ON QUASI-GALOIS EXTENSIONS OF DIVISION RINGS

Dedicated to Prof. Kinjiro Kunugi on his 60th birthday

By

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Throughout the present paper, R be always a division ring, and S a division subring of R. And, we use the following conventions: $C=V_R(R)$, $V=V_R(S)$, $H=V_R^2(S)=V_R(V_R(S))$, and further for any subrings $R_1\supseteq R_2$ of R the set of all R_2-(ring) isomorphisms of R_1 into R will be denoted as $\Gamma(R_1/R_2)$. As to other notations and terminologies used in this paper, we follow the previous one [3]. We consider here the following conditions:

(I) If S' is a subring of R properly containing S with $[S':S]_l<\infty$ then $\Gamma(S'/S)\neq 1$.

(I_0) If S' is a subring of R properly containing S with $[S':S]_r<\infty$ then $\Gamma(S'/S)\neq 1$.

(I′) H/S is Galois.

(II) If $S_1\supseteq S_2$ are intermediate rings of R/S with $[S_1:S]_l<\infty$ then $\Gamma(S_1/S)\,|\,S_2=\Gamma(S_2/S)$.

(II_0) If $S_1\supseteq S_2$ are intermediate rings of R/S with $[S_1:S]_r<\infty$ then $\Gamma(S_1/S)\,|\,S_2=\Gamma(S_2/S)$.

(II′) If $T_1\supseteq T_2$ are intermediate rings of R/H with $[T_1:H]_l<\infty$ then $\Gamma(T_1/S)\,|\,T_2=\Gamma(T_2/S)$[1].

(II_0') If $T_1\supseteq T_2$ are intermediate rings of R/H with $[T_1:H]_r<\infty$ then $\Gamma(T_1/S)\,|\,T_2=\Gamma(T_2/S)$.

Following [5], R/S is said to be *(left-)quasi-Galois* when (I) and (II) are fulfilled. Symmetrically, if (I_0) and (II_0) are done, we shall say R/S is *right-quasi-Galois*. In [5], we can find some fundamental theorems of quasi-Galois extensions. The purpose of the present paper is to expose several additional theorems concerning such extensions. At first, we shall recall the following lemmas which have been obtained in [4] and [5].

Lemma 1. *If S' is an intermediate ring of R/S then $[V:V_R(S')]_r\leqslant$*

1) In [5], the condition that if T is an intermediate ring of R/H with $[T:H]_l<\infty$ then $\Gamma(T/S)\,|\,H=\Gamma(H/S)$ was cited as (II′). However, it will be rather natural to alter it like above.

$[S':S]_l$, *and particularly in case* $V_R^2(S)=S$ *the equality holds (provided we do not distinguish between two infinite dimensions). If* $[S':S]_l<\infty$ *then* $V_R^2(S')=H[S']$, *and if* R/S *is (left-) locally finite then so is* R/H.

Lemma 2. *Let* R/S *be locally finite. In order that* R/S *is quasi-Galois it is necessary and sufficient that* (I′) *and* (II) *are satisfied, and if* (I′) *and* (II′) *are satisfied then* R/S *is quasi-Galois.*

Lemma 3. *Let* R *be locally finite and quasi-Galois over* S. *If* T *is an intermediate ring of* H/S *then* $T\Gamma(T/S)\subseteq H$, *whence it follows* $\Gamma(H/S)=\mathfrak{G}(H/S)$.

Lemma 4. *Let* R *be locally finite and quasi-Galois over* S. *If* S' *is an intermediate ring of* R/S *with* $[S':S]_l<\infty$ *then* R/S' *is quasi-Galois,* $V_R^2(S')/S'$ *is outer Galois and* $\mathfrak{G}(V_R^2(S')/S')\approx\mathfrak{G}(H/H\cap S')$ *by contraction, and* $\Gamma(V_R^2(S')/S)\,|S'=\Gamma(S'/S)$.

By Lemma 4, in the same way as in the proof of [3, Lemma 3.5], we can prove that *if* R *is locally finite and quasi-Galois over* S *and* R' *an intermediate ring of* R/S *with* $[H[R']:H]_l<\infty$ *then* $H[R']$ *is locally finite and outer Galois over* R' *and* $\mathfrak{G}(H[R']/R')\approx\mathfrak{G}(H/H\cap R')$ *by contraction.* Accordingly, we can apply the same argument as in the proof of [4, Lemma 4] to obtain the next

Theorem 1. *Let* R *be locally finite and quasi-Galois over* S. *If* R' *is an intermediate ring of* R/S, *and* H' *an intermediate ring of* H/S *that is Galois over* S, *then* $H'[R']$ *is locally finite and outer Galois over* R' *and* $\mathfrak{G}(H'[R']/R')\approx\mathfrak{G}(H'/H'\cap R')$ *(algebraically and topologically) by contraction.*

The proof of the next lemma will be easy from that of [3, Lemma 3.2].

