see easily $LN_\gamma=N_\gamma L\ (\gamma\in\Gamma)$ on $\mathfrak{L}$ and L is unique on $\mathfrak{L}$. Thus we have conclusion.

b).
$$L_1L_2(\sum_i N^*_{\gamma_i}x_i)=\sum_i N^*_{\gamma_i}B_1B_2x_i$$
$$=\sum_i N^*_{\gamma_i}B_2B_1x_i=L_2L_1(\sum_i N^*_{\gamma_i}x_i),$$

hence $L_1L_2=L_2L_1$.

c). As B is normal and commutative to all A_γ, B^* commutes with all A_γ $\gamma\in\Gamma$. And

$$\sum_{i,j}(A_{\gamma_i}B^*x_j,\ A_{\gamma_j}B^*x_i)=\sum_{i,j}(A_{\gamma_i}Bx_j,\ A_{\gamma_j}Bx_i)$$
$$\leqq C\sum_{i,j}(A_{\gamma_i}x_j,\ A_{\gamma_j}x_i)\ .$$

Therefore from a) B^* has a extension M on $\mathfrak{K}$ uniquely. From b) L and M are commutative and

$$(L(\sum_i N^*_{\gamma_i}x_i),\ \sum_i N^*_{\gamma_i}y_i)=\sum_{i,j}(A_{\gamma_j}Bx_i,\ A_{\gamma_i}y_j)$$
$$=\sum_{i,j}(A_{\gamma_j}x_i,\ A_{\gamma_i}B^*y_j)=(\sum_i N^*_{\gamma_i}x_i,\ M(\sum_i N^*_{\gamma_i}y_j)).$$

Thus we obtain $L^*=M$, consequently L is a normal operator on $\mathfrak{K}$.

Lemma 4. *Let* V_l $(l=1,2,\cdots,n)$ *be* n *commutative isometric operators on* $\mathfrak{H}$, *then we can extend them to* n *commutative unitary operators* U_l $(l=1,2,\cdots,n)$ *on* $\mathfrak{K}$ *containing* $\mathfrak{H}$.

Proof. In the case $n=1$, it is evident that V_1 can be extended to the unitary operator. Therefore the minimal normal extension of V_1

is an unitary operator. Let a semi-group generated by $V_1, V_2, \cdots, V_\nu$ $(\nu<n)$ have a minimal normal extension consisting of unitary operators on $\mathfrak{H}_0$ containing $\mathfrak{H}$ and W_l $(l=1,2,\cdots,\nu)$ be extensions of $V_l(l=1,2,\cdots,\nu)$. By Lemma 3, a) $V_{\nu+1}$ can be extended to the operator $W_{\nu+1}$ on $\mathfrak{H}_0$. It is easily proved $W_{\nu+1}$ is isometric operator on $\mathfrak{H}$. Hence putting $U_{\nu+1}$ a minimal normal extension of $W_{\nu+1}$ and $\mathfrak{K}$ the space on which $U_{\nu+1}$ is defined, again from Lemma 3, a) W_l $(l=1,2,\cdots,\nu)$ can be extended to U_l $(l=1,2,\cdots,\nu)$ defined on $\mathfrak{K}$. From Lemma 3, b) U_l $(l=1,2,\cdots,\nu+1)$ are commutative and U_l $(l=1,2,\cdots,\nu)$ are isometric, and normal from Lemma 3, c). Therefore U_l $(l=1,2,\cdots,\nu+1)$ are commutative unitary operators on $\mathfrak{K}$. By the induction the conclusion is obtained.

Proof of Theorem 3. From Lemma 4 V_γ $(\gamma\in\Gamma, \mathfrak{H})$ is positive definite. Therefore from Theorem 1 V_γ $(\gamma\in\Gamma, \mathfrak{H})$ has a minimal normal extension U_γ $(\gamma\in\Gamma, \mathfrak{K})$. If we replace A_γ with V_γ and N_γ with U_γ in the inequality (1.17), we have $\langle U_\rho\bar{x}, U_\rho\bar{x}\rangle=\langle\bar{x},\bar{x}\rangle$. Therefore U_γ $(\gamma\in\Gamma)$ are unitary operators on $\mathfrak{K}$.

Remark. In Lemma 3 and Lemma 4, Theorem 1 is not used essentially. And by Maximal theorem (or transfinite induction) and Lemma 4, Theorem 3 is proved independently of Theorem 1.

