A CHARACTERIZATION OF UNIFORMLY DISTRIBUTED SEQUENCES OF INTEGERS

By

Miyoko UCHIYAMA and Saburô UCHIYAMA

1. Introduction. Let m be an integer not less than two. An infinite sequence $A=(a_n)$ of integers is said to be *uniformly distributed modulo* m, if the limit

$$\lim_{N\to\infty}\frac{1}{N}A(N, j, m)=\frac{1}{m}$$

exists for all $j=0, 1, \cdots, m-1$, where $A(N, j, m)$ is the number of terms a_n $(1\leqq n\leqq N)$ which are $\equiv j \pmod{m}$. If the sequence A is uniformly distributed modulo m for every integer $m\geqq 2$, then we say that A is *uniformly distributed*.

The notion of uniform distribution of sequences of integers, which is in a sense dual to that of uniform distribution (mod 1) of sequences of real numbers, is due to I. Niven [4], who obtained a number of interesting results on uniformly distributed sequences of integers. And a criterion for a sequence $A=(a_n)$ of integers should be uniformly distributed has been given by one of the present authors (see [5]): thus, a necessary and sufficient condition that the sequence A be uniformly distributed modulo m, where $m\geqq 2$, is that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp\left(2\pi i a_n\frac{h}{m}\right)=0$$

for all $h=1, \cdots, m-1$. Hence, the sequence A is uniformly distributed if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp(2\pi i a_n t)=0$$

for all rational numbers t with $t\not\equiv 0 \pmod{1}$.

The main purpose of this note is to present another characterization of uniformly distributed sequences of integers, making use of a kind of integrals defined over the space of integers. This, as well as the criterion quoted above, will have some analogy with the well-known characterization of uniform distribution (mod 1) of sequences of real numbers (cf. [3; Chap. IV]).

We first define a (finitely additive) measure on the set of integers and then construct an integration theory of functions defined on the set of integers. The notion of uniform distribution of sequences of integers will be re-defined in terms of measures of certain sets of integers, and we shall formulate and prove our fundamental result on the characterization of uniformly distributed sequences.

The results of the present note can, of course, be easily generalized to the case of any finite dimensional product of the space of integers.

2. The measure of Banach-Buck. There is no essential loss of generality in restricting ourselves only to consider the set of non-negative integers. Let I denote throughout in the following the set of all non-negative integers. According to the result of S. Banach [1; p. 231] and R. C. Buck [2] a totally finite measure μ is defined on a class of subsets of the space I; μ is necessarily finitely additive since points are to have zero measure and the space on which the measure is defined is only countable.

Notation. If A and B are subsets of I, $A \dot{\subset} B$ means that there is a finite subset C of I for which $A-C \subset B-C$; $A \doteq B$ means that $A \dot{\subset} B$ and $B \dot{\subset} A$, thus $A \doteq \emptyset$ means that A itself is finite. (We shall use the symbol $\emptyset$ to denote the empty set.)

If A is a subset of I, A' denotes its complement, thus $A \cup A' = I$.

Definition 1. $\mathbf{R}$ is the Boolean ring generated by all finite subsets of I and all subsets of I which are infinite arithmetic progressions.

In particular, if $A \in \mathbf{R}$ and $A \doteq B$ then $B \in \mathbf{R}$. Clearly $\mathbf{R}$ contains the whole space I and hence is in fact a Boolean algebra.

Definition 2. If E is a finite set in $\mathbf{R}$, we define $\mu(E)=0$, in particular $\mu(\emptyset)=0$; if $E=(an+b)$ is an infinite arithmetic progression in $\mathbf{R}$, we define $\mu(E)=1/a$, in particular $\mu(I)=1$. If A and B are disjoint sets in $\mathbf{R}$ and if $\mu(A)$ and $\mu(B)$ are defined, then $\mu(A \cup B)$ is defined and is equal to $\mu(A)+\mu(B)$.