Lemme 5. *Let* T *be an intermediate division ring of* R/S, *and* $\mathfrak{H}$ *an automorphism group of* $H[T]$. *If* $J(\mathfrak{H}, H[T])=T$ *and* $H\mathfrak{H}=H$ *then* $[H^*\cdot T:H^*]_l$[2] $=[T:H\cap T]_l$ *and* $[T\cdot H^*:H^*]_r=[T:H\cap T]_r$ *for each intermediate division ring* H^* *of* $H/H\cap T$.

Now, Lemmas 4 and 5 enable us to apply the argument used in the proof of [3, Lemma 3.2] to obtain the next lemma.

Lemma 6. *Let* R *be locally finite and quasi-Galois over* S. *If* S' *is an intermediate ring of* R/S *with* $[S':S]_l<\infty$ *then* $[H^*[S']:H^*]_l=[R^*:H\cap R^*]_l=[S':H\cap S']_l$ *for each intermediate rings* H^* *of* $H/H\cap S'$ *and* R^* *of* $H[S']/S'$.

By the validity of Lemma 6, the proof of the next theorem proceeds evidently just like that of [3, Theorem 3.2] did.

2) $H^*\cdot T$ means the module product of H^* and T.

Theorem 2. *Let R be locally finite and quasi-Galois over S. If T is an f-regular intermediate ring of R/S then $[T:H\cap T]_l=[V:V_R(T)]_r<\infty$.*

Lemma 7. *If R/H is locally finite and R' is an intermediate ring of R/H with $[R':H]_l<\infty$ then R/H is right-locally finite and $[R':H]_r=[R':H]_l$.*

Proof. Although the first assertion is [2, Lemma 4] itself, we shall prove here both. Let X be an arbitrary finite subset of V that is linearly left-independent over $V'=V_R(R')$, and let $R_1=R'[X]$, that is evidently left-finite over H. We set here $V_1=V_{R_1}(H)$, $V_1'=V_{R_1}(R')$, and $C_1=V_{R_1}(R_1)$. Then, $[V_1:C_1]\leqslant[R_1:H]_l<\infty$ by Lemma 1, whence it follows $[V_1:V_1']_l=[V_1:V_1']_r<\infty$. On the other hand, Lemma 1 yields also $[V_1:V_1']_r\leqslant[R':H]_l$, whence we obtain $[V_1:V_1']_l\leqslant[R':H]_l$. Recalling here that $X\subseteq V_1$ and $V_1'\subseteq V'$, we obtain $\sharp X\leqslant[V_1:V_1']_l\leqslant[R':H]_l$, that is, $[V:V']_l\leqslant[R':H]_l$. Lemma 1 yields therefore $[R':H]_r=[V:V']_l\leqslant[R':H]_l$, because $V_R^2(V')=V'$. Now, the right-local finiteness of R/H is evident, and so it follows symmetrically $[R':H]_l\leqslant[R':H]_r$. We have proved therefore that $[R':H]_r=[R':H]_l$.

The next corollary has been stated in [2, Theorem 2], whose proof was essentially due to [1, Theorem 7.9.2]. However, we have recently found that the proof of [1, Theorem 7.9.2] would be open to doubt—we are afraid that the proof of [1, Theorem 7.8.1] was no longer efficient in that of [1, Theorem 7.9.2]. Because of this reason, we should like to present a new proof without making use of [1, Theorem 7.9.2] to our corollary.

