ON HAAR FUNCTIONS IN THE SPACE $L_{M(\xi,t)}$

By

Jyun ISHII and Tetsuya SHIMOGAKI

1. It is well known [6, 8, 12] that Haar functions constitute a (Schauder) basis in Banach spaces $L^p[0,1]$ $(1 \leqq p < +\infty)$ and Orlicz spaces $L_M[0,1]$ with the Δ_2-condition. Generalizing this fact to an arbitrary separable Banach function space E on a measure space, H. W. Ellis and I. Halperin showed in [3] that Haar system of functions (in an extended sense) composes a basis in E, if a norm of E satisfies a condition called *levelling length property*[1]. Although this condition is sufficiently general, yet it is not always a necessary one.

In this note we shall show a sufficient condition in order that Haar functions be a basis for the Banach function space $L_{M(\xi,t)}[0,1]$ or $L^{p(t)}[0,1]$. In fact, we shall establish, as for the space $L^{p(t)}$, that *if $p(t)$ satisfies the Lipschitz α-condition* $(0<\alpha \leqq 1)$ *then Haar functions constitute a basis in* $L^{p(t)}$ (Theorem 4). As a matter of course, the norms of these spaces do not satisfy the above condition given in [3] except some special cases.

In **2** we shall introduce Haar functions, the function spaces $L_{M(\xi,t)}$ and $L^{p(t)}$[2] with the notations used here. The main theorems shall be stated in **3**, and some remarks shall be presented in **4**.

2. A sequence of functions defined on $[0,1]$: $\{\chi_\nu(t)\}_{\nu=1}^{\infty}$ is called a *system of Haar functions*, if $\chi_1(t)=1$ for all $t \in [0,1]$ and for $\nu = 2^n + k$ $(n=0,1,2,\cdots;$ $k=1,2,\cdots,2^n)$[3]

$$(2.1)\qquad \chi_\nu(t) = \chi_{2^n+k}(t) = \begin{cases} \sqrt{2^n} & \text{for } t \in \left[\dfrac{2k-2}{2^{n+1}}, \dfrac{2k-1}{2^{n+1}}\right), \\ -\sqrt{2^n} & \text{for } t \in \left(\dfrac{2k-1}{2^{n+1}}, \dfrac{2k}{2^{n+1}}\right], \\ 0 & \text{otherwise in } [0,1]. \end{cases}$$

1) A norm $\|\cdot\|$ of E is called to have the *levelling length property*, if $\|f_e\| \leqq \|f\|$ holds for any $f \in E$ and measurable set e, where f_e coincides with f outside the e and on e, $f_e = \left\{\dfrac{1}{d(e)}\displaystyle\int_e f(t)dt\right\} C_e$ (C_e is the characteristic function of e). This property was first discussed by them in the earlier paper [4]. At the same time, G. G. Lorentz and D. G. Wertheim also found it independently and named it the *average invariant property* [9].

2) In the sequel, we eliminate $[0,1]$ and write simply $L_{M(\xi,t)}$ (or $L^{p(t)}$) in place of $L_{M(\xi,t)}[0,1]$ (resp. $L^{p(t)}[0,1]$). $L^{p(t)}$ was first discussed by W. Orlicz in [11], and was investigated precisely by H. Nakano [10].

3) This formulation of Haar functions is due to Z. Ciesielskii [2].

For any $a(t)\in \boldsymbol{L}^1[0,1]$ we denote by $S_n(a)=S_n(a;t)$ $(n=1,2,\cdots)$ the n-th partial sum of Haar Fourier series:

$$(2.2)\qquad S_n(a;t)=\sum_{\nu=1}^{n}\alpha_\nu\chi_\nu(t), \quad \text{where } \alpha_\nu=\int_0^1 a(t)\chi_\nu(t)\,dt.$$

It is a well-known fact that if $a(t)$ is *continuous* on $[0,1]$, $S_n(a;t)$ *converges uniformly to* $a(t)$ and

$$(2.3)\qquad S_{2^n}(a;t)=\left\{2^n\int_{(k-1)/2^n}^{k/2^n} a(s)\,ds\right\} \quad \left(t\in\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)\right)$$

holds for each $n\geqq 0$, $k=1,2,\cdots,2^n$.