It is easy to see that every finite union of arithmetic progressions is a finite union of disjoint arithmetic progressions. Thus μ is well defined for all sets in $\mathbf{R}$.

The set function μ defined on $\mathbf{R}$ is real valued, non-negative, monotone and finitely additive. If A and B belong to $\mathbf{R}$ then we have

$$\mu(A \cup B)+\mu(A \cap B)=\mu(A)+\mu(B).$$

DEFINITION 3. Let $\mathbf{S}$ be the class of all subsets of I. If E is a set in $\mathbf{S}$, we define

$$\mu^*(E)=\inf\{\mu(A):\ E\dot{\subset}A,\ A\in\mathbf{R}\}.$$

The set function μ^* defined on $\mathbf{S}$ is real valued, non-negative, monotone and finitely subadditive. Hence μ^* is an outer measure on $\mathbf{S}$.

DEFINITION 4. $\bar{\mathbf{S}}$ is the class of all sets E in $\mathbf{S}$ such that for all X in $\mathbf{S}$,

$$\mu^*(X)=\mu^*(X\frown E)+\mu^*(X\frown E').$$

A set E belonging to $\bar{\mathbf{S}}$ is called measurable. Clearly the class $\bar{\mathbf{S}}$ is a Boolean algebra containing the class $\mathbf{R}$. It is shown in [2] that $\bar{\mathbf{S}}$ strictly contains $\mathbf{R}$.

It is easily seen that every set of outer measure zero belongs to $\bar{\mathbf{S}}$ and that the set function $\bar{\mu}$, defined for E in $\bar{\mathbf{S}}$ by $\bar{\mu}(E)=\mu^*(E)$, is a complete, finitely additive measure on $\bar{\mathbf{S}}$. Since $\bar{\mu}(A)=\mu(A)$ for A in $\mathbf{R}$, there is no possibility of confusion in writing $\mu(E)$ instead of $\bar{\mu}(E)$ for sets E which are in $\bar{\mathbf{S}}$ but not necessarily in $\mathbf{R}$.

The following statements are mutually equivalent:

(i) E belongs to $\bar{\mathbf{S}}$;

(ii) $\mu^*(E)+\mu^*(E')=1$;

(iii) for any $\varepsilon>0$ there exist A and B in $\mathbf{R}$ with $A\dot{\subset}E\dot{\subset}B$ and $\mu(B-A)<\varepsilon$;

(vi) $\mu^*(E)=\mu_*(E)$, where $\mu_*(E)=\sup\{\mu(B):\ B\dot{\subset}E,\ B\in\mathbf{R}\}$.

Buck [2] has shown among others that the class $\bar{\mathbf{S}}$ contains infinite sets of measure zero and that the set of values of $\mu(E)$ for E in $\bar{\mathbf{S}}$ is exactly the closed unit interval.

3. Integration. By a partition $\Delta=(E_1,\cdots,E_r)$ of the set I is meant a finite, disjoint class $E_1,\cdots,E_r$ of sets in $\bar{\mathbf{S}}$ whose union is I.

Let $f(x)$ be a bounded, real valued function defined on the set I. If $\Delta=(E_1,\cdots,E_r)$ is a partition of I, we write

$$\underline{f}_j=\inf\{f(x):\ x\in E_j\},\qquad \bar{f}_j=\sup\{f(x):\ x\in E_j\}$$

for $j=1,\cdots,r$. We set

$$\underline{M}(f;\Delta)=\sum_{j=1}^{r}\underline{f}_j\mu(E_j),\qquad \bar{M}(f;\Delta)=\sum_{j=1}^{r}\bar{f}_j\mu(E_j).$$

Then

$$-K \leqq \underline{M}(f;\Delta) \leqq \overline{M}(f;\Delta) \leqq K,$$

if $|f(x)| \leqq K$.

