EXAMPLES OF NON MINIMAL POINTS ON RIEMANN SURFACES OF PLANER CHARACTER.

by

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The Martin's topologies on Riemann surfaces have been discussed by many authors and some examples of boundary points have been given. Prof. M. Brelot[1] gave a domain D in the z-plane such that there exist two sequences $q_{i,n}$ $(n=1, 2, \cdots)$ $(i=1, 2)$ in D tending to q_i as $n \to \infty$, $q_2 \neq q_1$ which determine the same K-Martin's boundary point to show that the K-Martin's topology is not necessarily finer than the euclidean topology. Also we constructed examples in the z-plane to show that N-Martin's topology is neither finer than the K-Martin's topology and K-Martin's topology is nor finer than the N-Martin's topology[2]. As for non minimal points, R. S. Martin presented an example of K-non minimal point in 3-dim. euclidean space[3] and we gave a Riemann surface of infinite genus contained in the classH. 2. P.[4] in which there exists atleast one K- and$N$-non minimal point[5]. But the examples of non minimal point in a domain of planer character have not been given. Mr. Ikegami proposed the following problem:

Does there exist a non minimal point on Riemann surfaces of planer character?

The purpose of the present paper is to discuss the relation between the classes of positive harmonic functions of some classes in R and R' when R varies to R' and also is to give examples of non minimal points of Riemann surfacs of planer character.

Lemma 1. (*An estimation of the harmonic measure of an arc on*

1) M. BRELOT: Sur le principe des singularités positives et la topologie de R. S. Martin. Ann. Univ. Grenoble 23, 113–138 (1948).

2) Z. KURAMOCHI: Relations among topologies on Riemann surfaces. I–IV. Proc. Japan. Acad. 38, 310–315, 457–472 (1953).

3) R. S. MARTIN: Minimal positive harmonic functions. Trans. Am. Math. Soc. 49, 137–172 (1941).

4) Z. KURAMOCHI: Examples of singular points J. Fac. Sci. Hokkaido Univ. 16, 149–187 (1962). H. 2. P. is the class of Riemann surfaces in which there exist two linearly independent positive harmonic functions.

5) Z. KURAMOCHI: Singular points of Riemann surfaces. J. Fac. Sci. Hokkaido Univ. 16, 80–148 (1962).

a curve). Let C be a unit circle, C: $|z|<1$ *and let* S' *be a straight in* C *with endpoints,* A' *and* B' *such that* S': $\operatorname{Im} z=0$, $a'\leqq \operatorname{Re} z\leqq b'$, *where* $a'\leqq -\frac{1}{3}$ *and* $b'\geqq \frac{1}{3}$. *Let* Γ *be a circle in* C: $|z|=\frac{1}{12}$. *Let* T *be a straight on* S' *such that* T: $\operatorname{Im} z=0$, $-\delta\leqq \operatorname{Re} z\leqq \delta$. *Suppose* $\delta\leqq \frac{\pi}{18}$. *Then* $w(T, z)$, H.M. *(harmonic measure) of* T *with respect to* $C-S'$ *satisfies*

$$w(T, z)\leqq \frac{0.664}{\pi}|\sin\theta|\ , \qquad z=\frac{e^{i\theta}}{12}.$$

Proof. Let S be a straight on the real axis, S: $\operatorname{Im} z=0$, $-\frac{1}{3}\leqq \operatorname{Re} z\leqq \frac{1}{3}$. Then $S'\supset S$. Let Ω_z be the complementary set of S in the z-plane. By brief consideration, when δ is sufficiently small, we see $w^*(T, z)$ has almost same value as $w(T, z)$ on Γ, where $w^*(T, z)$ is H.M. of T with respect to Ω_z. Clearly $w(T, z)\leqq w^*(T, z)$. Map Ω_z by $w=\dfrac{z+\frac{1}{3}}{z-\frac{1}{3}}$ onto Ω_w, where Ω_w is the complementary domain of the straight: $\operatorname{Im} w=0$, $0\leqq \operatorname{Re} w\leqq\infty$. Then $\Gamma\to$ a circle: $\left|w-\frac{6}{5}\right|<\frac{7}{15}$.

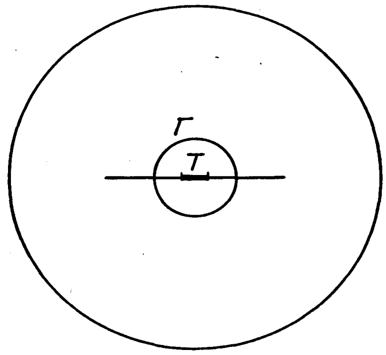

Fig. 1.

Map Ω_w by $\zeta=\sqrt{w}$ onto Ω_ζ: $\operatorname{Im}\zeta>0$. Also map Ω_ζ by $\xi=\frac{\zeta-i}{\zeta+i}$ onto $|\xi|<1$. Then T is mapped onto T_1+T_2 and Γ is mapped onto $\Gamma_1+\Gamma_2$ respectively, where T_1: $e^{i\left(-\frac{\pi}{2}+\varepsilon\right)}$ for $-\frac{3\delta}{2}\leqq\varepsilon\leqq\frac{3\delta}{2}$, T_2: $e^{i\left(\frac{\pi}{2}+\varepsilon\right)}$ for $-\frac{3\delta}{2}\leqq\varepsilon\leqq\frac{3\delta}{2}$, Γ_1 and Γ_2 are curves in the lower and upper semicircles respectively. Let $w^*(T_i, \xi)$ be H.M. of T_i. Then $w(T, z)\leqq\sum_{i=1}^{2} w^*(T_i, \xi)$. We consider $w^*(T_1, \xi)$ on Γ_1. Then $w(T, \xi)\leqq\dfrac{\Theta-\frac{3}{2}\delta}{\pi-\frac{3}{2}\delta}$,

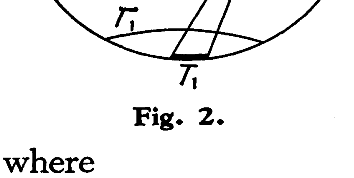

Fig. 2.

where

$$\Theta = \arg \frac{e^{i\left(-\frac{\pi}{2}+\frac{3\delta}{2}\right)}-\xi}{e^{i\left(-\frac{\pi}{2}-\frac{3\delta}{2}\right)}-\xi}\,. \tag{1}$$

We denote by p_ξ, the point ξ on Γ_1. We shall express p_ξ by z. Let $p_z = \frac{e^{i\theta}}{12}$ on Γ in the upper half plane. Then p_w, the image of p_z is given by Fig. 3 as

$$p_w = re^{i\varphi}\,,$$

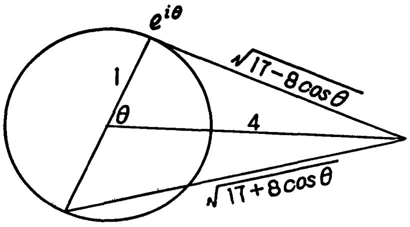

Fig. 3.

where $r=\sqrt{\dfrac{17+8\cos\theta}{17-8\cos\theta}}$ and

$\varphi=\cos^{-1}\dfrac{15}{\sqrt{17-8\cos\theta}}$.

Let p_ζ be the image of p_w. Then $\rho_\zeta=\rho e^{i\phi}$, where $\rho=r^{\frac{1}{2}}$, $\phi=\dfrac{\varphi}{2}$.

Let p_ξ be the image of p_ζ. Then $p_\xi = Re^{i\Phi}$, (2)

$$\text{where } R=\sqrt{\frac{1+\rho^2-2\rho\sin\phi}{1+\rho^2+2\rho\sin\phi}}\,,\quad \sin\Phi=\frac{2\rho\cos\varphi}{\sqrt{1+\rho^4+2\rho^2\cos 2\varphi}}\geqq 0\,,$$
$$\cos\Phi=\frac{\rho^2-1}{\sqrt{1+\rho^4+2\rho^2\cos 2\varphi}}\,. \tag{3}$$

By the shape of Γ_1 we see that R is minimal, when $\Phi=-\dfrac{\pi}{2}$, i.e. $\theta=\dfrac{\pi}{2}$,

$$\rho=1\,,\quad \cos\varphi=\frac{15}{17}\,,\quad \sin\phi=\frac{1}{\sqrt{17}}\quad \text{and}\quad R=\frac{\sqrt{17}-1}{4}\,. \tag{4}$$

Now by (1) $\Theta=\tan^{-1}\dfrac{2R\sin\Phi\sin\frac{3\delta}{2}+\sin 3\delta}{R^2+2R\sin\cos\frac{3\delta}{2}+\cos 3\delta}$. Put $\Psi=\Theta-\dfrac{3\delta}{2}$. Then

$$\tan\Psi=\frac{(1-R^2)\sin\frac{3}{2}\delta}{(1+R^2)\cos\frac{3}{2}\delta+2R\sin\Phi+\cos\frac{3}{2}\delta}\leqq\frac{(1-R^2)\sin\frac{3}{2}\delta}{(2+R^2)\cos\frac{3}{2}\delta}\,.$$

We have by (2) and (3) $(1-R^2)=\dfrac{2\sin\theta}{(17^2-8^2\cos^2\theta)^{\frac{1}{4}}\,((17^2-8^2\cos^2\theta)+15)^{\frac{1}{2}}}\leqq\dfrac{\sqrt{2}}{5}\sin\theta$. Hence by $\delta\leqq\dfrac{\pi}{18}$ and by (4)

$$\Psi \leqq \tan \Psi \leqq \frac{2\sin\theta\sin\frac{3}{2}\delta}{15(2+R^2)} \leqq \frac{8\sqrt{2}\sin\theta\sin\frac{3}{2}\delta}{15(25-\sqrt{17})\cos\frac{3}{2}\delta}$$
$$\leqq \frac{12\sqrt{2}\,\delta\sin\theta}{15(25-\sqrt{17})\cos\frac{\pi}{12}} < 0.28\delta\sin\theta\,. \tag{5}$$

Hence $w^*(T,\xi) \leqq \frac{0.332}{\pi}\delta\sin\theta$, $z=\frac{e^{i\theta}}{12}$ on Γ_1. Next clearly

$w(T_2, z) \leqq w^*(T,\xi)$ on Γ_1, whence

$$w(T,z) \leqq \frac{0.664\delta\sin\theta}{\pi}, \qquad z=\frac{e^{i\theta}}{12}, \qquad \pi \geqq \theta \geqq 0\,. \tag{6}$$

Similar result is obtained for $\pi \leqq \theta \leqq 2\pi$. Thus we have Lemma 1.