Now let $M(\xi,t)$ $(\xi\geqq 0,\ 0\leqq t\leqq 1)$ be a *convex function of* $\xi\geqq 0$ *for each* $t\in[0,1]$ *and a Lebesgue measurable function of* $t\in[0,1]$ *for each* $\xi\geqq 0$ *with the following properties*:

M. 1) $M(0,t)=0$ *for* a.e. $t\in[0,1]$;

M. 2) $\lim_{\xi\to\alpha-0} M(\xi,t)=M(\alpha,t)$ *for* a.e. $t\in[0,1]$ *and each* $\alpha>0$;

M. 3) $\lim_{\xi\to\infty} M(\xi,t)=+\infty$ *for* a.e. $t\in[0,1]$;

M. 4) *for any* $t\in[0,1]$ *there exists* $\alpha_t>0$ *such that* $M(\alpha_t,t)<+\infty$.

We denote by $\boldsymbol{L}_{M(\xi,t)}$ the set of all measurable functions $x(t)$ satisfying

$$\int_0^1 M(\alpha|x(t)|,t)\,dt<+\infty \qquad \text{for some} \quad \alpha=\alpha(x)>0.$$

Then $\boldsymbol{L}_{M(\xi,t)}$ is called a *modulared function space* and is considered as a *modulared semi-ordered linear space* [5,10] with a *modular* m:

$$(2.4)\qquad m(x)=\int_0^1 M(|x(t)|,t)\,dt \qquad (x\in\boldsymbol{L}_{M(\xi,t)}),$$

where $0\leqq x$, $x\in\boldsymbol{L}_{M(\xi,t)}$ means that $x(t)\geqq 0$ a.e. in $[0,1]$. Furthermore $\boldsymbol{L}_{M(\xi,t)}$ is a *Banach space* with a *norm* $\|\cdot\|$ defined by the modular:

$$(2.5)\qquad \|x\|=\inf_{m(\xi x)\leqq 1}\frac{1}{|\xi|}\ ^{4)} \qquad (x\in\boldsymbol{L}_{M(\xi,t)}),$$

and $\boldsymbol{L}_{M(\xi,t)}$ is *separable* if and only if $m(x)<+\infty$ *for every* $x\in\boldsymbol{L}_{M(\xi,t)}$, which is also equivalent to the fact that $M(\xi,t)$ *satisfies the generalized* Δ_2-*condition* [5,7], i.e.

(Δ_2) *there exist a positive number* $\gamma>0$ *and* $0\leqq a\in\boldsymbol{L}^1[0,1]$ *such that*

$$(2.6)\qquad M(2\xi,t)\leqq\gamma M(\xi,t)+a(t) \qquad \textit{for all}\ \ \xi\geqq 0\ \textit{and}\ \text{a.e. } t\in[0,1].$$

4) This norm is called the *modular norm* by the modular m. In the sequel, we consider $\boldsymbol{L}_{M(\xi,t)}$ with this norm.

This norm $\|\cdot\|$, as is easily seen, does not satisfy the levelling length property in general[5]. If there exists a convex function $M(\xi)$ such that $M(\xi,t)=M(\xi)$ holds for every $\xi\geqq 0$ and a.e. $t\in[0,1]$, then $L_{M(\xi,t)}$ is nothing but an *Orlicz space* L_M, and if $M(\xi,t)=\xi^{p(t)}$ holds, where $p(t)$ is a measurable function with $1\leqq p(t)$ $(t\in[0,1])$, $L_{M(\xi,t)}$ is denoted by $L^{p(t)}$ [10, 11]. $L^{p(t)}$ is separable if and only if $p(t)$ is bounded: $p(t)\leqq K$ for a.e. $t\in[0,1]$ and a constant $K>0$.