Corollary 1. *Let R be Galois over S and locally finite over H. If S' is an intermediate ring of R/S with $[S':S]_l<\infty$ then $[S':S]_r=[S':S]_l$.*

Proof. At first, if R/S is outer Galois, [3, Lemma 1.3] yields at once $\infty>[S':S]_l=[(\mathfrak{G}|S')R_r:R_r]_r$[3] $=[(\mathfrak{G}|S')C_r:C_r]_r=[(\mathfrak{G}|S')C_l:C_l]_r=[(\mathfrak{G}|S')R_l:R_l]_r$ $=[S':S]_r$. Next, for general case, R/S' is Galois by [2, Theorem 1][3] and there holds $\infty>[H[S']:H]_l=[H[S']:H]_r$ by Lemma 7. And so, by Lemmas 1 and 5, we obtain $\infty>[H[S']:H]_r\geqslant[S'\cdot H:H]_r=[S':H\cap S']_r\geqslant[V:V_R(S')]_l$ $=[H[S']:H]_r$ and $\infty>[H[S']:H]_l\geqslant[H\cdot S':H]_l=[S':H\cap S']_l\geqslant[V:V_R(S')]_r=$ $[H[S']:H]_l$. Accordingly, it follows $[S':H\cap S']_r=[H[S']:H]_r=[H[S']:H]_l$ $=[S':H\cap S']_l<\infty$. Recalling here that H/S is outer Galois, as is noted above, there holds $[H\cap S':S]_r=[H\cap S':S]_l<\infty$. Now, combining these equalities, our assertion $[S':S]_r=[S':S]_l$ will be evident.

Now, we shall prove the next theorem.

Theorem 3. *The following conditions are equivalent to each other:*

3) Since the division ring R is $\mathfrak{G}R_r$-irreducible and $V_{\mathrm{Hom}(R,R)}(\mathfrak{G}R_r)=S_l$, $\mathfrak{G}R_r$ is dense in $\mathrm{Hom}_{S_l}(R,R)$ by JACOBSON's density theorem [1, p. 28].

(1) *R/S is locally finite and quasi-Galois,* (1_0) *R/S is right-locally finite and right-quasi-Galois,* (2) *R/S is locally finite and* (I′), (II) *are fulfilled,* (2_0) *R/S is right-locally finite and* (I′), (II_0) *are fulfilled,* (3) *R/S is locally finite and* (I′), (II′) *are fulfilled, and* (3_0) *R/S is right-locally finite and* (I′), (II_0') *are fulfilled.*

Proof. In virtue of Lemma 2, one will readily see that only the implications (1)⇒(3) and (1)⇒(1_0) are left to be shown.

(1)⇒(3), Let $T_1 \supseteq T_2$ be intermediate rings of R/H with $[T_1:H]_l<\infty$. Choose an intermediate ring S_2' of T_2/S such that $[S_2':S]_l<\infty$ and $T_2=H[S_2']$ and an intermediate ring S_1 of T_1/S_2' such that $[S_1:S]_l<\infty$ and $T_1=H[S_1]=V_R^2(S_1)$. If we set $S_2=T_2\cap S_1(\supseteq S_2')$, then $[S_2:S]_l<\infty$ and $T_2=H[S_2]=V_R^2(S_2)$ evidently. As R/S_2 is quasi-Galois, $\mathfrak{G}(T_2/S_2)=\mathfrak{G}(T_1/S_1)|T_2$ by Lemma 4. Noting that $\Gamma(T_2/S)|S_2\subseteq\Gamma(S_1/S)|S_2=\Gamma(T_1/S)|S_2$ by Lemma 4, for each $\sigma\in\Gamma(T_2/S)$ we can find some $\rho\in\Gamma(T_1/S)$ such that $\sigma|S_2=\rho|S_2$. By Lemma 3, $T_2\sigma=H[S_2\sigma]\subseteq H[S_1\rho]=T_1\rho$ and $\sigma\rho^{-1}\in\Gamma(T_2/S_2)=\mathfrak{G}(T_2/S_2)=\mathfrak{G}(T_1/S_1)|T_2$. Accordingly, σ is contained in $\Gamma(T_1/S)|T_2$ obviously.

(1)⇒(1_0). Let S' be an intermediate ring of R/S with $[S':S]_l<\infty$. Since $\mathfrak{G}(H[S']/S')\approx\mathfrak{G}(H/H\cap S')$ by contraction (Lemma 4), Lemmas 1, 5 and 7 yield $[S':H\cap S']_r=[S'\cdot H:H]_r\leqslant[H[S']:H]_r<\infty$. On the other hand, recalling that H/S is outer Galois by Lemma 2, we obtain $[H\cap S':S]_r=[H\cap S':S]_l<\infty$. (See the proof of Corollary 1.) Combining those, we obtain $[S':S]_r<\infty$, which proves evidently the right-local finiteness of R/S. Now, our assertion will be obvious.