Lemma 2. (*An estimation of harmonic measure of an arc*). *Let Ω be the semicircle*: $|z| \leqq 1$, $\operatorname{Im} z \geqq 0$. *Let S be an arc on $|z|=1$ with endpoints A and B*, $A=1-\sqrt{2}+2\sqrt{2}\sqrt{\sqrt{2}-1}\,i$ *and* $B=\sqrt{2}-1+2\sqrt{2}\sqrt{\sqrt{2}-1}\,i$. *Let Γ be a semicircle*: $z=\frac{e^{i\theta}}{2}$, $0 \leqq \theta \leqq \pi$. *Let $w(S,z)$ be H.M. of S with respect to Ω. Then*

$$w(S,z) \geqq \frac{2}{3\pi}\min\left(\frac{\pi}{12},\ \frac{24}{25}\sin\theta\right) \quad \text{on} \quad \Gamma\,.$$

Map Ω by $w=\frac{z+1}{1-z}$ onto Ω_w, map Ω_w by $\zeta=w^2$ onto Ω_ζ and map Ω_ζ by $\xi=\frac{\zeta-i}{\zeta+i}$ onto $|\xi|<1$. Then A and B are mapped onto $e^{\frac{3}{4}\pi i}$ and $e^{\frac{\pi}{4}i}$ of ξ-plane respectively. Let $p=\frac{e^{i\theta}}{2}$: $\pi \geqq \theta \geqq 0$ and let p_w, p_ζ and p_ξ be the images of p in w, ζ and ξ-plane. Then we have $p_w = re^{i\varphi}$, where

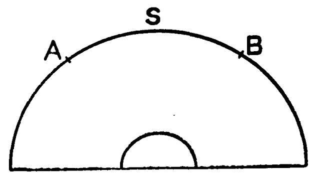

Fig. 4.

$$\cos\varphi = \frac{3}{\sqrt{25-16\cos^2\theta}}\cdot \quad r=\sqrt{\frac{5+4\cos\theta}{5-4\cos\theta}}\,,$$

$p_\zeta = \rho e^{i\phi}$,

where $\rho=r^2$ and $\phi=2\varphi$, $\sin\phi=\frac{24\sin\theta}{25-16\cos^2\theta} \geqq \frac{24\sin\theta}{25}$ and $\frac{1}{9} \leqq \rho \leqq 9$. (7)

$p_\xi = \mathrm{R}e^{i\Phi}$,

$$\text{where } R=\sqrt{\frac{1+\rho^2-2\rho\sin\phi}{1+\rho^2+2\rho\sin\varphi}}, \quad \sin\Phi = \frac{2\cos\varphi}{\sqrt{1+\rho^4+2\rho^2\cos^2\varphi}} \geqq 0 \tag{8}$$

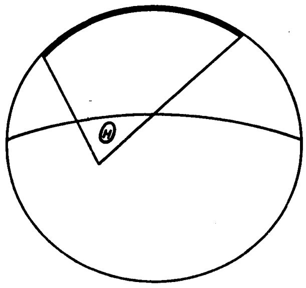

Fig. 5.

and R is minimal, when $\theta=0$, i.e. $r=1$, $\rho=1$, $\sin\phi=\frac{24}{25}$ and $R=\frac{1}{7}$.

Put $\Theta=\arg\frac{e^{\frac{3\pi}{4}i}-p_\xi}{e^{\frac{\pi}{4}i}-p_\xi}$. Then by (8) $\Theta=\left(1+\frac{1-R^2}{R^2-\sqrt{2}R\sin\Phi}\right)$. On the other hand, we see easily $\tan\left(\frac{\pi}{4}+\frac{s}{4}\right)\leqq 1+s$ for $0\leqq s\leqq\frac{\pi}{3}$, whence $\Theta\geqq\frac{\pi}{4}+\min\left(\frac{\pi}{12},\ \frac{1-R^2}{4(R^2-\sqrt{2}R\sin\Phi)}\right)$.

By (8) and (7) $(1-R^2)=\frac{4\rho\sin\phi}{1+\rho^2+2\rho\sin\phi}\geqq\frac{4\rho\sin\phi}{(1+\rho)^2}\geqq\frac{48}{25}\sin\theta$, becacse $\frac{4\rho}{1+\rho^2}\leqq 1$ for $\frac{1}{9}\leqq\rho\leqq 9$.

Also by (8) $\Theta\geqq\frac{\pi}{4}+\min\left(\frac{\pi}{12},\ \frac{24}{25}\sin\theta\right)$. Hence

$$w(T,z)\geqq\frac{\Theta-\frac{\pi}{4}}{\frac{3}{4}\pi}=\frac{4}{3\pi}\min\left(\frac{\pi}{12},\ \frac{24\sin\theta}{25}\right),\qquad z=\frac{e^{i\theta}}{2}.$$

Lemma 3. *Let C, C', C'' and C''' be circles, C: $|z|<1$, C': $|z|<\frac{1}{3}$, C'': $|z|<\frac{1}{6}$, C''': $|z|<\frac{1}{12}$. Let Γ_1 and Γ_2 be straights on the real axis such that Γ_1: $a\leqq Re\,z\leqq-\delta$, Γ_2: $-\delta\leqq Re\,z\leqq b$, where $a\leqq-\frac{1}{3}$, $b\geqq-\frac{1}{3}$ and let γ be a closed set in $C-C'$. Let T: $\mathrm{Im}\,z=0$, $-\delta\leqq Re\,z\leqq\delta$ and let $U(z)$ be a positive harmonic function in $C'-\Gamma_1-\Gamma_2$ vanishing on $\Gamma_1+\Gamma_2$ and let $w(T,z)$ be H.M. of T with respect to $C-T$. Then there exists a constant M such that*

$$M\delta U(z)\geqq(\sup_{z\in\partial C'''}U(z))\,w(T,z)\quad\text{on }\partial C'''.$$

Let $F=F_1+F_2$ and F_1: $\frac{e^{i\theta}}{6}$, $\frac{3\pi}{4}\geqq\theta\geqq\frac{\pi}{4}$, F_2: $\frac{e^{i\theta}}{6}$, $\frac{7\pi}{4}\geqq\theta\geqq\frac{5\pi}{4}$. Map $C'-\Gamma_1-\Gamma_2$ onto $|\xi|<1$ so that $z=0\to\xi=0$ and let Γ_ξ, F_ξ and C'''_ξ be the images of $\Gamma_1+\Gamma_2$, F_1+F_2 and C''' respectively. Let $C\Gamma_\xi$

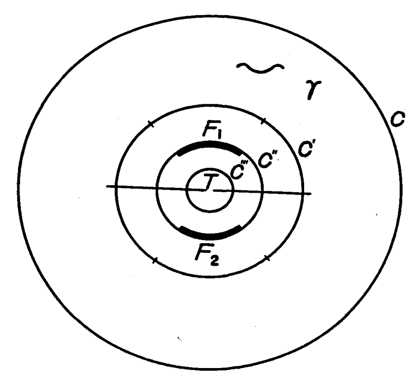

Fig. 6.

be the complementary periphery of Γ_ξ. Since $\mathrm{dist}(C\Gamma_\xi, F_\xi)>0$, there exists a const. K_1 such that $\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\geqq K_1$ for $\xi\in F_\xi$, $\xi=re^{i\theta}$, whence $\min_{z\in F_1+F_2} U(z)=\min_{\xi\in F_\xi} U(\xi)\geqq K_1U_\xi(0)$. Also since $\mathrm{dist}(\partial C_\xi''', C\Gamma_\xi)=0$, there exists a const. K such that $\frac{1}{1-2r\cos(\theta-\varphi)+r^2}\leqq K_2$ for $\xi\in\partial C_\xi'''$, whence $\sup_{\xi\in\partial C_\xi'''} U(\xi)\leqq K_2(1-r^2)U_\xi(0)$. Hence putting $M'=\frac{K_1}{K_2}$ we have $M'\min_{z\in F_1+F_2} U(z)\geqq \sup_{z\in\partial C'''} U(z)$. Put $A=\sup_{z\in\partial C'''} U(z)$ and $B=\min_{z\in F_1+F_2} U(z)$. Then $B\geqq\frac{A}{M'}$. Clearly $U(z)\geqq Bw(F, z)$ in C'' and by Lemma 2 $U(z)\geqq BC_2\sin\theta$ for $z=\frac{e^{i\theta}}{12}$, where $w(F, z)$ is H.M. of F_1+F_2 with respect to $C''-\Gamma_1-\Gamma_2-T$ and $C_2=\frac{4}{3\pi}\min\left(\frac{\pi}{12}, \frac{24}{25}\right)$. By Lemma 1, $w(T, z)\leqq C_1\delta\sin\theta$ for $z=\frac{e^{i\theta}}{12}$, where $w(T, z)$ is H.M. of T with respect to C and $C_1=\frac{0.664}{\pi}$. Hence $U(z)\geqq BC_2\sin\theta\geqq BC_2\frac{w(T, z)}{C_1\delta}\geqq\frac{AC_2}{M'C_1\delta}w(T, z)\frac{C_2}{M'C_1\delta}(\sup_{z\in\partial C'''} U(z))w(T, z)$ for $z=\frac{e^{i\theta}}{12}$. Put $M=\frac{M'C_1}{C_2}$. Then

$$M\delta U(z)\geqq(\sup_{z\in\partial C'''} U(z))w(T, z), \quad \text{for} \quad z=\frac{e^{i\theta}}{12}.$$

Let R be a Riemann surface with positive boundary and let $\{R_n\}$ be its exhaustion with compact relative boundary ∂R_n $(n=1, 2, \cdots)$. Let G be a subdomain (in this paper we suppose the relative boundary ∂G of G consists of enumerably infinite number of analytic curves clustering nowhere in R). Let P.H.(G) and $\mathring{\text{P}}$.H.(G) be the sets of positive harmonic function (is abbreviated to P.H.) in G and P.H. in G vanishing on ∂G. Let $U(z)\in\mathring{\text{P}}$.H.$(G)$ and let $U_n(z)$ be the least positive harmonic function (is abbreviated to L.P.H.) in $R-(G\cap(R-R_n))$ such that $U_n(z)=U(z)$ on $G\cap(R-R_n)$. Then $U_n(z)\uparrow$ a limit function denoted by $EU(z)$. Let $V(z)\in$P.H.(R) and let $V_n(z)$ be the L.P.H. in $G-(R-R_n)$ such that $V_n(z)\geqq V(z)$ on $G\cap(R-R_n)$. Then $V_n(z)\downarrow IV(z)$. Then we have the following

Theorem 1. a). *If* $EU(z)<\infty$, $IEU(z)=U(z)$[6].

b). *Let* $V(z)\in$P.H.(R). *Let D be a domain in R and let* $V_D(z)$ *be* L.P.H. *in $R-D$ such that $V_D(z)=V(z)$ on D. Let $U(z)\in\mathring{P}.H.(G)$ and $EU(z)<\infty$. Then* $\lim_n {}_{CG\cap(R-R_n)}EU(z)=0$.