3. For a system of convex N-functions $\{\varphi_\lambda(\xi)\}_{\lambda\in\Lambda}$, there exist always the *join* (the least upper bound function) and the *meet* (the greatest lower bound function) *as a convex function* in the family F of positive convex functions. In fact, put $\Phi(\xi)=\sup_{\lambda\in\Lambda}\varphi_\lambda(\xi)$ $(\xi\geqq 0)$, then Φ is a convex function which is the join of $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ in F (it is possible that $\Phi(\xi)=+\infty$ may hold for each $\xi>0$). Here we denote this join of $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ in F by $conv\text{-}\bigcup_{\lambda\in\Lambda}\varphi_\lambda$. As for the meet, put $\Psi=conv\text{-}\bigcup_{\lambda\in\Lambda}\phi_\lambda$, where $\phi_\lambda(\xi)$ is the complementary function to $\varphi_\lambda(\lambda\in\Lambda)$ in the sense of H. W. Young, and further let Φ_0 be the complementary function to Ψ in the above sense too, then Φ_0 comes to be a convex function which is the meet of $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ in F (it is possible also that $\Phi_0(\xi)=0$ may hold for every $\xi\geqq 0$). We denote the meet of $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ in F by $conv\text{-}\bigcap_{\lambda\in\Lambda}\varphi_\lambda$ as well. From the definitions, it is clear that $conv\text{-}\bigcap_{\lambda\in\Lambda}\varphi_\lambda(t)\leqq\varphi_\lambda(t)\leqq conv\text{-}\bigcup_{\lambda\in\Lambda}\varphi_\lambda(t)$ holds for each $t\geqq 0$ and $\lambda\in\Lambda$.

Now let $L_{M(\xi,t)}$ be a modulared function space. Since $M(\xi,t)$ is convex N-functions of $\xi\geqq 0$ for all $t\in[0,1]$ by M. 1)–M. 4) in **2**, we can define convex functions $\overline{M}_{n,k}(\xi)$ and $\underline{M}_{n,k}(\xi)$ as follows:

$$\overline{M}_{n,k}(\xi)=conv\text{-}\bigcup_{t\in I_{n,k}}M(\xi,t)^{6)}\qquad(\xi\geqq 0),\tag{3.1}$$

$$\underline{M}_{n,k}(\xi)=conv\text{-}\bigcap_{t\in I_{n,k}}M(\xi,t)\qquad(\xi\geqq 0),\tag{3.2}$$

where $I_{n,k}=\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)$ $(n=0,1,2,\cdots;\ k=1,2,\cdots,2^n)$. We put also

$$\omega_n=\operatorname*{Max}_{k=1,2,\cdots,2^n}\left\{\frac{\overline{M}_{n,k}(2^n)}{\underline{M}_{n,k}(2^n)}\right\}\qquad(n=1,2\cdots),\tag{3.3}$$

if it has a sense.

With these preparations, we have

Theorem 1. *Haar functions* $\{\chi_\nu\}_{\nu=1}^\infty$ *constitute a basis for a modulared function space* $L_{M(\xi,t)}$, *if* $M(\xi,t)$ *satisfies the following conditions*:

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5) Indeed, we can show that if the norm $\|\cdot\|$ on $L_{M(\xi,t)}$ fulfils the requirement of the levelling length property, $L_{M(\xi,t)}$ reduces to an Orlicz space L_M.

6) Since $M(\xi,t)$ satisfies M.1)–M.4), $M(\xi,t)$ is considered as convex N-functions for a.e. $t\in[0,1]$.

C. 1) *the Δ_2-condition in* **2** *holds true for* $M(\xi, t)$;

C. 2) *there exists a positive number* δ *such that* ess. $\inf_{t\in[0,1]} M(\delta, t) \geqq 1$;

C. 3) *there exists a positive number* κ *such that* $\overline{\lim_{n\to\infty}}\, \omega_n \leqq \kappa$.[7)]

Before entering into the proof of Theorem 1, we first prove the auxiliary lemmas.