Corollary 2. *Let R be locally finite and quasi-Galois over S. If S' is an intermediate ring of R/S finitely generated over S then $[S':S]_r=[S':S]_l$.*

Proof. As R/H is locally finite by Lemma 1 and R is right-locally finite and right-quasi-Galois over S by Theorem 3, Lemmas 6 and 7 together with their symmetries yield $[S':H\cap S']_l=[H[S']:H]_l=[H[S']:H]_r=[S':H\cap S']_r$. Hence, we readily obtain $[S':S]_r=[S':S]_l$. (Cf. the proof of Corollary 1.)

The following corollary is [3, Corollary 3.5] itself. However, its proof contained a gap. In fact, in order to be able to apply the argument used in the proof of [3, Lemma 3.9], we had to prove the validity of (II′). This fact requested is now secured by Theorem 3.

Corollary 3. *If R is locally finite and quasi-Galois over S and $[R:H]_l\leqslant\aleph_0$, then R/S is Galois.*

Further, for the sake of completeness, we shall give here the proof of the following theorem [5, Theorem 2].

Theorem 4. *If R/S is locally finite and quasi-Galois then so is R/T for each f-regular intermediate ring T of R/S.*

Proof. Obviously, by Lemma 4, we may restrict our proof to the case that $T\subseteq H$. Let F be an arbitrary finite subset of R, and set $S'=S[F]$, $H'=T[H\cap S']$, $R'=H'[S']=T[F]$. Then, $[R':H']_l=[S':H\cap S']_l<\infty$ by Lemma 6. On the other hand, noting that H is locally finite and outer Galois over S, there holds $[H':T]<\infty$ by [3, Conclusion 2.1]. Hence, we have $[T[F]:T]_l=[R':H']_l\cdot[H':T]<\infty$, which means evidently the local finiteness of R/T. Moreover, as $V_R^2(T)=H$ and the condition (II′) holds by Theorem 3, our assertion is a consequence of Theorem 3.

Lemma 8. *Let R be locally finite and quasi-Galois over S. If T is an f-regular intermediate ring of R/S then $\Gamma(V_R^2(T)/S)|T=\Gamma(T/S)$.*

Proof. Take an intermediate ring S' of T/S such that $[S':S]_l<\infty$ and $V_R(S')=V_R(T)$. Then, $T'=V_R^2(T)=V_R^2(S')=H[S']$ and $[T':H]_l<\infty$ by Lemma 1. As $\mathfrak{G}(T'/S')\approx\mathfrak{G}(H/H\cap S')$ by contraction (Lemma 4), [3, Conclusion 2.1] will yield at once $T=(H\cap T)[S']$. Now, let σ be an arbitrary element of $\Gamma(T/S)$. Then, by Lemma 4 $\sigma|S'=\tau|S'$ for some $\tau\in\Gamma(T'/S)$, and by Lemma 3 we see that $T\sigma=((H\cap T)\sigma)[S'\sigma]\subseteq H[S'\sigma]=H[S'\tau]=T'\tau$. And so, $\sigma\tau^{-1}\in\Gamma(T/S')=\mathfrak{G}(T'/S')|T$ by Lemmas 3, 4 and [3, Conclusion 2.1], whence we have $\sigma=(\sigma\tau^{-1})\tau\in\Gamma(T'/S)|T$.

By the light of Lemma 8, we can prove the following extension theorem of isomorphisms that corresponds to [3, Theorem 3.5].

Theorem 5. *Let R be locally finite and quasi-Galois over S, and $T_1\supseteq T_2$ intermediate rings of R/S. If T_1 is f-regular then $\Gamma(T_2/S)=\Gamma(T_1/S)|T_2$.*

Proof. Setting here $T_i'=V_R^2(T_i)$ $(i=1,2)$, we have $T_1'\supseteq T_2'\supseteq H$, $T_1'\supseteq T_1\supseteq T_2$, $[T_1':H]_l<\infty$ by Lemma 1, and so $\Gamma(T_i/S)=\Gamma(T_i'/S)|T_i (i=1,2)$ and $\Gamma(T_2'/S)=\Gamma(T_1'/S)|T_2'$ by Lemma 8 and Theorem 3 respectively. It follows therefore $\Gamma(T_2/S)=(\Gamma(T_1'/S)|T_2')|T_2=\Gamma(T_1'/S)|T_2=(\Gamma(T_1'/S)|T_1)|T_2=\Gamma(T_1/S)|T_2$.

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