6) Z. KURAMOCHI: Relations between harmonic dimensions. Proc. Japan Acad. 30, 576–580 (1954).

This means $EU(z)$ *tends to zero as* $z\to B\cap CG$ ($B\cap CG$ *means the ideal boundary determined by a domain* CG) *except a set of harmonic measure zero*).

Let $V(z)\in$ P.H. (R). If $\lim_n V_{G\cap(\hat{R}-R_n)}(z)=V(z)$, we say $V(z)=0$ a.e. on $(B\cap CG)$. Next let $w(z)$ be L.P.H. in G such that $w(z)\geqq V(z)$ on ∂G. Let $\tilde{w}_n(z)$ be L.P.H. in $R-(G\cap(R-R_n))$ such that $\tilde{w}_n(z)=w(z)$ on $G\cap(R-R_n)$. If $\lim_n \tilde{w}_n(z)=0$, we say $V(z)$ is *regular relative to* G. Then we have

c). *Let* $V(z)\in P.H.(R)$. *Suppose* $V(z)=0$ *a.e. on* $CG\cap B$ *and* $V(z)$ *is regular relative to* G. *Then if* $IV(z)>0$, *then* $EIV(z)=V(z)$.

d). *If* $EU(z)<\infty$, *then* $EU(z)$ *is regular relative to* G.

e). *Let* $V(z)\in P.H.(R)$. *If there exists at least one* $U(z)$[7] *in* $\mathring{P}.H.(\mathfrak{G})$ *such that* $V(z)\leqq EU(z)$ *and if* $IV(z)>0$, *then* $EIV(z)=V(z)$.

Proof of a) is given in the previous paper[8].

Proof of b). Let $\check{V}_{n,n+i}(z)$ be a P.H. in $R_n+((R_{n+i}-R_n)\cap G)$ such that $\check{V}_{n,n+i}(z)=EU(z)$ on $\partial R_{n+i}\cap G$, $\check{V}_{n,n+i}(z)=0$ on $(\partial R_n-G)+(\partial G\cap(R_{n+i}-R_n))$. Let $\hat{V}_{n,n+i}(z)$ be a P.H. in $R_n+((R_{n+i}-R_n)\cap G)$ such that $\hat{V}_{n,n+i}(z)=EU(z)$ on $(\partial R_n\cap CG)+(\partial G\cap(R_{n+i}-R_n))$ and $\hat{V}_{n,n+i}(z)=0$ on $\partial R_{n+i}\cap G$. Then

$$\hat{V}_{n,n+i}(z)+\check{V}_{n,n+i}(z)=EU(z)\,.$$

where $G^{\sum_{n_0}^{\infty}\gamma_n}(z,p)$ is the Green's function of R', because $R''-\sum_{n_0}^{\infty}\gamma_n=R+\sum_{1}^{n_0-1}\gamma_n=R'$.

In the following we suppose K-Martin's topologies are defined in R'' and R' by using $K''(z,p)$ and $K'(z,p)$, where $K''(z,p)=\dfrac{G''(z,p)}{G''(z_0,p)}$, $K'(z,p)=\dfrac{G'(z,p)}{G'(z_0,p)}$ and z_0 is a fixed point in $R-\sum^{\infty}C_n$. Put $\bar{R}'=R'+B'$ and $\bar{R}''=R''+B''$, where B' and B'' are ideal boundaries of R' and R'' respectively.

Let $\{p_i\}$ be a sequence determining a point $p'\in B'$ such that $p_i\notin\sum^{\infty}C_n$. Then by (10)

$$\frac{1}{2}K''(z,p_i)\leqq K'(z,p_i)\leqq 2K''(z,p_i)\qquad\text{for}\quad z\notin\sum^{\infty}C_n''.\qquad(11)$$

Then we can find a subsequence $\{p_i'\}$ of $\{p_i\}$ such that $\{p_i'\}$ converges to a point $p''\notin B''$. Then

$$\frac{1}{2}K''(z,p'')\leqq K'(z,p')\leqq 2K''(z,p'')\qquad\text{for}\quad z\notin\sum^{\infty}C_n',\quad(12)$$

Proof of d). Let $U(z) \in \mathring{P}.H.(G)^{r}$ and $EU(z) < \infty$. Let $w_n(z)$ be a P.H. in $G \cap R_n$ such that $w_n(z)=0$ on $\partial R_n \cap G$, $w_n(z)=EU(z)$ on $\partial G \cap R_n$. Then $w_n(z)+U(z) \leqq EU(z)$ on $(\partial G \cap R_n)+(\partial R_n \cap G)$. Let $n \to \infty$. Then $\lim_n w_n(z)=w(z)$ is L.P.H. in G such that $w(z) \geqq EU(z)$ on ∂G and $w(z)+U(z) \leqq EU(z)$ in G. Let $\tilde{w}_{n,n+i}(z)$ be a P.H. in $R_{n+i}-(G \cap (R_{n+i}-R_n))$ such that $\tilde{w}_{n,n+i}(z)=w(z)$ on $\partial R_n \cap G$, $\tilde{w}_{n,n+i}(z)=0$ on $(\partial G \cap (R_{n+i}-R_n))+\partial R_{n+i}-G$. Let $T_{n,n+i}(z)$ be a P.H. in $R_{n+i}-(G \cap (R_{n+i}-R_n))$ such that $T_{n,n+i}(z)=U(z)$ on $\partial R_n \cap G$, $T_{n,n+i}(z)=0$ on $(\partial G \cap (R-R_n))+(\partial R_{n+i}-G)$. Then $\tilde{w}_{n,n+i}(z)+T_{n,n+i}(z) \leqq w(z)+U(z)$ on $(\partial R_{n+i} \cap G)+(\partial G \cap (R_{n+i}-R_n))$, $\tilde{w}_{n,n+i}(z)+T_{n,n+i}(z) \leqq EU(z)$. Let $i \to \infty$. Then $\tilde{w}_{n,n+i}(z) \to \tilde{w}_n(z)$ and $T_{n,n+i}(z) \to T_n(z)$. Now $T_n(z)$ is L.P.H. in $R-((R-R_n) \cap G)$ such that $T_n(z) \geqq U(z)$ on $\partial G \cap R_n$, whence $\lim_n T_n(z)=EU(z)$. Also $\tilde{w}_n(z)$ is L.P.H. in $R-((R-R_n) \cap G)$ such that $\tilde{w}_n(z) \geqq w(z)$ on $\partial G \cap R_n$. Let $n \to \infty$. Then $\lim_n \tilde{w}_n(z)+EU(z) \leqq EU(z)$ and $\lim_n \tilde{w}_n(z)=0$. Whence $EU(z)$ is regular relative to G.

Proof of e). By $V(z) \leqq EU(z)$, $\lim_n V_{CG \cap (R-R_n)}(z)=0$ by (b). Clearly

$$\lim_n V_{CG \cap (R-R_n)}(z)+\lim_n V_{G \cap (R-R_n)}(z) \geqq \lim_n V_{R-R_n}(z) = V(z) \geqq \lim_n V_{G \cap (R-R_n)}(z) .$$

Hence $V(z)=\lim_n V_{G \cap (R-R_n)}(z)$, i.e. $V(z)=0$ a.e. on $CG \cap B$. By $V(z) \leqq EU(z)$ and by (d) $V(z)$ is regular relative to G. Hence by (c) we have (e).

We denote by $\overset{*}{P}.H.(R)$ the subclass of $V(z)$ in P.H.(R) such that $V(z)=0$ a.e. on $CG \cap B$, $V(z)$ is regular relative to G and $IV(z)>0$ and let $\overset{*}{P}.H.(G)$ be the subclass of $U(z)$ in P.H.(G) such that $EU(z)<\infty$. Then by a) and c) we have at once the following

Corollarly. *$\overset{*}{P}.H.(R)$ and $\overset{*}{P}.H.(G)$ are isomorphic with respect to E and I operations.*

Let $G \supset G'$ be subdomains and let Γ and Γ' be sets consisting of arcs in ∂G and $\partial G'$ respectively, where $\Gamma' \subset \Gamma$. Let $\mathring{P}.H.^{\Gamma}(G)$ be the set of P.H.'s $U(z)$ in G such that $U(z)=0$ on $\partial G-\Gamma$, $\frac{\partial}{\partial n}U(z)=0$ on Γ. Let $U_n(z)$ be L.P.H. in G' such that $U_n(z) \geqq U(z)$ on $G' \cap (R-R_n)$, $\frac{\partial}{\partial n}U_n(z)=0$ on Γ. Put $\overset{N}{E}U(z)=\lim_n U_n(z)$. Similarly we define $\overset{N}{I}V(z)$ from $V(z)$ in P.H.$^{\Gamma}(G)$. Let $\hat{G}$ be the symmetric surface of G with respect to ∂G and identify Γ and $\hat{\Gamma}$, the symmetric image of Γ. Then we have a doubled surface $G+\hat{G}$ which has its relative boundary $(\partial G-\Gamma)+(\partial\hat{G}-\hat{\Gamma})$, where $\hat{G}$ is the symmetric image of G. Let $\hat{z}$ be the symmetric image of z and put $U(\hat{z})=U(z)$ in $\hat{G}$. Then $\tilde{U}(z)$ $(=U(z)$ in G and $=U(\hat{z})$ in $\hat{G}) \in$ P.H.$(G+\hat{G})$. Let $\hat{G}'$ to the symmetric image of G' with respect to ∂G and identify Γ' and $\hat{\Gamma}'$. Then we have $(G+\hat{G}')$. Now $(G'+\hat{G}')$ is contained in $(G+\hat{G})$ by $G'\subset G$ and $\Gamma'\subset\Gamma$, $(G'+\hat{G}')$ has relative boundary $(\partial G'-\Gamma'+\partial\hat{G}'-\hat{\Gamma}')$ in $(G+\hat{G})$. Hence Theorem 1 is valid for $\overset{N}{E}$ and $\overset{N}{I}$. To avoid repitition we do not state the Theorem 1 for $\overset{N}{E}$ and $\overset{N}{I}$ and we denote it simply by N-Theorem 1.