Lemma 1. *If* $M(\xi, t)$ *satisfies* C. 2), *then* $\|x\| \leqq 1$ $(x \in \boldsymbol{L}_{M(\xi,t)})$ *implies* $\int_0^1 |x(t)|dt \leqq 2\delta$, *hence* $\boldsymbol{L}_{M(\xi,t)} \subseteq \boldsymbol{L}^1$.

Proof. If $\|x\| \leqq 1$ $(x \in \boldsymbol{L}_{M(\xi,t)})$, then the formulas (2.4) and (2.5) imply $m(x) \leqq 1$. Hence we get

$$1 \geqq \int_0^1 M(|x(t)|, t)\,dt = \int_{e_\delta} M(|x(t)|, t)\,dt + \int_{e'_\delta} M(|x(t)|, t)\,dt$$
$$\geqq \frac{1}{\delta} \int_{e_\delta} |x(t)|\,dt\,,$$

where $e_\delta = \{t : |x(t)| \geqq \delta\}$ and e'_δ is the complement of e_δ, since $M(\xi, t) \geqq \frac{\xi}{\delta} M(\delta, t) \geqq \frac{\xi}{\delta}$ holds for every ξ with $\xi \geqq \delta > 0$ and a.e. $t \in [0,1]$ by virtue of convexity of $M(\xi, t)$ and C. 2). Therefore we obtain $\int_0^1 |x(t)|dt \leqq 2\delta$. Q.E.D.

Lemma 2. *If* $M(\xi, t)$ *satisfies* C. 3), *then* $\mathbf{1}\,(\mathbf{1}(t) = 1$ *for all* $t \in [0,1])$ *belongs to* $\boldsymbol{L}_{M(\xi,t)}$.

Proof. As $\operatorname{Max}_{k=1,\cdots,2^n} \left\{\frac{\overline{M}_{n,k}(2^n)}{\underline{M}_{n,k}(2^n)}\right\} = \omega_n$, we have for some n

$$\overline{M}_{1,i}(1) \leqq \operatorname{Max}_{k=1,2,\cdots,2^n} \{\overline{M}_{n,k}(2^n)\} < +\infty \qquad (i=1,2)\,,$$

whence

$$m(\mathbf{1}) = \int_0^1 M(1, t)\,dt \leqq \int_0^{1/2} \overline{M}_{1,1}(1)\,dt + \int_{1/2}^1 \overline{M}_{1,2}(1)\,dt < +\infty\,.$$

which implies $\mathbf{1} \in \boldsymbol{L}_{M(\xi,t)}$. Q.E.D.

Proof of Theorem 1: Putting for $n = 1, 2, \cdots$; $k = 1, 2, \cdots, 2^n$

$$T_n^k x = 2^n \left\{\int_{I_{n,k}} x(t)\,dt\right\} c_n^k \qquad (x \in \boldsymbol{L}_{M(\xi,t)})\,,$$

where c_n^k is the characteristic function of the interval $I_{n,k} = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$, we

7) In the definition of ω_n, we can substitute *conv*- $\bigcup_{t\in I_{n,k}} M(\xi, t)$ (or *conv*- $\bigcap_{t\in I_{n,k}} M(\xi, t)$) by *conv*- $\bigcup_{t\in I_{n,k}-e} M(\xi, t)$ (resp. *conv*- $\bigcap_{t\in I_{n,k}-e} M(\xi, t)$) in the formulae (3.1) and (3.2), where e is a set of measure zero, as the proof shows below.