3. A continuous one parameter semi-group. In this section we shall study a continuous one parameter semi-group consisting of subnormal operators and give two types of characterization of subnormal operators. One parameter family of bounded operators A_t $(t\geqq0)$ on $\mathfrak{H}$ is called continuous one parameter semi-group when

$$
(3.1)\qquad \begin{array}{ll} \text{(i)} & A_{t_1}A_{t_2}=A_{t_1+t_2}\quad (t_1\geqq0,\ t_2\geqq0),\ A_0=I, \\ \text{(ii)} & \text{weakly continuous on } t\geqq0. \end{array}
$$

Lemma 5. *Continuous one parameter semi-group $A_t(t\geqq0, \mathfrak{H})$ of subnormal operators is positive definite.*

Proof. For an arbitrary finite number of $t_i\geqq0$ $(i=1,2,\cdots,n)$ we find siquences of positive rational numbers $r_{\nu,i}$ $(\nu=1,2,\cdots;\ i=1,2,\cdots,n)$ such that $\lim_{\nu\to\infty} r_{\nu,i}=t_i$ $(i=1,2,\cdots,n)$. We can put $r_{\nu,i}=b_{\nu,i}/a_\nu$ $(\nu=1,2,\cdots;\ i=1,2,\cdots,n)$, where $b_{\nu,i}$, a_ν are positive integers. Since one parameter semi-group is weakly continuous if and only if strongly continuous ([6]), we have

$$
(3.2)\qquad \sum_{i,j}(A_{t_i}x_j,\ A_{t_j}x_i)=\lim_{\nu\to\infty}\sum_{i,j}(A_{r_{\nu,i}}x_j,\ A_{r_{\nu,j}}x_i) =\lim_{\nu\to\infty}\sum_{i,j}((A_{1/a_\nu})^{b_{\nu,i}}x_j,\ (A_{1/a_\nu})^{b_{\nu,j}}x_i)
$$

On the other hand every A_{1/a_ν} is subnormal by assumption, hence

$$\sum_{i,j}((A_{1/a_\nu})^{b_{\nu,i}}x_j,\ (A_{1/a_\nu})^{b_{\nu,j}}x_i) \geqq 0$$

Therefore $A_t(t \geqq 0, \mathfrak{H})$ is positive definite.

Theorem 4. *Continuous one parameter semi-group $A_t\,(t \geqq 0,\ \mathfrak{H})$ of subnormal operators can be extended to continuous one parameter semi-group $N_t\,(t \geqq 0, \mathfrak{K})$ consisting of normal operators on $\mathfrak{K}$ containing $\mathfrak{H}$.*

Proof. From Lemma 5 $A_t\,(t \geqq 0, \mathfrak{H})$ has a minimal normal extension $N_t\,(t \geqq 0, \mathfrak{K})$. We shall show the continuity of N_t about the parameter. It is evident N_t is continuous about $t \geqq 0$ on the linear manifold $\mathfrak{L}$ (cf. Def. 3) of $\mathfrak{K}$, and for any $t_0 \geqq 0$ and a sequence of rational numbers r_ν $(\nu=1,2,\cdots)$ such that $\lim\limits_{\nu\to\infty} r_\nu = t_0$ $\{\|N_{r_\nu}\|;\ \nu=1,2,\cdots\}$ is uniformly bounded, because $\|N_{r_\nu}\| = \|N_1\|^{r_\nu}(\nu=1,2,\cdots)$. Therefore easily we can see $\lim\limits_{\nu\to\infty} N_{r_\nu} = N_{t_0}$ strongly by observing that $\mathfrak{L}$ is dense in $\mathfrak{K}$ and $\{\|N_{r_\nu}\|;\ \nu=1,2,\cdots\}$ is uniformly bounded. Thus N_t is strongly continuous about $t \geqq 0$.

Remark. (i) If in Theorem 4 $\mathfrak{H}$ is separable, the space $\mathfrak{K}$ of the minimal normal extension $N_t\,(t \geqq 0,\ \mathfrak{K})$ is also separable.

(ii) From Theorem 2 we have $\|A_t\| = \|A_1\|^t (t \geqq 0)$ for every continuous one parameter semi-group $A_t\,(t \geqq 0)$ of subnormal operators.