Lemma 4. *Let R^* be a Riemann surface with positive boundary. Let R be a subsurface in R^* and let $\sum_{n=1}^{n_0}\gamma_n$ be a compact set on ∂R, which is compact in R^*. Put $R'=R+\sum_{n=1}^{n_0}\gamma_n$. Then $\mathring{P}.H.(R)$ and $\mathring{P}.H.(R')$ are isomorphic.*

In fact, let $U(z)\in\mathring{P}.H.(R)$. Since $\sum^{n_0}\gamma_n$ is compact, there exists a number n' such that $\sum^{n_0}\gamma_n\subset R^*_{n'}$, where R^*_n $(n=1,2,\cdots)$ is an exhaustion of R^* with compact relative boundary ∂R^*_n. Put $M=\sup_{z\in R^*_{n'}}U(z)$. Then $M<\infty$. Let $w(\sum^{n_0}\gamma_n, z, R)$ be H.M. of $\sum^{n_0}\gamma_n$. Then $U(z)+Mw(\sum^{n_0}\gamma_n, z, R)$ is superharmonic in R' and $\geqq U(z)$, whence $EU(z)\leqq U(z)+Mw(\sum^{n_0}\gamma_n, z, R)<\infty$, where E is from R to R'. Let $V(z)\in\mathring{P}.H.(R')$. Let $V'(z)$ be L.P.H. in R such that $V'(z)\geqq V(z)$ on $\sum^{n_0}\gamma_n$. Then $V'(z)-V(z)>0$, because if $V(z)=V'(z)$, then $V'(z)$ is harmonic in R' and by the maximum principle $V'(z)=V(z)=0$. Whence $IV(z)\geqq V(z)-V'(z)>0$, where I is from R' to R. Next we see easily that any $V(z)$ is regular relative to R and $V(z)=0$ a.e. on $(CR\cap B)$ by $R\cap(R^*-R^*_n)=R'\cap(R^*-R^*_n)$ for $n>n'$, where $R^*_{n'}\supset\sum^{n_0}\gamma_n$.

Theorem 2. *Let R^* be a Riemann surface with positive boundary. Let R be a sub Riemann surface of R^* with relative boundary ∂R in R^*. Let $C_n\supset C'_n\supset C''_n$ $(n=1,2,\cdots)$ be discs in R^* such that $\partial C_n+\partial C'_n$ and $\partial C''_n$ may intersect ∂R and $C'_{n,s}$ are disjoint each other. Let γ_n be a continuum in $\partial R\cap C''_n$. Suppose there exists a const. M_n such that*

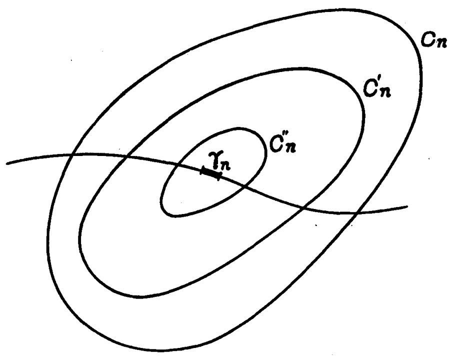

Fig. 7.

$$(\sup_{z\in\partial C'_n}U(z))\,w(\gamma_n, z)\leqq M_nU(z) \quad on \quad \partial C''_n$$

for any P.H. $U(z)$ in $C_n\cap R^{+\gamma_n}$ vanishing on $\partial R,$ $-\gamma_n$; where $w(\gamma_n, z)$ is H.M. of γ_n with respect to $R''=R+\sum_{n=1}^{\infty}\gamma_n$.

If $$\sum_{n=1}^{\infty}M_n<\infty,$$

then $\mathring{P}.H.(R)$ and $\mathring{P}.H.(R'')$ are isomorphic, where R'' is the surface obtained from R by *adding* $\sum^{\infty}\gamma_n$ *to* R, *i.e.* $R''-R=\sum^{\infty}\gamma_n$ *and* $\partial R''=\partial R-\sum\gamma_n$.

Proof. Let n_0 be a number such that $\prod_{n_0}^{\infty}(1-2M_n)\geqq\frac{1}{4}$. Let $R'=R+\sum^{n_0-1}\gamma_n$ and $R''=R+\sum^{\infty}\gamma_n$. Then $R\subset R'\subset R''$. Since $R'-R=\sum^{n_0-1}\gamma_n$ is compact, $\mathring{\mathrm{P}}.\mathrm{H}.(R)$ and $\mathring{\mathrm{P}}.\mathrm{H}.(R')$ are isomorphic by Lemma 4. We shall prove $\mathrm{P}^{\circ}.\mathrm{H}.(R')$ and $\mathrm{P}^{\circ}.\mathrm{H}.(R'')$ are isomorphic. Let $G''(z,p)$ be the Green's function of R''. Suppose $p\notin\sum C_n$. Let $M=\max_{z\in\partial C_n''}G''(z,p)$. Consider $G''(z,p)-2Mw(\gamma_n,z)$ in C_n''. Since $G''(z,p)$ is a P.H. in $C_n\cap R''$ vanishing on $(\partial R-\gamma_n)\cap C_n$

$$G''(z,p)-2Mw(\gamma_n,z)\geqq\frac{M}{M_n}w(\gamma_n,z)-2Mw(\gamma_n,z)$$
$$=\frac{M}{M_n}(1-2M_n)w(\gamma_n,z)>0 \qquad \text{on} \quad \partial C_n''.$$

Put $\tilde{U}(z)=G''(z,p)-2Mw(\gamma_n,z)$ for $z\in C_n''$ and $\tilde{U}(z)=\max(G''(z,p)-2Mw(\gamma_n,z),0)$ in C_n''. Since $G''(z,p)\leqq M$ and $w(\gamma_n,z)=1$ on γ_n, the open set $\mathfrak{G}=E[z: G''(z,p)-2Mw(\gamma_n,z)<0]$ contains γ_n and does not tend to $\partial C_n''$, whence $\tilde{U}(z)$ is continuous in $R''-\gamma_n$ except endpoints of γ_n and p, $\tilde{U}(z)$ is subharmonic except p and $\tilde{U}(z)=0$ on γ_n. Let $G^{\gamma_n}(z,p)$ be the Green's function of $R''-\gamma_n$. Then $G^{\gamma_n}(z,p)$ is L.P.H. except p where $G^{\gamma_n}(z,p)$ has a logarithmic singularity. Hence

$$G''(z,p)\geqq G^{\gamma_n}(z,p)\geqq G''(z,p)-2Mw(\gamma_n,z).$$

Also $G''(z,p)\geqq\frac{M}{M_n}w(\gamma_n,z)$ on $\partial C_n''$. By the maximum principle $2Mw(\gamma_n,z)\leqq 2M_nG''(z,p)$ for $z\sim C_n''$. Hence

$$G''(z,p)\geqq G^{\gamma_n}(z,p)\geqq G''(z,p)(1-2M_n) \qquad \text{for} \quad z\notin C_n''.$$

Replace $G''(z,p)$ and $G^{\gamma_n}(z,p)$ by $G^{\gamma_n}(z,p)$ and $G^{\gamma_n+\gamma_{n+1}}(z,p)$, where $G^{\gamma_n+\gamma_{n+1}}(z,p)$ is the Green's function of $R''-\gamma_n-\gamma_{n+1}$. Then as above

$$\begin{aligned}G''(z,p)\geqq G^{\gamma_n}(z,p)\geqq G^{\gamma_n+\gamma_{n+1}}(z,p)\geqq G^{\gamma_n}(z,p)(1-2M_{n+1})\\ \geqq G''(z,p)(1-2M_n)(1-2M_{n+1}) \qquad \text{for} \quad z\notin C_n''+C_{n+1}''.\end{aligned}$$

In this way we have

$$G''(z,p)\geqq G^{\sum_{n_0}^{\infty}\gamma_n}(z,p)\geqq G''(z,p)\prod_{n_0}^{\infty}(1-2M_n)\geqq\frac{G''(z,p)}{4} \qquad \text{for} \quad z\notin\sum_{n_0}^{\infty}C_n'', \tag{10}$$

where $G^{\sum_{n_0}^{\infty}\gamma_n}(z,p)$ is the Green's function of R', because $R''-\sum_{n_0}^{\infty}\gamma_n=R+\sum_{1}^{n_0-1}\gamma_n=R'$.

In the following we suppose K-Martin's topologies are defined in R'' and R' by using $K''(z,p)$ and $K'(z,p)$, where $K''(z,p)=\dfrac{G''(z,p)}{G''(z_0,p)}$, $K'(z,p)=\dfrac{G'(z,p)}{G'(z_0,p)}$ and z_0 is a fixed point in $R-\sum^{\infty}C_n$. Put $\bar{R}'=R'+B'$ and $\bar{R}''=R''+B''$, where B' and B'' are ideal boundaries of R' and R'' respectively.