obtain by Lemma 2 a linear operator T_n^k of $L_{M(\xi,t)}$ into itself for each n, k. Then (2.3) can be written as

$$S_{2^n}(x)=\sum_{k=1}^{2^n} T_n^k x\,. \tag{3.4}$$

Now let $x\in L_{M(\xi,t)}$ with $\|x\|\leqq 1$. From Lemma 1 we have $|(T_n^k x)(t)|\leqq \delta 2^{n+1}c_n^k(t)$ $(t\in[0,1])$. According to C.3) we can find a natural number n_0 such that $n\geqq n_0$ implies $\omega_n\leqq 2\kappa$. Then we get for any n with $n>n_0$ and any k $(1\leqq k\leqq 2^n)$

$$\begin{aligned} m\Big(\frac{1}{2\delta}T_n^k x\Big) &\leqq \int_{I_{n,k}} \overline{M}_{n,k}\Big(\frac{1}{2\delta}\big|(T_n^k x)(t)\big|\Big)dt \\ &\leqq \mathrm{Max}\Big\{2\kappa\int_{I_{n,k}} \underline{M}_{n,k}\Big(\frac{1}{\delta}\big|(T_n^k x)(t)\big|\Big)dt,\quad m(2^{n_0}c_n^k)\Big\}\,. \end{aligned} \tag{3.5}$$

Because, if $2^{n_0}\leqq 2^\nu<\frac{1}{2\delta}\big|(T_n^k x)(t)\big|\leqq 2^{\nu+1}\leqq 2^n$ $(t\in I_{n,k})$ holds, it follows from the definitions (3.1), (3.2) of $\overline{M}_{n,k}$ and $\underline{M}_{n,k}$ that

$$\begin{aligned} \overline{M}_{n,k}\Big(\frac{1}{2\delta}\big|(T_n^k x)(t)\big|\Big) &\leqq \overline{M}_{\nu+1,k'}(2^{\nu+1})\leqq 2\kappa \underline{M}_{\nu+1,k'}(2^{\nu+1}) \\ &\leqq 2\kappa\underline{M}_{\nu+1,k'}\Big(2\cdot\frac{1}{2\delta}\big|(T_n^k x)(t)\big|\Big)\leqq 2\kappa\cdot\underline{M}_{n,k}\Big(\frac{1}{\delta}\big|(T_n^k x)(t)\big|\Big)\,, \end{aligned}$$

where k' is a suitable natural number such that $I_{\nu+1,k'}\supset I_{n,k}$.

Now, applying the Jensen's inequality to the last term of (3.5), we get

$$\begin{aligned} m\Big(\frac{1}{2\delta}T_n^k x\Big) &\leqq \mathrm{Max}\Big\{2\kappa\int_{I_{n,k}}\underline{M}_{n,k}\Big(\frac{1}{\delta}\big|x(t)\big|\Big)dt,\quad m(2^{n_0}c_n^k)\Big\} \\ &\leqq 2\kappa\int_{I_{n,k}} M\Big(\frac{1}{\delta}\big|x(t)\big|,t\Big)dt+m(2^{n_0}c_n^k)\leqq 2\kappa m\Big(\frac{1}{\delta}x\cdot c_n^k\Big)+m(2^{n_0}c_n^k)\,. \end{aligned}$$

Consequently (3.4) gives for $n>n_0$

$$\begin{aligned} m\Big(\frac{1}{2\delta}S_{2^n}(x)\Big) &= m\Big(\frac{1}{2\delta}\sum_{k=1}^{2^n}T_n^k x\Big)=\sum_{k=1}^{2^n} m\Big(\frac{1}{2\delta}T_n^k x\Big) \\ &\leqq 2\kappa\sum_{k=1}^{2^n} m\Big(\frac{1}{\delta}x\cdot c_n^k\Big)+\sum_{k=1}^{2^n} m(2^{n_0}c_n^k)=2\kappa m\Big(\frac{1}{\delta}x\Big)+m(2^{n_0}\mathbf{1})\,. \end{aligned}$$

Therefore, $\|x\|\leqq 1$ implies $m\Big(\frac{1}{2\delta}S_{2^n}x\Big)\leqq\kappa'$ $(n>n_0)$[8] for a suitable chosen $\kappa'>0$, which also shows $\sup_{\|x\|\leqq 1,\ n>n_0}\|S_{2^n}x\|<+\infty$.

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8) Since $M(\xi,t)$ satisfies the Δ_2-condition, we have $\sup_{\|x\|\leqq 1} m\Big(\frac{1}{\delta}x\Big)<+\infty$.