Theorem 5. *A bounded operator A on $\mathfrak{H}$ is subnormal if and only if one parameter semi-group* $\exp(tA)$ $(t \geqq 0)$ *is positive definite.*

Proof. Necessity. Let N be a minimal normal extension of A. Then we have

$$(3.3)\qquad \sum_{i,j}(\exp(t_iA)x_j,\ \exp(t_jA)x_i) = \sum_{i,j}(\exp(t_jN^*)x_j,\ \exp(t_iN^*)x_i)$$
$$= \|\sum_i \exp(t_iN^*)x_i\|^2 \geqq 0\,.$$

Sufficiency. Let $N_t(t \geqq 0, \mathfrak{K})$ be a minimal normal extension of $\exp(tA)(t \geqq 0,\ \mathfrak{H})$. Then N_t is continuous about $t \geqq 0$ from Theorem 4. If we put $N = \left.\dfrac{dN_t}{dt}\right|_{t=0}$, that is, the infinitesimal operator of $N_t\,(t \geqq 0)$ ([6]), N is regular normal operator on $\mathfrak{K}$ ([8]) (generally non-bounded). Since $\left.\dfrac{d\exp(tA)}{dt}\right|_{t=0} = A$ on $\mathfrak{H}$, we have $\bigcap\limits_{n=1}^{\infty} D_{N^n} \supset \mathfrak{H}$, where D_{N^n} denotes the domain of N^n. Therefore

$$(3.4)\qquad \sum_{n,m}(A^u x_m,\ A^m x_n) = \sum_{n,m}(N^n x_m,\ N^m x_n) = \sum_{n,m}(N^{*m}x_m,\ N^{*n}x_n)$$
$$= \|\sum_n N^{*n}x_n\|^2 \geqq 0\,.$$

Hence A is subnormal.

Remark. Naturally A is subnormal if and only if $\exp(tA)$ $(-\infty<t<+\infty)$ is positive definite.

Let Γ be a group and e be an identity element of Γ. An operator valued function $\psi(\gamma)$ from Γ into bounded operators on a HILBERT space $\mathfrak{H}$ is called a positive definite function in NAGY's sense ([11]) if $\psi(e)=I$; identity operator on $\mathfrak{H}$, and $\sum_{i,j}(x_i, \psi(\gamma_i^{-1}\gamma_j)x_j)\geqq 0$ for every finite number of x_i in $\mathfrak{H}$ and γ_i in Γ.

Lemma 6. *Let Γ be a group, $\varphi(\gamma)$ $(\gamma\in\Gamma, \mathfrak{H})$ an operator representation of Γ and $\psi(\gamma)=\varphi(\gamma^{-1})^*\varphi(\gamma)$. Then $\varphi(\gamma)(\gamma\in\Gamma, \mathfrak{H})$ is positive definite in the sense of Definition* 1 *if and only if $\psi(\gamma)$ is a positive definite function on Γ in NAGY's sense.*

Proof. Because

$$\begin{aligned}\sum_{i,j}(x_i,\psi(\gamma_i^{-1}\gamma_j)x_j) &= \sum_{i,j}(\varphi(\gamma_j^{-1}\gamma_i)x_i,\ \varphi(\gamma_i^{-1}\gamma_j)x_j)\\ &= \sum_{i,j}(\varphi(\delta_j)y_i,\ \varphi(\delta_i)y_j)\end{aligned} \tag{3.5}$$

where $\delta_i=\gamma_i^{-1}$ and $y_i=\varphi(\gamma_i)x_i$ for all i. And hence the conclusion is clear.

From Theorem 5 and Lemma 6 we obtain the following theorem which is a generalization of BRAM's theorem ([1] Theorem 2).

Theorem 6. *A bounded operator A on a HILBERT space $\mathfrak{H}$ is subnormal if and only if $\exp(-tA^*)\exp(tA)$ $(-\infty<t<+\infty)$ is a positive definite function in NAGY's sense.*

Remark. If for an arbitrary positive definite function on an abelian group $\psi(\gamma)$ in NAGY's sense it is possible to find an operator representation $\varphi(\gamma)$ such that $\psi(\gamma)=\varphi(-\gamma)^*\varphi(\gamma)$, we shall obtain from Theorem 1 NAGY's result ([11] Theorem III) in the case of abelian groups. But in general it is impossible. For example for continuous one parameter semi-group $T_t(t\geqq 0)$ of contractions; $\|T_t\|\leqq 1$, if we put $\psi(t)=T_t$ for $t\geqq 0$ and $\psi(t)=T^*_{-t}$ for $t\leqq 0$, then $\varphi(t)$ exists for such $\psi(t)$ if and only if all T_t are unitary operators.