Let $\{p_i\}$ be a sequence determining a point $p'\in B'$ such that $p_i\notin\sum^{\infty}C_n$. Then by (10)

$$\frac{1}{2}K''(z,p_i)\leqq K'(z,p_i)\leqq 2K''(z,p_i)\qquad\text{for}\quad z\notin\sum^{\infty}C''_n. \tag{11}$$

Then we can find a subsequence $\{p'_i\}$ of $\{p_i\}$ such that $\{p'_i\}$ converges to a point $p''\notin B''$. Then

$$\frac{1}{2}K''(z,p'')\leqq K'(z,p')\leqq 2K''(z,p'')\qquad\text{for}\quad z\notin\sum^{\infty}C'_n, \tag{12}$$

i.e. there exists at least one point $p''\in B''$ corresponding to any $p'\in B'$.

Suppose $p_i\in C_m$. Then $K'_{C_m}(z,p_i)=K'(z,p_i)$ for $z\notin C_m$ and $K'_{C_m}(z,p)$ is representable by a positive mass distribution $\mu_{p_i}(q)$ on ∂C_m such that $\int d\mu_{p_i}(q)=1$ by $K'(z_0,p_i)=1$ and $K'_{C_m}(z,p_i)=\int_{\partial C_m}K(z,p)d\mu_{p_i}(q)$ for $z\notin C_m$, where $K'_{C_m}(z,p)$ is L.P.H. in R' larger than $K'(z_i,p)$ on C_m. Now since $q\notin C_m$, by (10) $\frac{1}{2}K''(z,q)\leqq K'(z,q)\leqq 2K''(z,q)$, whence

$$\frac{1}{2}\int K''(z,q)d\mu_{p_i}(q)\leqq K'(z,p_i)=\int K'(z,q)d\mu_{p_i}(q)\leqq 2\int K''(z,q)d\mu_{p_i}(q)\qquad\text{for}\quad z\notin C_m. \tag{13}$$

Let $\{p_i\}$ be a sequence in R' determining a point $p'\in B'$. Then by (11) and (13) we can find an weak limit $\mu_{p'}(q)$ on B' of $\{\mu_{p_i}(q)\}$ such that

$$\frac{1}{2}\int K(z,q)d\mu_{p'}(q)\leqq K'(z,p')\leqq 2\int K''(z,q)d\mu_{p'}(q)\quad\text{for}\ z\notin\sum C''_n. \tag{14}$$

Let $U(z)\in\mathring{\mathrm{P}}.\mathrm{H}.(R')$ and $U(z_0)=1$. Then $U(z)$ is representable by a positive mass μ on B' such that $\mu=0$ on $\partial R'=\partial R''+\sum\gamma_n$, $\int d\mu=1$ and

$$U(z)=\int_{B'}K(z,p')d\mu(p').$$

Hence by (14) there exists a function $V(z) \in \mathring{P}.H.(R'')$ such that

$$\frac{1}{2}V(z) \leqq U(z) \leqq 2V(z) \qquad \text{for} \quad z \notin \sum C''_n, \qquad (15)$$

where

$$V(z) = \Big(\int_{B''} K''(z, q) d\mu_{p'}(q)\Big) d\mu(p') \quad \text{and} \quad V(z_0) = 1 .$$

Similarly for any $V(z) \in \mathring{P}.H.(R'')$, $V(z_0)=1$ we can find a $U(z) \in P_0.H.(R')$, $U(z_0)=1$ such that

$$\frac{1}{2}U(z) \leqq V(z) \leqq 2U(z) \qquad \text{for} \quad z \notin \sum C_n . \qquad (16)$$

Proof of the theorem. As for non constant positive function $A(z)$, we can suppose without loss of generality $A(z_0)=1$. To define E (from R' to R'') and I (from R'' to R') operations we can use decreasing sequence $\{v_n\}$: $v_n = (R^* - R_n^* - \sum_{m=n_0}^{\infty} C_m)$ instead of $(R^* - R_n^*)$ by the maximum principle, where $\{R_n^*\}$ is an exhaustion of R^*. For example $EU(z) = \lim_n U_n(z)$ for $U(z) \in \mathring{P}.H.(R')$, where $U_n(z)$ is L.P.H. in $R' - v_n$ such that $U_n(z) \geqq U(z)$ on v_n. $IV(z)$ is defined similarly.

Let $U(z) \in P.H.(R')$. Then by (15) there exists a $V(z) \in \mathring{P}.H.(R'')$ such that $U(z) \leqq 2V(z)$ on v_n. Hence

$$EU(z) \leqq 2V(z) < \infty . \qquad \text{(a)}$$

Let $V(z) \in \mathring{P}_0.H.(R'')$. Then by (16) there exists an $U(z) \in \mathring{P}.H.(R')$ such that

$$\frac{U(z)}{2} \leqq V(z) \leqq U(z) \qquad \text{on} \quad v_n ,$$

hence

$$0 < \frac{U(z)}{2} \leqq IV(z) . \qquad \text{(b, 1)}$$

Also by (15) there exists another $\tilde{V}(z) \in \mathring{P}.H.(R'')$ depending on $U(z)$ such that $2\tilde{V}(z) \geqq U(z)$. Hence $V(z) \leqq 2U(z) \leqq 4\tilde{V}(z)$, whence $V(z) \leqq 2EU(z) \leqq 4\tilde{V}(z) < \infty$ and

$$V(z) \leqq 2EU(z) . \qquad \text{(b, 2)}$$

Thus the conditions, a), b, 1) and b, 2) are verified. Hence by the corollarly of Theorem 1 $\mathring{P}.H.(R')$ and $\mathring{P}.H.(R'')$ are isomorphic and $\mathring{P}.H.(R)$ and $\mathring{P}.H.(R'')$ are isomorphic.

Corollarly. *Under the condition of Theorem 2 any positive minimal harmonic function in R'' vanishing on $\partial R''$ is the image of a uniquely determined minimal function in R vanishing on ∂R by the operation E and* *conversely any minimal function in R vanishing on ∂R is the image of a uniquely determined minimal function in R'' vanishing on $\partial R''$ and the correspondence is one-to-one manner.*

Example 1. Let R^* be a unit circle, R^*: $|z|<1$ and let I be a straight on the real axis, I: $\operatorname{Im} z=0$, $0\leqq \operatorname{Re} z\leqq 1$. Put $R=R^*-I$. Let C_n $(n=1, 2, \cdots)$ be a hyperbolic circle with centre at q_n: $1-\frac{1}{4^n}$ and with hyperbolic radius $\frac{1}{3}$, i.e. C_n: $\left|\dfrac{z-1+\frac{1}{4^n}}{1-\left(1-\frac{1}{4^n}\right)z}\right|<\frac{1}{3}$. Then C_n intersects I at $1-\dfrac{\frac{4}{3}}{4^n\left(\frac{2}{3}+\frac{1}{3\times 4^n}\right)}$ and $1-\dfrac{\frac{2}{3}}{4^n\left(\frac{4}{3}-\frac{1}{3\times 4^n}\right)}$ and $\{C'_{n,s}\}$ are disjoint each other.

Let s_n be a straight on I of hyperbolic length $2\delta_n<\frac{1}{12}$ with its middle point at q_n, s_n: $\operatorname{Im} z=0$, $1-\dfrac{1+\delta_n}{4^n\left(1-\delta_n+\frac{\delta_n}{4^n}\right)}\leqq \operatorname{Re} z\leqq 1-\dfrac{1-\delta_n}{4^n\left(1+\delta_n-\frac{\delta_n}{4^n}\right)}$. Let $R''=R+\sum_{n=1}^{\infty} s_n$. Suppose K-Martin's topology is defined in R'' and

$$\sum_{n=1}^{\infty}\delta_n<\infty .$$

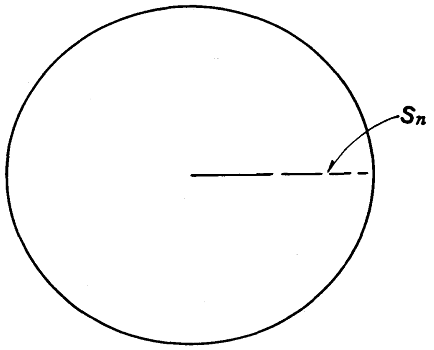

Fig. 8.

Then R'' has the following properties.

1). *There exist only two minimal points p^U and p^L on $z=1$.*

2). *Let $\{p_i^U\}$ $(i=1, 2, \cdots)$ $(\{p_i^L\})$ be a sequence in $R''-\sum_{n=1}^{\infty} C_n$ tending to $z=0$ such that the imaginary part of p_i^U, $\operatorname{Im} p_i^U>0$ $(\operatorname{Im} p_i^L<0)$. Then $K''(z, p_i^U)\to K''(z, p^U)$ and $K''(z, p_i^L)\to K''(z, p^L)$ as $i\to\infty$, where $K''(z, p^U)$ and $K''(z, p^L)$ are minimal functions corresponding to p^U and p^L and $K''(z, p_i)$ are normalized as $K''\left(-\frac{1}{2}, p_i\right)=1$.*

3). *Let $p_i\in\sum^{\infty} s_n$. Then $K(z, p_i)\to\frac{1}{2}(K''(z, p^U)+K''(z, p^L))$, i.e. any sequence $\{p_i\}$ on $\sum^{\infty} s_n$ determines a non minimal point. Let D_n $(n=1, 2, \cdots)$ be a domain $\ni q_n$ and $\{(D_n-q_n)\}$ are all conformally equivalent. Let $\{p_i\}$*

be a fundamental sequence in $\sum_n^{\infty} D_n$. *Then* $\{p_i\}$ *determines a non minimal boundary point on* $z=1$.