As the result of the above, we can see directly that the operator norms of S_ν $(\nu=1,2,\cdots)$ are *uniformly bounded.* Since the set of all continuous functions is dense in $\boldsymbol{L}_{M(\xi,t)}$, in case $M(\xi,t)$ satisfies the (Δ_2)-condition, and uniform convergence implies the norm convergence in $\boldsymbol{L}_{M(\xi,t)}$, our assertion is obtained.

Q. E. D.

Next we shall replace the condition C. 3) in Theorem 1 by a somewhat simpler one. For this purpose we define from $M(\xi,t)$

$$L(\xi,t)=\log M(\xi,t)\big/\log\xi\,, \tag{3.6}$$

if $M(\xi,t)$ and ξ are both greater than 1, and

$$L(\xi,t)=0$$

otherwise, where $t\in[0,1]$ and $\xi\geqq 0$.

Using $L(\xi,t)$ we shall prove

Theorem 2. *Suppose that* C. 1) *and* C. 2) *hold for* $M(\xi,t)$. *If* $L(\xi,t)$ *(defined by* (3.6) *from* $M(\xi,t)$) *satisfies the Lipschitz* α*-condition* $(0<\alpha\leqq 1)$ *with a constant* $\gamma>0$ *for all* $\xi\geqq\xi_0$, *i.e.*

C. 3′) $|L(\xi,t)-L(\xi,t')|\leqq\gamma|t-t'|^\alpha$ for all $t,t'\in[0,1]$, $\xi\geqq\xi_0$, *where* α,γ *and* $\xi_0\geqq 0$ *are all certain fixed constants*[9].
Then Haar functions $\{\chi_\nu(t)\}_{\nu=1}^\infty$ *constitute a basis in* $\boldsymbol{L}_{M(\xi,t)}$.

Proof. It follows by C. 2) $L(\xi,t)=\log M(\xi,t)/\log\xi$ for all ξ with $\xi\geqq\mathrm{Max}(\delta,1)$, which implies also for sufficiently large $\xi_0>1$

$$M(\xi,t)\leqq\xi^{\gamma|t-t'|^\alpha}M(\xi,t') \qquad (t,t'\in[0,1],\ \xi\geqq\xi_0) \tag{3.7}$$

by virtue of C. 3′).

Let n_0 be a natural number such that both $\dfrac{n_0\gamma}{2^{n_0\alpha}}\leqq 1$ and $2^{n_0}>\xi_0$ hold.

Then for any $n>n_0$, the inequality (3.7) gives $M(\xi,t)\leqq\xi^{\frac{\gamma}{2^{n\alpha}}}M(\xi,t')$ for all $t,t'\in I_{n,k}$ and $\xi\geqq\xi_0$, where $k=1,2,\cdots,2^n$. Recalling the definition of $\overline{M}_{n,k}(\xi)$, we obtain for every $t,t'\in I_{n,k}$ and $\xi\geqq\xi_0$

$$M(\xi,t')\leqq\overline{M}_{n,k}(\xi)\leqq\xi^{\frac{\gamma}{2^{n\alpha}}}M(\xi,t)\,. \tag{3.8}$$

Now we put

9) In view of the proof of Theorem 2, we can see that Theorem 2 remains to be true if we replace C. 3′) by a somewhat general condition : C. 3″) $|L(\xi,t)-L(\xi,t')|\leqq\omega(|t-t'|)$ $(\xi\geqq\xi_0,$ $t,t'\in[0,1])$, where $\omega(\delta)$ is a function defined on $[0,\infty)$ satisfying $\overline{\lim_{\delta\to 0}}\left(\dfrac{1}{\delta}\right)^{\omega(\delta)}<+\infty$.