If $\Delta_1 = (F_1, \cdots, F_s)$ is another partition of I, then it is clear that

$$\underline{M}(f;\Delta) \leqq \overline{M}(f;\Delta_1),$$
$$\underline{M}(f;\Delta_1) \leqq \overline{M}(f;\Delta).$$

Consequently, if we put

$$\underline{M}(f) = \sup_{\Delta} \underline{M}(f;\Delta), \qquad \overline{M}(f) = \inf_{\Delta} \overline{M}(f;\Delta),$$

then we have

$$\underline{M}(f) \leqq \overline{M}(f).$$

DEFINITION 5. A bounded, real valued function $f(x)$ defined on I is said to be integrable, if $\underline{M}(f) = \overline{M}(f)$, in which case this common value determined by f is the integral of f over I and is denoted by $\int f(x)\,d\mu(x)$ or simply by $\int f d\mu$.

A bounded, complex valued function $f(x)$ defined on I is integrable, if the real valued functions $\operatorname{Re} f(x)$ and $\operatorname{Im} f(x)$ are integrable, and then the integral of f over I is

$$\int f(x)\,d\mu(x) = \int \operatorname{Re} f(x)\,d\mu(x) + i\int \operatorname{Im} f(x)\,d\mu(x).$$

Remark. For an unbounded real or complex valued function $f(x)$ defined on I we may define the integrability of f by the existence and finiteness of the limit

$$\lim_{K\to\infty} \int f_K(x)\,d\mu(x),$$

where $f_K(x) = f(x)$ if $|f(x)| \leqq K$ and $= K$ otherwise. But in what follows we shall concern, for our purpose, only with bounded functions $f(x)$ defined on I.

Our integrals thus defined possess many elementary properties in common with ordinary Riemann integrals. However, it should be noted that there is an integrable function on I whose lower and upper Darboux sums do not converge, so that the Darboux theorem for integrals does not hold in the present situation.

4. Measurable functions. A real valued function $f(x)$ defined on I is said to be *measurable*, if for all real c we have

$$\{x : f(x) < c\} \in \overline{\mathbf{S}}.$$

Since $\bar{\mathbf{S}}$ is a Boolean algebra, the measurability of $f(x)$ implies that

$$\{x:\ f(x)\geqq c\}\in\bar{\mathbf{S}}, \qquad \{x:\ c_1\leqq f(x)<c_2\}\in\bar{\mathbf{S}}$$

for all real c, c_1 and c_2.

A complex valued function $f(x)$ defined on I is measurable, if the functions $\operatorname{Re} f(x)$ and $\operatorname{Im} f(x)$ are both measurable.

It is easy to prove the following propositions:

PROPOSITION 1. *If a real or complex valued function $f(x)$ defined on I is bounded and measurable, then $f(x)$ is integrable.*

PROPOSITION 2. *If $f(x)$ is a real valued, bounded and measurable function defined on I, then we have*

$$\int f(x)\,d\mu(x)=\lim_{r\to\infty}\sum_{k=-Kr}^{Kr-1}\frac{k}{r}\mu(E_{r,k}),$$

where $|f(x)|<K$, K being an integer, and

$$E_{r,k}=\left\{x:\ \frac{k}{r}\leqq f(x)<\frac{k+1}{r}\right\} \qquad (-Kr\leqq k\leqq Kr-1).$$

Examples. The characteristic function $\chi_E(x)$ of a measurable set E is measurable and hence is integrable: we have in fact

$$\int \chi_E(x)\,d\mu(x)=\mu(E).$$

A less trivial example is the following. It can be shown that the function

$$e_\alpha(x)=\exp(2\pi i\alpha x)$$

defined on I, α being a real valued parameter, is measurable if and only if α is a rational number (see Proposition 4 below).