Map C_n conformally onto $|w|<\frac{1}{3}$. Then by Lemma 3), there exists a const. M such that for any positive harmonic function $U(w)$ in $|w|<\frac{1}{3}$ vanishing on the image of $I-s_n$ and $M\delta_n U(w) \geqq \sup_{|w|=\frac{1}{12}} U(w)w(s'_n, w)$, where $w(s'_n, w)$ is H.M. of the image of s_n relative to $|w|<1$. Put $M_n=M\delta_n$, then $\sum^{\infty} M_n<\infty$. Let $\mathring{\mathrm{P}}.\mathrm{H}.(R'')$ and $\mathring{\mathrm{P}}.\mathrm{H}.(R)$ be the sets of P.H. functions in R'' and R vanishing on $1-\sum_{n=1}^{\infty} s_n$ and I respectively. The by Theorem 2 $\mathring{\mathrm{P}}.\mathrm{H}.(R'')$ and $\mathring{\mathrm{P}}.\mathrm{H}.(R)$ are isomorphic. It is more convenient to consider the relation between $\mathring{\mathrm{P}}.\mathrm{H}.(R'')$ and $\mathring{\mathrm{P}}.\mathrm{H}.(R')$, where $R'=R+\sum_{n=1}^{n_0} s_n$ and n_0 is a number such that

$$\prod_{n_0}^{\infty}(1-2M_n) \geqq \frac{1}{4}. \tag{17}$$

Suppose K-Martin's topologies are defined in R'' and R' by $K''(z, p)$ and $K'(z, p)$ respectively, where they are normalized as $K''\left(-\frac{1}{2}, p\right)=K'\left(-\frac{1}{2}, p\right)=1$. Let v be a sufficiently small euclidean neighbourhood of $z=1$. Then since $v \cap R'$ consists of two simply connected domains, there exist only two points p^U and p^L on $z=1$ which are minimal and the corresponding functions, $K'(z, p^U)$ and $K'(z, p^L)$ have the properties. 1). $K'(z, p^U)=K'(\bar{z}, p^L)$, where $\bar{z}$ is the conjugate of z. 2). $K'(z, p^U) \to \infty$ as $z \to 1$ in the upper half plane and $K'(z, p^U)<L<\infty$ in the lower half plane. Let $v_n=E\left[z:\ |z|>1-\frac{1}{n}\right]$. Then E and I operations between R' and R'' are defined with respect to $\{v_n\}$. Then by the condition (17)

$$\begin{aligned} &EU(z)<\infty \quad \text{and} \quad IEU(z)=U(z) \qquad \text{for} \quad U(z) \in \mathring{\mathrm{P}}.\mathrm{H}.(R'), \\ &IV(z)>0 \quad \text{and} \quad EIV(z)=V(z) \qquad \text{for} \quad V(z) \in \mathring{\mathrm{P}}.\mathrm{H}.(R''), \end{aligned} \tag{18}$$

$$\frac{1}{2}K'(z, p) \leqq K''(z, p) \leqq 2K'(z, p) \qquad \text{for} \quad z \not\in \sum_{n_0}^{\infty} C_n. \tag{19}$$

We see at once $\mathring{1}$). $EK'(z, p^U)=EK'(\bar{z}, p^L)$ and by (19) $EK'(z, p^U) \to \infty$ as $z \to 1$ outside of $\sum^{\infty} C_n$ in the upper half plane and $EK'(z, p)<2L$ for $z \in \sum C_n$ and $\operatorname{Im} z<0$. $\mathring{2}$). $EK'(z, p^U)$ and $EK'(z, p^L)$ are minimal by the minimality of $K'(z, p^U)$ and $K'(z, p^L)$. Let $\{p_i\}$ be a sequence in R'' determining an ideal boundary point on $z=1$, i.e. $K''(z, p_i)$ converges to a $K(z, p)$ as $p_i \to z=1$. Then clearly $K''(z, p)=0$ on $I-\sum_{n_0}^{\infty} s_n$ and $|z|=1$ except $z=1$. By (18) and since there exist only two points of the boundary points of R' on $z=1$, $IK''(z, p)$ has the form

$$\begin{aligned} IK''(z, p) &= \alpha K'(z, p^U) + \beta K'(z, p^L) \quad \text{and} \\ K''(z, p) &= EIK''(z, p) = E(\alpha K'(z, p^U) + \beta K'(z, p^L)) . \end{aligned} \tag{20}$$

Hence there exist only two minimal boundary points of R'' on $z=1$.

Let $\{p_i\}$ be any sequence in $R''-\sum_{n_0}^{\infty} C_n$ such that $p_i \to z=1$ and p_i lies in the upper half plane. Let $\{p_i'\}$ be a subsequence of $\{p_i\}$ such that $K''(z, p_{i'}) \to K''(z, p)$. Then by (19) $K''(z, p) < 2L$ in the lower half plane outside of $\sum_{n_0}^{\infty} C_n$. Whence the representation (20) of $K''(z, p)$ has the form $K''(z, p) = \alpha EK'(z, p^U)$, where α is given by $K''\left(-\frac{1}{2}, p\right) = \alpha EK'\left(-\frac{1}{2}, p^U\right)$ and does not depend on the subsequence $\{p_{i'}\}$. Hence $\{p_i\}$ converges to p on $z=1$ of R'' (p corresponds to $K''(z, p^U)$). Similarly for any sequence $\{p_i\}$ in $R''-\sum_{n_0}^{\infty} C_n$ in the lower half plane, $K''(z, p_i) \to K''(z, p^L)$. Thus we have the property ($\mathring{1}$).

Let $\{p_i\}$ be a sequence on $\sum_{n_0}^{\infty} s_n$. Then by $K''(z, p_i) = K''(\bar{z}, p_i)$, $K''(z, p_i) \to \frac{1}{2}(K''(z, p^U) + K''(z, p^L))$ as $i \to \infty$. Let $\{p_i'\}$ be a sequence of $\{p_i\}$ in $\sum^{\infty} D_n$ such that $p_i \to z=1$ and $K''(z, p_i)$ converges to $K''(z, p)$ as $i \to \infty$. Let q_n be the middle point of s_n. Then as above $K''(z, q_i) \to \frac{1}{2}(K''(z, p^U) + K''(z, p^L))$ as $i \to \infty$. Since $G''(z, p)$ of R'' is a harmonic function of p for fixed $z \notin \sum_{n=n_0}^{\infty} D_n$, there exists by Harnack's theorem a const. M depending on D_n but non on n by the conformal equivalency of D_n such that

$$\frac{1}{M} \leqq \frac{G''(z, p_i)}{G''(z, q_i)} \leqq M \quad \text{and} \quad \frac{1}{M} \leqq \frac{G''\left(-\frac{1}{2}, p_i\right)}{G''\left(-\frac{1}{2}, q_i\right)} \leqq M .$$

Hence $$\frac{1}{M^2} \leqq \frac{K''(z, p_i)}{K''(z, q_i)} \leqq M^2 \qquad \text{for} \quad z \notin \sum_{n=n_0}^{\infty} C_n .$$

Let $i \to \infty$. Then $K''(z, p_i) \to K''(z, p)$ and

$$\frac{1}{M^2} \leqq \frac{\frac{1}{2}(K''(z,p^U)+K''(z,p^L))}{K''(z,p)} \leqq M^2, \quad z \notin \sum^{\infty} C_n. \quad (21)$$

Consider $K''(z,p)$ and $\frac{1}{2}(K''(z,p^U)+K''(z,p^L))$ in the upper half plane outside of $\sum_{n=n_0}^{\infty} C_n$. Then since $K''(z,p) \leqq 2L$, $\frac{M^2}{2} \geqq \alpha \geqq \frac{1}{2M^2} > 0$ in the form, $K''(z, p) = \alpha K''(z,p^U) + \beta K''(z,p^L)$. Similarly we have $\beta \geqq \frac{1}{2M^2}$. Hence $K''(z,p)$ is non minimal and $\{p_i'\}$ determines a non minimal point of R''. Thus we have the property $(\overset{\circ}{2})$.

Remark. In Example 1, we can take a circle $\tilde{C}_n$ of hyperbolic radius r_n with centre at q_n (r_n and q_n depend on n) instead of C_n. Let $I_n = E[z: 1 + r_n e^{i\theta_n}]$ $(n=1,2,\cdots)$, $0 < r_n < -2l_n \cos\theta_n$, $\pi > \theta_n > \frac{\pi}{2}$. Let $s_{n,m}$ $(m=1,2,\cdots)$ be a straight on I_n. Then we can choose $s_{n,m}$ so small that there exist infinitely many K-Martin's minimal boundary points of R''' on $z=1$, where $R''' =$ unit rcle$-\sum_{n}^{\infty} I_n + \sum_{n,m}^{\infty} s_{n,m}$.

In Example 1 we discussed $\mathring{P}.H.(R'')$ by $\mathring{P}.H.(R)$, when R increased to R''. We show an example to consider $\mathring{P}.H.(R'')$ by $P.H.(R)$, when R decreases to R''.

Example 2. Let R: $|z|<1$ and let C_n be a hyperbolic circle with hyperbolic radius $\frac{1}{3}$ and its centre at q_n also let S_n be a concentric hyperbolic circle with hyperbolic radius δ_n. Suppose C_n's are disjoint each other and

$$\sum^{\infty} \frac{1}{-\log\delta_n} < \infty.$$

Put $R'' = R - \sum_{n=1}^{\infty} S_n$. Let $\mathring{P}.H.(R)$ be the class of P.H. functions in $|z|<1$ and let $\mathring{P}.H.(R'')$ be of P.H. functioons vanishing on $\sum^{\infty} S_n$. Then $\mathring{P}.H.(R)$ and $\mathring{P}.H.(R'')$ are isomorphic. Especially there exist only one K-Martin's boundary point of R'' which is minimal at $e^{i\theta}$.

In fact, map C_n onto $|\zeta| < \frac{1}{3}$ and let $U(\zeta)$ be a P.H. in $|\zeta| < \frac{1}{6}$. Then there exists a const. M such that $\max_{|\zeta|=\frac{1}{12}} U(\zeta) \leqq M \min_{|\zeta|=\frac{1}{6}} U(\zeta)$. Let C_n' and C_n'' be concentric circles as C_n with hyperbolic radius $\frac{1}{6}$ and $\frac{1}{12}$ respectively. Then $w(S_n, z) = \frac{\log 12}{-\log\delta_n}$ on $\partial C_n''$, where $w(S_n, z)$ is H.M. of S_n relative to $|z|<1$.