(3.9) $$\beta_n^k = \text{ess.}\inf_{t\in I_{n,k}} \varphi(2^n, t),$$

where $\varphi(2^n, t) = \lim_{\varepsilon\to 0} \frac{M(2^n,t)-M(2^n-\varepsilon,t)}{\varepsilon}$ $(t\in I_{n,k},\ k=1,2,\cdots,2^n)$. From the condition C. 2) and the fact that $M(\xi,t)$ is a convex function for each $t\in[0,1]$, we have $\beta_n^k>0$ for every $n>n_0$ and $1\leqq k\leqq 2^n$. Thus, for each n, k $(n>n_0, 1\leqq k\leqq 2^n)$ there exists $t_0\in I_{n,k}$ such that

$$\frac{1}{2}\varphi(2^n, t_0)\leqq \beta_n^k .$$

For this t_0, we put

(3.10) $$\phi_{n,k}(\xi, t) = \begin{cases} \frac{1}{2} M(\xi, t_0) & \text{for } 0\leqq\xi\leqq 2^n; \\ \frac{\varphi(2^n, t_0)}{2}(\xi-2^n)+\frac{M(2^n,t_0)}{2}, & \text{for } 2^n\leqq\xi . \end{cases}$$

Then, according to (3.8), $\phi_{n,k}(\xi,t)$ is a convex N-function satisfying for each $\xi\geqq 2^{n_0}$

(3.11) $$\phi_{n,k}(\xi)\leqq M(\xi, t) \qquad \text{for a.e. } t\in I_{n,k}$$

and

(3.12) $$\overline{M}_{n,k}(\xi)\leqq 4\phi_{n,k}(\xi) \qquad \text{for all } 2^n\geqq\xi\geqq 2^{n_0},$$

because, $\xi^{\frac{\gamma}{2^{n\alpha}}}\leqq 2^{\frac{n\gamma}{2^{n\alpha}}}\leqq 2$ holds for $\xi\leqq 2^n$.

Then, substituting $\underline{M}_{n,k}$ in the proof of Theorem 1 by $\phi_{n,k}$, we can prove by (3.11) and (3.12) that $m\left(\frac{1}{2\alpha}S_{2^n}x\right)\leqq 4m(2^{n_0}1)+4m\left(\frac{1}{2\delta}x\right)$ holds for all $n\geqq n_0$, hence $\sup_{\|x\|\leqq 1,\ n>n_0}\|S_{2^n}x\|<+\infty$ $(n>n_0)$, in the quite same way. From this the proof is immediately established. Q. E. D.

For the $L^{p(t)}$ spaces ($1\leqq p(t)$ for a. e. $t\in[0,1]$), the matters in question come to be quite simple. In this case, $M(\xi, t)=\xi^{p(t)}$ satisfies C. 2) always, and C. 1) is also implied from the condition corresponding to C. 3) or C. 3'). In fact, we obtain

Theorem 3. *Haar functions* $\{\chi_\nu(t)\}_{\nu=1}^\infty$ *constitute a basis in* $L^{p(t)}$, *if*

C. 4) $$\overline{\lim_{\delta\to 0}}\ \omega(\delta)\log\frac{1}{\delta}<+\infty$$

holds, where $\omega(\delta)=\text{ess.}\sup_{t,t'\in[0,1],\ |t-t'|\leqq\delta}|p(t)-p(t')|$.

Furthermore from Theorem 2 we have

Theorem 4. *If $p(t)$ satisfies the Lipschitz α-condition* $(0<\alpha\leqq 1)$, *then Haar functions* $\{\chi_\nu(t)\}_{\nu=1}^{\infty}$ *constitute a basis in* $\boldsymbol{L}^{p(t)}$.

In these cases, the conditions C. 3) and C. 3') hold respectively, whence C. 1) is also necessarily fulfiled, as easily verified. Therefore the assertion is obtained from Theorem 1 and 2.

4. In this section some remarks concerning the theorems in **3** shall be presented.

Remark 1. The condition C. 2) in Theorems 1 and 2 can not be erased. Indeed, in case $M(\xi, t)$ satisfies only C. 1) and C. 3), it may occur, as an easy example shows, that $\alpha_\nu=\int_0^1 a(t)\chi_\nu(t)\,dt=+\infty$ holds for some ν (hence necessarily for an infinite number of ν), where $a(t)$ is an element of $\boldsymbol{L}_{M(\xi,t)}$.