5. Uniform distribution. Let H be a fixed subset of the set I. An infinite sequence $A=(a_n)$ of non-negative integers is said to be *uniformly distributed in the set H,* if for every set E in $\mathbf{R}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown E}(a_n)=c(A,\,H)\,\mu(E)$$

where $c(A,H)$ is a constant depending only on A and H. In the special case of $H=I$ we say simply that A is *uniformly distributed.*

It follows from the definition that we have

$$0\leqq c(A,H)\leqq 1$$

for every set H in which there is a uniformly distributed sequence A and, in particular, $c(A,I)=1$ for every sequence A which is uniformly distributed. In the following we shall restrict our considerations, when we speak of uniform distribution of sequences of integers in certain sets, to the sets H and sequences A for which $c(A, H)>0$.

It is clear that this new definition of uniform distribution of sequences of integers (in the whole space I) coincides with the previous one (without the reference to modulus m) given in the Introduction, though **R** contains sets other than mere arithmetic progressions.

PROPOSITION 3. *If $A=(a_n)$ is a sequence of integers uniformly distributed in a set H, then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown E}(a_n)=c(A, H)\mu(E)$$

for all sets in $\bar{\mathbf{S}}$.

Proof. Put $c=c(A, H)$. Let E be any set in **S**. We have, by definition,

$$\mu^*(E)\leqq\mu(B) \quad \text{for every } B \text{ with } E\dot{\subset}B,\ B\in\mathbf{R};$$
$$\mu_*(E)\leqq\mu(C) \quad \text{for every } C \text{ with } C\dot{\subset}E,\ C\in\mathbf{R}.$$

Hence

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown E}(a_n)\leqq\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown B}(a_n)=c\mu(B),$$

so that

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown E}(a_n)\leqq c\inf\mu(B)=c\mu^*(E).$$

Similarly

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown E}(a_n)\geqq c\sup\mu(C)=c\mu_*(E).$$

If E belongs to $\bar{\mathbf{S}}$, then, by (iv) in § 2, we have $\mu^*(E)=\mu_*(E)=\mu(E)$. The result follows at once.

We can now state our main theorem.

THEOREM. *Let H be a subset of I and let $A=(a_n)$ be a sequence of non-negative integers. A necessary and sufficient condition that the sequence A be uniformly distributed in the set H is that*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_H(a_n)f(a_n)=c(A, H)\int f(x)\,d\mu(x)$$

for all real or complex valued integrable functions $f(x)$ defined on I, where $c(A, H)>0$ is a constant depending only on A and H.

Proof. The sufficiency of the condition is obvious. To prove the necessity, suppose that A is uniformly distributed in H and let $f(x)$ be an integrable function defined on I. There is no loss of generality in assuming that $f(x)$ is real valued and non-negative.

Given a positive number ε, there exist partitions

$$\Delta=(E_1,\cdots,E_r) \quad \text{and} \quad \Delta_1=(F_1,\cdots,F_s)$$

of I such that

$$\int f\,d\mu-\varepsilon \leqq \sum_{j=1}^{r} \underline{f}_j\,\mu(E_j) \leqq \sum_{k=1}^{s} \bar{f}_k\,\mu(F_k) \leqq \int f\,d\mu+\varepsilon,$$

where

$$\underline{f}_j=\inf\{f(x):\ x\in E_j\} \qquad (1\leqq j\leqq r),$$
$$\bar{f}_k=\sup\{f(x):\ x\in F_k\} \qquad (1\leqq k\leqq s).$$

Here, we may suppose without loss of generality that

$$\mu(E_j)>0 \quad \text{for all } j=1,\cdots,r, \quad \text{and}$$
$$\mu(F_k)>0 \quad \text{for all } k=1,\cdots,s.$$