Put $M_n = M\dfrac{\log 12}{-\log \delta_n}$. Then $M_n \min_{z\in\partial C_n'} U(z) \geqq \max_{z\in\partial C_n''} U(z) w(S_n, z)$. Hence by Theorem 1, $\mathring{\mathrm{P}}.\mathrm{H}.(R)$ and $\mathring{\mathrm{P}}.\mathrm{H}.(R'')$ are isomorphic and there exists only one boundary point R at $e^{i\theta}$ which is minimal.

Let R be a Riemann surface with positive boundary and let R_0 be a compact disc. Let $N(z, p)$ be an N-Green's function in $R-R_0$ such that $N(z, p)=0$ on ∂R_0, $N(z, p)$ has a logarithmic singularity at p and $N(z, p)$ has minimal Dirichlet integral (Dirichlet integral is taken about $N(z, p)+\log|z-p|$ in a neighbourhood of p). Suppose the N-Martin's[8] topology is defined on $R-R_0$. We shall construct a Riemann surface of planer character in which there exist non N-minimal points.

Example 3. Let C be a unit circle, C: $|z|<1$. Let s_n $(n=1, 2, \cdots)$ be a straight, s_n: $\mathrm{Im}\, z=0$, $0<a_n \leqq \mathrm{Re}\, z \leqq b_n$, $a_n<b_n \cdots <a_1<b_1=1$ in C and let t_n: $\mathrm{Im}\, z=0$, $b_{n+1} \leqq \mathrm{Re}\, z \leqq a_n$. Let $\hat{s}_n$ and $\hat{t}_n$ be symmetric images of s_n and t_n with respect to the imaginary axis. Let I be a straights such that I: $\mathrm{Re}\, z=0$, $0<\mathrm{Im}\, z<\dfrac{1}{2}$ and $0>\mathrm{Im}\, z>-\dfrac{1}{2}$.

Condition. *$z=0$ is contained in the closure of $\sum^{\infty} s_n$* and *$\sum^{\infty} s_n$ is so thinly distributed as $z=0$ is an irregular point for the Dirichlet problem in $C-\sum^{\infty}(s_n+\hat{s}_n)$. Put $R-R_0=C-I-\sum^{\infty}(t_n+\hat{t}_n)$. Then there exist four N-minimal points and non N-minimal points of $R-R_0$ on $z=0$.*

Let $p_i=r_i e^{i\theta_i}$. Suppose $r_i \to 0$ as $i\to\infty$ and $\dfrac{\pi}{2}>\theta_i>\delta>0$. Then p_i determines an N-minimal point[9]. *Similar fact occurs in each sector S_j: $0<|z|<1$, $\dfrac{\pi(j-1)}{2} \leqq \arg z \leqq \dfrac{\pi j}{2}$, $(j=1, 2, 3, 4)$.*

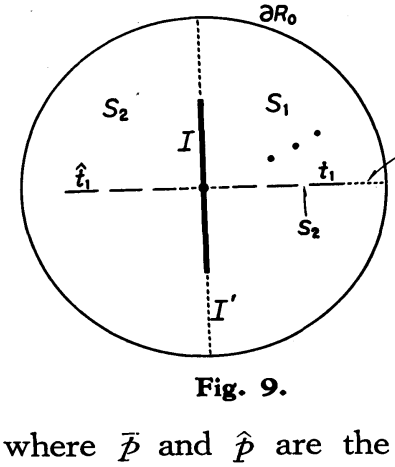

Fig. 9.

Let $N(z, p)$ be an N-Green's function of $R-R_0$. Then $N(z, p)=0$ on ∂R_0, $\dfrac{\partial}{\partial n}N(z, p)=0$ on $\sum^{\infty}(t_n+\hat{t}_n)+I$.

Let $G(z, p)$ be the Green's function of C. Then

$$N(z, p_i)+N(z, \bar{p}_i)+N(z, \hat{p}_i)+N(z, \hat{\bar{p}}_i) = G(z, p_i)+G(z, \bar{p}_i)+G(z, \hat{p}_i)+G(z, \hat{\bar{p}}_i),$$

where $\bar{p}$ and $\hat{p}$ are the conjugate of p and the symmetric image of p with respect to I.

Let $p_i \to p$ on $z=0$. Then

9) See (8).

$$N(z,p)+N(z,\bar{p})+N(z,\hat{p})+N(z,\hat{\bar{p}})=-4\log|z|\,.$$

Let C' be a circle, C': $|z|<\frac{2}{3}$ and put $\Omega_s=C'-\sum^{\infty}(s_n+\hat{s}_n)$. Let $G'(z,p)$ be the Green's function of Ω_s. By the condition, $z=0$ is an irregular point for also the Dirichlet problem in Ω_s. Hence there exists a sequence $\{p_i\}$ such that $\lim_i G'(z,p_i)=G'(z,p)>0$. Let $v_n=\left[z:\ |z|<\frac{1}{n}\right]$. We define E and I operations between C and Ω_s with respect to $\{v_n\}$. Then by $G(z,p_i)\geqq G'(z,p_i)$, $EG'(z,p)<\infty$. Now $EG'(z,p)$ is harmonic in C except $z=0$, whence $EG'(z,p)$ must be $\alpha(-\log|z|)$ (α is a positive constant $\leqq 1$). By $IEG'(z,p)=G'(z,p)$, $I(-\log|z|)$ is a harmonic function in Ω_s such that $I(-\log|z|)\leqq(-\log|z|)$, $I(-\log|z|)=0$ on $\sum^{\infty}(s_n+\hat{s}_n)$, $\frac{\partial}{\partial n}I(-\log|z|)=0$ on $I+\sum^{\infty}(t_n+\hat{t}_n)$. Hence for any p on $z=0$: $p=\lim_i p_i$,

$$-4\log|z|=N(z,p)+N(z,\bar{p})+N(z,\hat{p})+N(z,\hat{\bar{p}})\geqq 4I(-\log|z|)>0\,. \quad (22)$$

Let R' be the part of $R-R_0$ over $|z|<\frac{2}{3}$. Put $R_s=R'-\sum^{\infty}(s_n+\hat{s}_n)$. Then R_s consists of four components, R''_j $(j=1,2,3,4)$, where $R''_j\subset S_j$. Suppose $p_i\in R''_j$. Let $N^s(z,p_i)$ be an N-Green's function in R_s such that $N^s(z,p_i)$ is harmonic in R_s, $N^s(z,p_i)$ has a logarithmic singularity at p_i, $N^s(z,p_i)=0$ on $\sum^{\infty}(s_n+\hat{s}_n)$ and on $|z|=\frac{2}{3}$, $\frac{\partial}{\partial n}N^s(z,p_i)=0$ on $\sum^{\infty}(t_n+\hat{t}_n)+I$ and $N^s(z,p_i)=0$ in $R''_{j'}$ for $j'\neq j$. Then

$$\begin{aligned}-4\log|z| &\geqq \lim_i\left(N^s(z,p_i)+N^s(z,\bar{p}_i)+N^s(z,\hat{p}_i)+N^s(z,\hat{\bar{p}}_i)\right)\\ &=\lim_i\left(G'(z,p_i)+G(z,\bar{p}_i)+G(z,\hat{p}_i)+G(z,\hat{\bar{p}}_i)\right)\\ &=4\alpha I(-\log|z|)\,, \qquad (23)\end{aligned}$$

α depends on $\{p^i\}$ and $1\geqq\alpha\geqq 0$.

We shall use $I^{N,\Gamma}$ and $E^{N,\Gamma}$ operations between $R-R_0$ and R_s, where $\Gamma=I+\sum^{\infty}(t+\hat{t}_n)$. Since $I(-\log|z|)=0$ on $\sum^{\infty}(s_n+\hat{s}_n)$, $\frac{\partial}{\partial n}I(-\log|z|)=0$ on $I+\sum^{\infty}(t_n+\hat{t}_n)$, $I^{N,\Gamma}$ can be defined and we have by (22)

$$I^{N,\Gamma}(\lim_i N(z,p_i))>0 \qquad \text{for any } p_i\to p \text{ on } z=0\,. \quad (24)$$

Suppose $\lim_i N^s(z,p_i)$ exists and >0. Then by (23)

$$E^{N,\Gamma}(\lim_i N^s(z,p_i))>0\,. \quad (25)$$

We can find a sequence $\{p_i\}$ such that $\lim_i G'(z, p_i)=\beta I(-\log|z|)>0$. Then by (23) $E^{N,r}(\lim_i (N^s(z, p_i)+N^s(z, \bar{p}_i)+N^s(z, \hat{p}_i)+N^s(z, \hat{\bar{p}}_i))=4\beta EI(-\log|z|)$. On the other hand, since $\frac{\partial}{\partial n}I(-\log|z|)=0$ on $I+\sum^{\infty}(t_n+\hat{t}_n)$, $E^{N,r}(I(-\log|z|)$ $=EI(-\log|z|)=-\log|z|$. Put $U(z)=\frac{4}{\beta}\lim_i (N^s(z, p_i)+N^s(z, \bar{p}_i)+N^s(z, \hat{p}_i)+N^s(z, \hat{\bar{p}}_i))>0$. Then $U(z)=0$ on $\sum(s_n+\hat{s}_n)$ and on $|z|=\frac{2}{3}$, $\frac{\partial}{\partial n}U(z)=0$ on $I+\sum(t_n+\hat{t}_n)$, $E^{N,r}U(z)=-4\log|z|$ and for any $p_j\to p$ on $z=0$

$$E^{N,r}U(z) = -4\log|z| = N(z,p)+N(z,\bar{p})+N(z,\hat{p})+N(z,\hat{\bar{p}})\,.$$

Hence by Theorem 1, N, a) and e) and (24)

$$E^{N,r}I^{N,r}N(z,p)=N(z,p)>0 \qquad \text{for any } p \text{ on } z=0. \tag{26}$$

Let $\mathring{\text{P}}$.H.N.(R) be the set of positive harmonic function $U(z)$ of the form $N(z,p)$ in $R-R_0$ except $z=0$ such that $U(z)=0$ on ∂R_0, $\frac{\partial}{\partial n}U(z)=0$ on $\sum(t_n+\hat{t}_n)+I$ and let $\mathring{\text{P}}$.H.N.(R_s) be the set of positive harmonic function $V(z)$ of the form $N^s(z,p)$ in R_s except $z=0$ such that $V(z)=0$ on $|z|=\frac{2}{3}$ and $\sum^{\infty}(s_n+\hat{s}_n)$ and $\frac{\partial}{\partial n}V(z)=0$ on $I+\sum^{\infty}(t_n+\hat{t}_n)$. Then $\mathring{\text{P}}$.H.N.(R) and $\mathring{\text{P}}$.H.N.(R_s) are isomorphic by (25) and (26).