Remark 2. Theorems 3 and 4 have the direct extensions, without adding any assumption, to the following special modulared function spaces: $\boldsymbol{L}_{M^{\boldsymbol{p}}}$ or $\boldsymbol{L}_{\boldsymbol{p}^M}$, where $M^{\boldsymbol{p}}$ and $\boldsymbol{p}^M$ are defined such that

$$M^{\boldsymbol{p}}(\xi, t)=M(\xi)^{p(t)} \quad \text{and} \quad \boldsymbol{p}^M(\xi, t)=M(\xi^{p(t)}) \tag{4.1}$$

hold respectively for all $\xi\geqq 0$ and $t\in[0,1]$ with a convex N-function $M(\xi)$ satisfying the Δ_2-condition and $p(\cdot)\geqq 1$.

For the proof based on Theorems 1 and 2, we only note here that if a convex N-function $M(\xi)$ satisfy the Δ_2-condition $M(\xi)\leqq\gamma\xi^p$ holds $(\xi\geqq\xi_0)$ for some $\gamma>0$, $p\geqq 1$ and $\xi_0\geqq 0$.

Remark 3. In Theorems 3 and 4 we can not weaken the assumption by the *continuity* of $p(t)$ without failing to hold the validity, as the following example shows.

Example: Let $\{\nu_i\}_{i=1}^{\infty}$ be a sequence of natural numbers such that and $(2^{\nu_i+1})^{\frac{1}{2^{i+1}}}>i$ and $\nu_1<\nu_2\cdots$ for all $i\geqq 1$, and $I_i=\left(\frac{1}{2^{\nu_i+1}}, \frac{1}{2^{\nu_i}}\right)$.

Now we put

$$p_0(t)=\begin{cases} 1+\dfrac{1}{2^n} & \text{for } t\in\left[\dfrac{1}{2^{\nu_n}}-\dfrac{1}{3\cdot 2^{\nu_n+1}}, \dfrac{1}{2^{\nu_n}}\right]; \\ 1+\dfrac{1}{2^{n+1}} & \text{for } t\in\left[\dfrac{1}{2^{\nu_n+1}}, \dfrac{1}{2^{\nu_n+1}}+\dfrac{1}{3\cdot 2^{\nu_n+1}}\right]; \\ \text{linear} & \text{otherwise.} \end{cases} \tag{4.2}$$

for all $n\geqq 1$, and also

$$f_n(t)=\begin{cases} \beta_n & \text{for } t\in\left[\dfrac{1}{2^{\nu_n+1}}, \dfrac{1}{2^{\nu_n+1}}+\dfrac{1}{3\cdot 2^{\nu_n+1}}\right]; \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

where $\beta_n=(3\cdot 2^{\nu_n+1})^{\frac{2^{n+1}}{2^{n+1}+1}}$ for all $n\geqq 1$. Then $p_0(t)$ is continuous, $1\leqq p_0(t)\leqq 2$ for all $t\in[0,1]$, $f_n\in L^{p_0(t)}$ and $\|f_n\|=1$ for all $n\geqq 1$.

Now, suppose that the set of Haar functions be a basis for the space $L^{p_0(t)}$, and by virtue of the Banach's Theorem [1], the norms $\|S_\nu\|$ ($\nu\geqq 1$) must be bounded from above. On the other hand, we can deduce without difficulty that for a sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ suitable chosen we have $m(S_{k_n}f_n)\geqq\sqrt{n}/9$ for all $n\geqq 1$, whence we have $\sup_{n\geqq 1}\|S_nf_n\|=\sup_{n\geqq 1}\|S_n\|=+\infty$. Therefore we obtain a contradiction. Consequently, we conclude that *Haar functions* $\{\chi_\nu(t)\}_{\nu=1}^{\infty}$ *do not compose a basis in the space* $L^{p_0(t)}$ *thus constructed.*

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Department of Mathematics,
Hirosaki University

Department of Mathematics,
Hokkaido University

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