Now, since $A=(a_n)$ is uniformly distributed in H, there is an N_0 depending only on A, H, ε, Δ, Δ_1 such that for all $N>N_0$ we have

$$\left|\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown E_j}(a_n)-c\mu(E_j)\right|<\varepsilon c\mu(E_j)$$

for $j=1,\cdots,r$, and

$$\left|\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown F_k}(a_n)-c\mu(F_k)\right|<\varepsilon c\mu(F_k)$$

for $k=1,\cdots,s$, where we have put $c=c(A,H)>0$. Hence, for $N>N_0$ we have

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\chi_H(a_n)f(a_n)&=\sum_{j=1}^{r}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown E_j}(a_n)f(a_n)\\&\geqq\sum_{j=1}^{r}\underline{f}_j\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown E_j}(a_n)\\&\geqq\sum_{j=1}^{r}\underline{f}_j\,c\mu(E_j)(1-\varepsilon)\\&\geqq c(1-\varepsilon)\left(\int f\,d\mu-\varepsilon\right),\end{aligned}$$

and similarly

$$\frac{1}{N}\sum_{n=1}^{N}\chi_H(a_n)f(a_n)=\sum_{k=1}^{s}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown F_k}(a_n)f(a_n)$$

$$\leqq \sum_{k=1}^{s} \bar{f}_k \frac{1}{N} \sum_{n=1}^{N} \chi_{H \frown F_k}(a_n)$$
$$\leqq \sum_{k=1}^{s} \bar{f}_k c\mu(F_k)(1+\varepsilon)$$
$$\leqq c(1+\varepsilon)\Big(\int f d\mu + \varepsilon\Big).$$

It now follows from the arbitrariness of ε that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_H(a_n) f(a_n) = c \int f d\mu,$$

which is the desired result.

COROLLARY 1. *Let $A=(a_n)$ be a sequence of integers uniformly distributed. If the set A is measurable, then $\mu(A)=1$. Conversely, if a set $A=(a_n)$ of integers is measurable and $\mu(A)=1$ then the sequence A is uniformly distributed.*

Proof. Put $H=I$ and take $f(x)=\chi_A(x)$ in the theorem. Then, since $c(A, I)=1$, we have

$$1 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_A(a_n) = \int \chi_A \, d\mu = \mu(A).$$

To prove the converse, suppose that A is measurable and $\mu(A)=1$. Then the complement A' of A is also measurable and $\mu(A')=0$. Since the sequence $I=(n)$ is uniformly distributed, we find that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_E(a_n) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{E\frown A}(n)$$
$$= \mu(E \frown A)$$
$$= \mu(E \frown A) + \mu(E \frown A')$$
$$= \mu(E)$$

for every E in $\bar{\mathbf{S}}$, and *a fortiori* for every E in $\mathbf{R}$.

This completes the proof of Corollary 1.

COROLLARY 2. *If a uniformly distributed sequence $A=(a_n)$ of integers is uniformly distributed in a measurable set H, then $c(A, H)=1$, $\mu(H)=1$. Conversely, if a sequence $A=(a_n)$ of integers is uniformly distributed in a measurable set H with $\mu(H)=1$, then $c(A, H)=1$ and the sequence A is uniformly distributed.*

Proof. Put $H=I$ and take $f(x)=\chi_H(x)$ in the theorem. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_H(a_n) = \mu(H).$$

On the one hand, if we apply the theorem to $H=H$ and $f(x)=\chi_I(x)\equiv 1$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_H(a_n)=c(A,H).$$

On the other hand, if we apply the theorem to $H=H$ and $f(x)=\chi_H(x)$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_H(a_n)=c(A,H)\mu(H).$$

Thus

$$\mu(H)=c(A,H)=c(A,H)\mu(H),$$

whence follows that $c(A,H)=\mu(H)=1$, since $c(A,H)>0$.

Conversely, if A is uniformly distributed in H, then for every set E in $\bar{\mathbf{S}}$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\cap E}(a_n)=c(A,H)\mu(E).$$

Since H is measurable and $\mu(H)=1$, we have $\mu(H')=0$ and, for any set E,

$$0\leqq\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H'\cap E}(a_n)\leqq\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H'}(n)=\mu(H')=0,$$

i.e.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H'\cap E}(a_n)=0.$$

It now follows that, for every E in $\bar{\mathbf{S}}$,

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_E(a_n)&=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(\chi_{H\cap E}(a_n)+\chi_{H'\cap E}(a_n))\\&=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\cap E}(a_n)\\&=c(A,H)\mu(E).\end{aligned}$$

Since this holds for every measurable H with $\mu(H)=1$, we must have $c(A,H)=c(A,I)=1$, and the sequence A is uniformly distributed.