Let $N^s(z,p)=\lim_i N^s(z,p_i)>0$: $p_i\in R''_j$ and $p_i\to p$ on $z=0$. We shall show that $N^s(z,p)$ is N-minimal in R_s. Let $U(z)$ be a $\overline{\text{super}}$harmonic function in R_s such that $N^s(z,p)-U(z)>0$ is also $\overline{\text{super}}$harmonic in R_s. Now $\frac{\partial}{\partial n}N^s(z, p)=0$ on $I+\sum^{\infty}(t_n+\hat{t}_n)$. This means $N^s(z,p)$ has no mass on $I+\sum^{\infty}(t_n+\hat{t}_n)$, whence by the $\overline{\text{super}}$harmonicity of $N^s(z,p)-U(z)$, $\frac{\partial}{\partial n}U(z)=0$ on $1+\sum^{\infty}(t_n+\hat{t}_n)$. Put $\mathring{V}(z)=N^s(z, p)=N^s(z, \hat{p})=N^s(z, \bar{p})=N^s(z, \hat{\bar{p}})$ in each R''_j. Also define $\mathring{U}(z)$ similarly from $U(z)$ into $R-R_0$. Then $\mathring{U}(z)$ and $\mathring{V}(z)$ is harmonic in $C'-\sum^{\infty}(s_n+\hat{s}_n)$, where C': $|z|<\frac{2}{3}$. Now there exists only one linearly independent harmonic function $I(-\log|z|)$ in this surface. Whence $\mathring{U}(z)=\alpha\mathring{V}(z)=\beta I(-\log|z|)$ and $U(z)=\alpha N^s(z,p)$ and $N^s(z,p)$ is N-minimal in R_s. On the other hand, in every R''_j, $N^s(z,p)$ exists denoted by $N^s(z,p^j)$. Then clearly $N^s(z,p^j)$ are linearly independent. Because $N^s(z,p_j)=0$ in $R''_{j'}$, for $j'\neq j$. Let $N(z,p)=\lim_i N(z,p_i)$. Then by (26) $I^{N,r}N(z,p)>0$, whence $I^{N,r}(z,$

$p)=\sum_{j=1}^{4}\alpha_j N^s(z,p^j)$. Next also by (26) $N(z,p)=\sum_{j}^{4}\alpha_j E^{N,\Gamma}N^s(z,p^j)$ and there exist exact four N-minimal points of $R-R_0$ on $z=0$.

Property of $E^{N,\Gamma}N^s(z,p^j)$. $N^s(z,p^j)=0$ in $R''_{j'}$ for $j'\neq j$. Hence

$$E^{N,\Gamma}N^s(z,p^j)=E^{N,\Gamma}_{R''_j}N^s(z,p^j),$$

where $E^{N,\Gamma}_{R''_j}$ is from R''_j to $R-R_0$ and $E^{N,\Gamma}$ is from R_s to $R-R_0$.
By Theorem $N.1$). b)

$$\begin{aligned}&I^{N,\Gamma}_{R''_k}E^{N,\Gamma}N^s(z,p^j)\leqq E^{N,\Gamma}_{R''_k}E^{N,\Gamma}N^s(z,p^j)\leqq E^{N,\Gamma}_{CR''_j}E^{N,\Gamma}_{R''_j}N^s(z,p^j)=0 \quad\text{and}\\ &I^{N,\Gamma}_{R''_j}E^{N,\Gamma}_{R''_j}N^s(z,p^j)=N^s(z,p^j) \qquad \text{for}\quad k\neq j.\end{aligned}\tag{27}$$

We shall show $p_i=r_ie^{i\theta_j}\left(\frac{\pi}{2}\geqq\theta_j>\delta>0\right)\in S_1$ determines an N-minimal point p^1 of $R-R_0$. $N(z,p_i)+N(z,\hat{p}_i)=N(z,\bar{p}_i)+N(z,\hat{\bar{p}}_i)=U(z,p_i)$ for $\operatorname{Im}z=0$, where $U(z,p_i)=\frac{1}{2}(G(z,p_i)+G(z,\bar{p}_i)+G(z,\hat{p}_i)+G(z,\hat{\bar{p}}_i))$ and $G(z,q)=\log\frac{|1-\bar{q}z|}{|z-q|}$.
$N(z,p_i)$ is harmonic in S_4 (has no singularity) such that $\frac{\partial}{\partial n}N(z,p_i)=0$ on $I+\sum^{\infty}(t_n+\hat{t}_n)$ and $N(z,p_i)\leqq U(z,p_i)$ on $\sum^{\infty}(s_n+\hat{s}_n)+I'\left(=E\left[z:\ \operatorname{Re}z=0,\ -1\leqq\operatorname{Im}z\leqq-\frac{1}{2}\right]\right)$ and $N(z,p_i)=\frac{1}{2\pi}\int_{\Sigma S_n+I'}N(\zeta,p_i)\frac{\partial}{\partial n}N^S(\zeta,z)ds$ in S_4, where $N^S(\zeta,z)$ is a harmonic function in S_4 such that $N^S(\zeta,z)$ has a logarithmic singularity at z, $\frac{\partial}{\partial n}N^S(\zeta,z)=0$ on $I+\sum^{\infty}t_n$ and $N^S(\zeta,z)=0$ on $|z|=1$ and on $\sum^{\infty}s_n+I'$.
Put $p_i=r_ie^{i\theta_i}$. Then by $\frac{\pi}{2}>\theta_i>\delta$, $\frac{5}{4}>|1-p_i\zeta|>\frac{3}{4}$ for $r_i<\frac{1}{4}$ and $|\zeta-p_i|\geqq\zeta\sin\zeta$. Hence there exists a const. A such that

$$U(\zeta,p_i)\leqq-2\log|\zeta|+A \qquad \text{for}\quad |r|\leqq\frac{1}{4},$$

whence $N(\zeta,p_i)\frac{\partial}{\partial n}N^S(\zeta,z)$ is uniformly integrable on $\sum^{\infty}s_n+I'$ for $|p_i|\leqq\frac{1}{4}$.
Hence
$$N(z,p^1)=\frac{1}{2\pi}\int_{\Sigma s_n+I'}N(\zeta,p^1)\frac{\partial}{\partial n}N^S(\zeta,z)ds=U(z)\ \text{in}\ S_4,$$
where $N(z,p^1)=\lim_{i}N(z,p_i)$.

$I^{N,\Gamma}_{R''_4}N(z,p^1)=\lim_{n}V_n(z)$, where $V_n(z)$ is a harmonic function in $R''_4\cap E\left[|z|>\frac{1}{n}\right]$ such that $V_n(z)=U(z)$ on $|z|=\frac{1}{n}$, $\frac{\partial}{\partial n}V_n(z)=0$ on $I+\sum^{\infty}t_n$, $V_n(z)=0$ on

$I'+\sum^{\infty} s_n$ and on $|z|=\frac{2}{3}$. Let $w_n(z)$ be a harmonic function in $R''_4\cap E\left[|z|>\frac{1}{n}\right]$ such that $w_n(z)=0$ on $|z|=\frac{1}{n}$ and $\frac{\partial}{\partial n}w_n(z)=0$ on I and $w_n(z)=U(z)$ on $\sum^{\infty} s_n+I'$ and on $|z|=\frac{2}{3}$. Then $w_n(z)=\frac{1}{2\pi}\int_{\Sigma s_n+I'} N(\zeta,p)\frac{\partial}{\partial n}N^S_n(\zeta,z)ds$, $U(z)=w_n(z)+V_n(z)$ and $\lim_n w_n(z)=U(z)$ by $\frac{\partial}{\partial n}N^S_n(\zeta,z)\uparrow\frac{\partial}{\partial n}N^S(\zeta,z)$, where $N^S_n(\zeta,z)$ is a function in $R''_4\cap E\left[|z|>\frac{1}{n}\right]$ such that $N^S_n(\zeta,z)$ has singularity at z, $N^S_n(\zeta,z)=0$ on $I'+\sum^{\infty} s_n$ and $|z|=\frac{1}{n}$ and $\frac{\partial}{\partial n}N^S_n(\zeta,z)=0$ on $I+\sum^{\infty} t_n$. Whence $\lim_n V_n(z)=I^{N,\Gamma}_{R''_4}N(z,p)=0$. Similarly we have

$$I^{N,\Gamma}_{R''_k}N(z,p^1)=0 \qquad \text{for} \quad k\neq 1. \tag{28}$$

Hence in the form $N(z,p)=\sum_j^4 \alpha_j E^{N,\Gamma}N^s(z,p^j)$ $\alpha_k=0$: $k\neq 1$ and $N(z,p^1)=\alpha_1 E^{N,\Gamma}N^s(z,p^1)$. Now by $\int_{|z|=1}\frac{\partial}{\partial n}N(z,p)ds=2\pi$, α_1 does not depend on special sequence $\{p_i\}$ if $\frac{\pi}{2}\geqq\arg p_i>\delta>0$. Hence any $\{p_i\}$ determines an N-minimal point on $z=0$ as $p_i\to z=0$, if $\frac{\pi}{2}>\arg p_i>\delta>0$. Similar fact occurs in S_k, $k=2,3,4$.

Let $p_i\in\sum^{\infty} s_n$. Then we have at once $\lim_i N(z,p_i)=\frac{1}{2}(N(z,p^1)+N(z,p^4))$ and $\{p_i\}$ *determines a non N-minimal point on* $z=0$.

Remark. It is easy to construct a boundary point $z=0$ on which infinitely many N-minimal points exist as the remark of Example 1.

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