4. A weak closure of $A_\gamma(\gamma\in\Gamma,\mathfrak{H})$. Let $A_\gamma(\gamma\in\Gamma,\mathfrak{H})$ be a positive definite operator representation and $A_\omega(\omega\in\Omega,\mathfrak{H})$ be the weakly closed algebra (not necessary self-adjoint) generated by $A_\gamma(\gamma\in\Gamma)$. In this section we shall give a theorem which shows the relation between the minimal normal extension $N_\gamma(\gamma\in\Gamma,\mathfrak{K})$ of $A_\gamma(\gamma\in\Gamma,\mathfrak{H})$ and that of A_ω $(\omega\in\Omega,\mathfrak{H})$. This theorem is a generalization of [1] Theorem 9 but our proof seems to be simpler than that of [1].

Theorem 7. *If* $A_\gamma\ (\gamma\in\Gamma,\ \mathfrak{H})$ *is positive definite, then the weak closed algebra* $A_\omega\ (\omega\in\Omega,\ \mathfrak{H})$ *is positive definite also. And let* $N_\gamma\ (\gamma\in\Gamma,\ \mathfrak{K})$ *and* L_ω $(\omega\in\Omega,\ \mathfrak{M})$ *be respectively the minimal normal extensions of* $A_\gamma\ (\gamma\in\Gamma,\ \mathfrak{H})$ *and* $A_\omega\ (\omega\in\Omega,\ \mathfrak{H})$*. Then we may consider* $\mathfrak{M}=\mathfrak{K}$ *and* $L_\omega\in\mathfrak{R}\{N_\gamma(\gamma\in\Gamma)\}\ (\omega\in\Omega)$*, where* $\mathfrak{R}\{N_\gamma(\gamma\in\Gamma)\}$ *is the operator ring (weakly closed self-adjoint algebra) generated by* $N_\gamma(\gamma\in\Gamma)$ *on* $\mathfrak{K}$.

Proof. Since $A_\gamma\ (\gamma\in\Gamma,\mathfrak{H})$ is positive definite, the algebra $\mathfrak{A}$ generated by $A_\gamma\ (\gamma\in\Gamma)$ is evidently positive definite. By the definition of positive definite naturally the strong closure of $\mathfrak{A}$ is also positive definite. Therefore $A_\omega(\omega\in\Omega,\mathfrak{H})$ is positive definite, because the strong closure of a linear set of operators is the same as its weak closure.

Let $L_\omega\ (\omega\in\Omega,\mathfrak{M})$ be the minimal normal extension of $A_\omega\ (\omega\in\Omega,\mathfrak{H})$, $L_\gamma(\gamma\in\Gamma)$ be its part which is a extension of $A_\gamma(\gamma\in\Gamma)$ onto $\mathfrak{M}$ and $\mathfrak{R}\{L_\gamma\ (\gamma\in\Gamma)\}$ be the operator ring on $\mathfrak{M}$ generated by $L_\gamma\ (\gamma\in\Gamma)$.

First we shall show that for any $L_\omega\ (\omega\in\Omega)$ (fixed) and for any $\varepsilon>0$ and $x_i\in\mathfrak{H}\ (i=1,2,\cdots,n)$ there exists an operator L such that

$$(4.1)\qquad \begin{cases} L\in\mathfrak{R}\{L_\gamma\ (\gamma\in\Gamma)\}\,, \\ \|L\|\leqq\sqrt{2}\,\|L_\omega\|\,, \\ \|L_\omega x_i-Lx_i\|\leqq\varepsilon \qquad\qquad (i=1,2,\cdots,n)\,. \end{cases}$$