As an application of Corollary 2 to the theorem we shall prove the following

PROPOSITION 4. *The function*

$$e_\alpha(x)=\exp(2\pi i\alpha x)$$

defined on I is measurable if and only if α is a rational number. If α is rational then $e_\alpha(x)$ is integrable and we have

$$\int e_\alpha(x)\,d\mu(x)=\begin{cases}1 & \text{for } \alpha\equiv 0 \pmod 1),\\ 0 & \text{for } \alpha\not\equiv 0 \pmod 1).\end{cases}$$

Proof. It is easy to see that $e_\alpha(x)$ is measurable for rational α. The result for the integral of $e_\alpha(x)$ over I is obvious from the relation

$$\int e_\alpha(x)\,d\mu(x)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e_\alpha(n).$$

Suppose now that α is irrational. Since $e_\alpha(x)$ is a periodic function of α with period 1, we may assume without loss of generality that $0<\alpha<1$. Put $\beta=1/\alpha$. To show that $e_\alpha(x)$ is not measurable, it suffices to prove that the set H of integers $[\beta n]$ with

$$c_1 \leqq \{\beta n\} < c_2$$

is not measurable for all real numbers c_1, c_2 satisfying $0\leqq c_1<c_2\leqq 1$, where $\{t\}$ denotes the fractional part of t.

Consider the sequence $A=(a_n)$, where $a_n=[\beta n]$. Since β is irrational, A is uniformly distributed (see [4]). Moreover, A is uniformly distributed in the set H. For it is easily verified that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{H\frown E}(a_n)=(c_2-c_1)\mu(E)$$

for every E in $\mathbf{R}$; thus $c(A,H)=c_2-c_1>0$. (To see this, note that if m, j are integers with $m\geqq 2$, $0\leqq j\leqq m-1$ then $[\beta n]\equiv j \pmod m$ if and only if

$$\left\{\frac{\beta n}{m}\right\}=\frac{j+\{\beta n\}}{m}.\quad)$$

Now, if the set H were measurable, then we would have

$$c(A,H)=1,\qquad \mu(H)=1,$$

by Corollary 2. Suppose that $c(A,H)=c_2-c_1=1$. Then $c_1=0$, $c_2=1$, and hence the sets A and H are identical. However, every measurable set has (natural) density equal to its measure. Thus

$$\frac{1}{\beta}=\mu(A)=\mu(H)=1,$$

which is a contradiction since $1/\beta=\alpha<1$. This completes the proof of Proposition 4.

Remark. If α is irrational then the function $e_\alpha(x)$, defined in Proposition 4, is not integrable. To show this it will be sufficient to prove that

$$\begin{matrix}\inf\\ \sup\end{matrix}\{\cos 2\pi\alpha x:\ x\in E\}=\begin{cases}-1\\ \ \ 1\end{cases}$$

for every measurable set E of positive measure. But this is in substance contained in the proof of Proposition 4. (Of course, a similar result holds for $\sin 2\pi\alpha x$.)

References

[1] S. BANACH: Théorie des opérations linéaires, Warszawa, 1932.
[2] R.C. BUCK: The measure theoretic approach to density, Amer. Journ. of Math., vol. 68 (1946), pp. 560–580.
[3] J.W.S. CASSELS: An introduction to Diophantine approximation, Cambridge, 1957.
[4] I. NIVEN: Uniform distribution of sequences of integers, Trans. Amer. Math. Soc., vol. 98 (1961), pp. 52–61.
[5] S. UCHIYAMA: On the uniform distribution of sequences of integers, Proc. Japan Acad., vol. 37 (1961), pp. 605–609.

Department of Mathematics,
Hokkaidô University

(Received August 20, 1962)