Because for A_ω there exists $B\in\mathfrak{A}$ such that $\|A_\omega x_i-Bx_i\|\leqq\varepsilon\ (i=1,2,\cdots,n)$, putting M the extension of B onto $\mathfrak{M}$, then $M\in\mathfrak{R}\{L_\gamma(\gamma\in\Gamma)\}$ and $\|L_\omega x_i-Mx_i\|\leqq\varepsilon\ (i=1,2,\cdots,n)$, it follows that $\|R(L_\omega)x_i-R(M)x_i\|\leqq\varepsilon$, $\|I(L_\omega)x_i-I(M)x_i\|\leqq\varepsilon\ (i=1,2,\cdots,n)$, where $R(T)$ and $I(T)$ denote respectively the real part and the imaginary part of T, evidently $R(M),\ I(M)\in\mathfrak{R}\{L_\gamma(\gamma\in\Gamma)\}$, we can find a polynomial $P(t)$ (cf. von NEUMANN [9] p. 399) such that

$$(4.2)\qquad \begin{aligned} &\|P(R(M))\|\leqq\|L_\omega\|, \quad \|P(I(M))\|\leqq\|L_\omega\|\,, \\ &\|P(R(M))x_i-R(L_\omega)x_i\|\leqq\varepsilon\,, \\ &\|P(I(M))x_i-I(L_\omega)x_i\|\leqq\varepsilon \qquad\qquad i=1,2,\cdots,n\,. \end{aligned}$$

If we put $L=P(R(M))+iP(I(M))$, then L satisfies (4.1).

Putting $\mathfrak{L}=\{\sum_i L^*_{\gamma_i}x_i;$ for every finite number of $x_i\in\mathfrak{H}$ and $\gamma_i\in\Gamma\}$ and $\mathfrak{K}=\overline{\mathfrak{L}}$ (closure of $\mathfrak{L}$ in $\mathfrak{M}$). Then $\mathfrak{K}$ reduces every $L_\gamma\ (\gamma\in\Gamma)$ and therefore $\mathfrak{K}$ reduces $\mathfrak{R}\{L_\gamma(\gamma\in\Gamma)\}$. We can see by using (4.1) $\mathfrak{K}$ reduces also every $L_\omega\ (\omega\in\Omega)$. Because, for any $f\in\mathfrak{K}$ and $\varepsilon>0$, we can find $\sum_{i=1}^n L^*_{\gamma_i}x_i\in\mathfrak{L}$ and $L\in\mathfrak{R}\{L_\gamma(\gamma\in\Gamma)\}$ such that $\|f-\sum_{i=1}^n L^*_{\gamma_i}x_i\|\leqq\varepsilon/\sqrt{2}\,\|L_\omega\|$,

$\|L\| \leqq \sqrt{2}\,\|L_\omega\|$, $\|L_\omega x_i - Lx_i\| \leqq \varepsilon \Big/ \sum_{i=1}^{n} \|L_{\gamma_i}\|$. Hence $\|L_\omega f - Lf\| \leqq \|L_\omega f - L_\omega(\sum_{i=1}^{n} L^*_{\gamma_i} x_i)\| + \|L_\omega(\sum_{i=1}^{n} L^*_{\gamma_i} x_i) - L(\sum_{i=1}^{n} L^*_{\gamma_i} x_i)\| + \|L(\sum_{i=1}^{n} L^*_{\gamma_i} x_i) - Lf\| \leqq 3\varepsilon$, that is, we have $\|L_\omega f - Lf\| \leqq 3\varepsilon$. Since L_ω and L are commutative we have $\|L^*_\omega f - L^* f\| \leqq 3\varepsilon$ also. As $Lf, L^*f \in \mathfrak{K}$ and ε is arbitrary, it follows that $L_\omega f$, $L^*_\omega f \in \mathfrak{K}$ namely $\mathfrak{K}$ reduces L_ω. Since L_ω $(\omega \in \Omega, \mathfrak{M})$ is a minimal normal extension, we must have $\mathfrak{K} = \mathfrak{M}$. Likeweise above we can see for arbitrary L_ω, $f_i \in \mathfrak{K}$ $(i=1,2,\cdots,n)$ and $\varepsilon > 0$ there exists $L \in \mathfrak{R}\{L_\gamma(\gamma \in \Gamma)\}$ such that $\|L_\omega f_i - Lf_i\| \leqq 3\varepsilon$ $(i=1,2,\cdots,n)$. Therefore L_ω belongs to the strong closure of $\mathfrak{R}\{L_\gamma(\gamma \in \Gamma)\}$ on $\mathfrak{K}$, that is, $L_\omega \in \mathfrak{R}\{L_\gamma(\gamma \in \Gamma)\}$. $L_\gamma(\gamma \in \Gamma, \mathfrak{K})$ is evidently the minimal normal extension of $A_\gamma(\gamma \in \Gamma, \mathfrak{H})$. Thus the proof is complete.

Remark. As A_ω $(\omega \in \Omega, \mathfrak{H})$ is positive definite, by using Lemma 3 a) A_ω can be extended uniquely to an operator L_ω acting on $\mathfrak{K}$ which is the space of the minimal normal extension N_γ $(\gamma \in \Gamma, \mathfrak{K})$ of A_γ $(\gamma \in \Gamma, \mathfrak{H})$. In other words Theorem 7 is as follows A_ω, an element of the weakly closed algebra generated by A_γ $(\gamma \in \Gamma)$, can be extended uniquely to an operator L_ω on $\mathfrak{K}$ such that $L_\omega \in \mathfrak{R}\{N_\gamma(\gamma \in \Gamma)\}$ and $\|L_\omega\| = \|A_\omega\|$.

5. A spectrum of subnormal operators. Halmos [4] has shown that if N is the minimal normal extension of the subnormal operator A, then the resolvent set $\rho(A)$ of A is contained in the resolvent set $\rho(N)$ of N. Let $\rho_n(N)(n=1,2,\cdots)$ be all connected components of $\rho(N)$. Then Bram [1] has shown $\rho(A) = \sum_{n \in J} \rho_n(N)$, where J is a subset of the positive integers. In this section we shall show simpler another proof of this theorem, in our proof the theory of complex variable functions is not necessary.

Denoting $\rho(A)$ the resolvent set of a operator A, $\mathfrak{P}$ the whole of polynomials $P(\lambda)$ on the complex plane, $\mathfrak{F}_A$ the whole of rational functions $f(\lambda) = P_1(\lambda)/P_2(\lambda)$ which are regular on the spectrum of A. We can define $P(A)$ for $P(\lambda) \in \mathfrak{P}$ and $f(A) = P_1(A)\,P_2(A)^{-1}$ for $f(\lambda) \in \mathfrak{F}_A$.

Lemma 7. *The following conditions are equivalent each other*

a) A *is subnormal,*

b) $P(A)$ $(P(\lambda) \in \mathfrak{P}, \mathfrak{H})$ *is positive definite,*

c) $f(A)$ $(f(\lambda) \in \mathfrak{F}_A, \mathfrak{H})$ *is positive definite,*

d) $(A-\lambda)^{-1}$ *for some* $\lambda \in \rho(A)$ *is subnormal.*

Proof. It is evident that a) implies b) and c) implies d). We shall prove b) implies c). Because for arbitrary finite number of $f_i(\lambda) \in \mathfrak{F}_A$

$(i=1,2,\cdots n)$, $f_i(\lambda)=P_i(\lambda)/Q_i(\lambda)$, we have

$$(5.1)\quad \sum_{i,j=1}^{n}(f_i(A)x_j,\ f_j(A)x_i)=\sum_{i,j=1}^{n}(P_i(A)Q_i(A)^{-1}x_j,\ P_j(A)\,Q_j(A)^{-1}x_i)$$
$$=\sum_{i,j=1}^{n}(P_i(A)R_i(A)y_j,\ P_j(A)R_j(A)y_i)\geqq 0\ ,$$

where $y_i=Q_1(A)^{-1}Q_2(A)^{-1}\cdots Q_n(A)^{-1}x_i(i=1,2,\cdots,n)$ and $R_i(\lambda)=Q_1(\lambda)Q_2(\lambda)\cdots Q_n(\lambda)/Q_i(\lambda)\,(i=1,2,\cdots,n)$. And likewise d) implies a).

Remark. Let $f(A)\,(f\in\mathfrak{F}_A,\ \mathfrak{H})$ be positive definite and $N_f(f\in\mathfrak{F}_A,\ \mathfrak{K})$ be its minimal extension. Then by remenbering the proof of Theorem 1 we obtain $N_{f_1}N_{f_2}=N_{f_1f_2}$ and $N_{f_1+f_2}=N_{f_1}+N_{f_2}$. Therefore we have from Theorem 2 $\|f_1(A)+f_2(A)\|_{\mathfrak{H}}=\|N_{f_1}+N_{f_2}\|_{\mathfrak{K}}$ for every $f_1,f_2\in\mathfrak{F}_A$ (this fact will be used in the proof of Theorem 8).

Theorem 8. (Halmos, Bram). *Let A be subnormal and N be the minimal normal extension of A. Then we have*

$$(5.2)\qquad \rho(A)=\sum_{n\in J}\rho_n(N)$$

where $\rho_n(N)\,(n=1,2,\cdots)$ are all connected components of $\rho(N)$ and J is a suitable subset of the positive integers.

Proof. From Lemma 9 $f(A)\,(f(\lambda)\in\mathfrak{F}_A,\ \mathfrak{H})$ is positive definite, hence it has a minimal normal extension $N_f(f(\lambda)\in\mathfrak{F}_A,\ \mathfrak{K})$. We shall denote by N the extension of A onto $\mathfrak{K}$. For any $\lambda_0\in\rho(A)$ if we put $P(\lambda)=\lambda-\lambda_0$, then $P(\lambda)^{-1}\in\mathfrak{F}_A$ and $P(\lambda)\,P(\lambda)^{-1}=1$, hence $N_PN_{P^{-1}}=N_{P^{-1}}N_P=I$. Therefore $(N-\lambda_0)\,N_{P^{-1}}=N_{P^{-1}}(N-\lambda_0)=I$ namely $\lambda_0\in\rho(N)$ and $N_{(\lambda-\lambda_0)^{-1}}=(N-\lambda_0)^{-1}$. Furthermore we obtain $N_f=f(N)$ for every $f\in\mathfrak{F}_A$. On the other hand N is a minimal normal extension of A. Because, if a subspace $\mathfrak{K}_0$ of $\mathfrak{K}$ contains $\mathfrak{H}$ and reduces N, then evidently $\mathfrak{K}_0$ reduces every $f(N)=N_f(f\in\mathfrak{F}_A)$, therefore $\mathfrak{K}_0=\mathfrak{K}$. Thus we obtain Halmos' theorem, that is, $\rho(A)\subseteqq\rho(N)$.

Next we shall prove that $\rho_n(N)\frown\rho(A)=\phi$ or $=\rho_n(N)$ for every n. If $\rho_n(N)\frown\rho(A)\neq\phi$ for some n, then $\rho_n(N)\frown\rho(A)$ is non-empty open set. On the other hand $\rho_n(N)\frown\rho(A)$ is a closed set in $\rho_n(N)$. Because, for every sequence $\lambda_\nu\in\rho_n(N)\frown\rho(A)$ such that $\lim_{\nu\to\infty}\lambda_\nu=\lambda_0\in\rho_n(N)$ we have $\lim_{\nu\to\infty}\|(N-\lambda_\nu)^{-1}-(N-\lambda_0)^{-1}\|_{\mathfrak{K}}=0$, by the Remark after Lemma 9 $\|(A-\lambda_\nu)^{-1}-(A-\lambda_\mu)^{-1}\|_{\mathfrak{H}}=\|(N-\lambda_\nu)^{-1}-(N-\lambda_\mu)^{-1}\|_{\mathfrak{K}}$, therefore we find an operator B on $\mathfrak{H}$ such that $\lim_{\nu\to\infty}\|(A-\lambda_\nu)^{-1}-B\|_{\mathfrak{H}}=0$, and hence for every $\lambda\in\rho(A)$ we have $(A-\lambda)^{-1}-B=\lim_{\nu\to\infty}(A-\lambda)^{-1}-(A-\lambda_\nu)^{-1}=\lim_{\nu\to\infty}(\lambda-\lambda_\nu)(A-\lambda)^{-1}(A-\lambda_\nu)^{-1}=(\lambda-\lambda_0)(A-\lambda)^{-1}B$, that is, $(A-\lambda)^{-1}-B=(\lambda-\lambda_0)(A-\lambda)^{-1}B$. Therefore it follows that $\lambda_0 \in \rho(A)$ and $B=(A-\lambda_0)^{-1}$. Consequently $\rho_n(N) \frown \rho(A)$ is open and closed in $\rho_n(N)$, hence $\rho_n(N) \frown \rho(A) = \rho_n(N)$. The proof is complete.

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