ON A DIFFERENTIAL-DIFFERENCE EQUATION

By

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In connexion with the study of certain incomplete sums of multiplicative functions, N. G. de Bruijn and J. H. van Lint [3] have introduced the function $f_s(x)$ $(s \geqq 0)$ satisfying the set of conditions:

(i) $f_s(x) = 0$ for $x < 0$,

(ii) $f_s(x)$ is continuous for $x > 0$,

(iii) $f_s(x) = x^{s-1}$ for $0 < x \leqq 1$,

(iv) $x f_s'(x) = (s-1) f_s(x) - s f_s(x-1)$ for $x > 1$.

(The function $f_s(x)$ is originally defined in [3; II, §2] only for $x > 0$; it will be convenient, however, to define $f_s(x) = 0$ for $x < 0$ for our purpose.)

On the other hand, N. G. de Bruijn [1 and 2] has investigated in detail the property and behaviour of $f_s(x)$ for $s = 1$. In particular, there he obtained an explicit formula for $f_1(x)$:

$$f_1(x) = \frac{e^C}{2\pi i} \int_{-i\infty}^{i\infty} \exp\left(-xt + \int_0^t \frac{e^z - 1}{z} dz\right) dt \qquad (x > 0),$$

where C is Euler's constant,

$$C = \lim_{n \to \infty} \left(\sum_{m=1}^{n} \frac{1}{m} - \log n \right).$$

In the present note we shall prove an analogous formula for $f_s(x)$ with general $s > 0$.

Remark. For $s = 0$ it is easy to see that $f_s(x) = f_0(x) = x^{-1}$ $(x > 0)$. We may suppose, therefore, that $s > 0$ throughout in the following.

1. Lemmata. We require two lemmas independent of one another.

Lemma 1. *If $\phi(s)$ is a (complex valued) continuous function defined for $s > 0$ and satisfying the functional equation*

$$\phi(s + r) = \phi(s)\phi(r) \qquad (s > 0, \ r > 0),$$

then there is an integer A independent of s such that

$$\phi(s) = e^{2\pi i A s} \left(\phi(1)\right)^s \qquad (s > 0).$$

Proof. We may assume without loss of generality that $\phi(s)$ does not vanish for $s>0$ and hence that $\phi(1)\neq 0$. Consider the continuous function

$$\psi(s)=\frac{\phi(s)}{(\phi(1))^{s}}\qquad (s>0),$$

where $z^{s}=\exp(s\log z)$ and the branch of $\log z$ is taken in such a way that $\log z$ is real for real $z>0$. We have for any $s>0$

$$\psi(s+1)=\frac{\phi(s+1)}{(\phi(1))^{s+1}}=\frac{\phi(s)}{(\phi(1))^{s}}=\psi(s).$$

Thus, if we put

$$\alpha(s)=\frac{1}{1+|\log|\psi(s)||},$$

then

$$\int_0^1\alpha(s)\,ds=\int_0^1\alpha(2s)\,ds. \tag{1}$$

Indeed, we have

$$\begin{aligned}\int_0^1\alpha(s)\,ds&=\int_0^1\alpha(s+1)\,ds=\int_1^2\alpha(s)\,ds\\&=2\int_{1/2}^1\alpha(2s)\,ds=2\int_0^1\alpha(2s)\,ds-\int_0^1\alpha(s)\,ds,\end{aligned}$$

which is equivalent to (1). Since $\psi(2s)=(\psi(s))^2$, we deduce from (1) that

$$\int_0^1\frac{|\log|\psi(s)||}{(1+|\log|\psi(s)||)(1+2|\log|\psi(s)||)}\,ds=0,$$

and this implies that $\log|\psi(s)|=0$ almost everywhere on $(0,1)$. It follows that $|\psi(s)|=1$ *everywhere* on $(0,\infty)$. This means that, if we set

$$\frac{\phi(s)}{(\phi(1))^{s}}=e^{2\pi i\,\theta(s)}, \tag{2}$$

then $\theta(s)$ is a real valued continuous function of $s>0$ satisfying the congruence

$$\theta(s+r)\equiv\theta(s)+\theta(r)\pmod 1\qquad (s>0,\ r>0).$$

Hence, there is a constant $c\equiv 0 \pmod 1$ such that

$$\theta(s+r)=\theta(s)+\theta(r)+c\qquad (s>0,\ r>0),$$

and it follows from this that the limit

$$\lim_{s\to+0}\theta(s)=-c$$

exists. Thus, if we put

$$\theta^*(s) = \theta(s) + c\,,$$

then $\theta^*(s)$ satisfies the equation

$$\theta^*(s+r) = \theta^*(s) + \theta^*(r) \qquad (s>0,\ r>0).$$

Since $\theta^*(s)$ is continuous for $s>0$, we find by a well-known theorem (which is in fact easy to prove) that $\theta^*(s)=As$ $(s>0)$ for some real constant A, so that $\theta(s)=\theta^*(s)-c=As-c$. But, in view of (2), we may take $c=0$. Finally, the constant A must be integral, since $e^{2\pi i A}=1$. This completes the proof of the lemma.

Lemma 2. *We have*

$$f_{s+r}(x) = \frac{\Gamma(s+r)}{\Gamma(s)\,\Gamma(r)} \int_0^x f_s(y) f_r(x-y)\,dy \qquad (s>0,\ r>0).$$

Proof. Put

$$f(x) = \int_0^x f_s(y) f_r(x-y)\,dy\,.$$

Apparently, $f(x)=0$ for $x<0$ and $f(x)$ is continuous for $x>0$. For $0<x\leqq 1$ we have

$$\begin{aligned} f(x) &= x^{s+r-1}\int_0^1 z^{s-1}(1-z)^{r-1}\,dz \\ &= f_{s+r}(x)\frac{\Gamma(s)\,\Gamma(r)}{\Gamma(s+r)}\,. \end{aligned}$$

Suppose now that $x>1$ and write

$$\begin{aligned} xf(x) &= \int_0^x y f_s(y) f_r(x-y)\,dy + \int_0^x f_s(y)(x-y) f_r(x-y)\,dy \\ &= \int_0^x (x-y) f_s(x-y) f_r(y)\,dy + \int_0^x f_s(y)(x-y) f_r(x-y)\,dy \\ &= I_1 + I_2\,, \end{aligned}$$

say. We have

$$\begin{aligned} \frac{dI_1}{dx} &= \int_0^x \Big((x-y) f_s'(x-y) + f_s(x-y)\Big) f_r(y)\,dy \\ &= \int_0^x (x-y) f_s'(x-y) f_r(y)\,dy + \int_0^x f_s(x-y) f_r(y)\,dy \\ &= (s-1)f(x) - sf(x-1) + f(x) \\ &= sf(x) - sf(x-1)\,, \end{aligned}$$

and, by symmetry,

$$\frac{dI_2}{dx}=rf(x)-rf(x-1).$$

Since $\bigl(xf(x)\bigr)'=xf'(x)+f(x)$, we thus obtain

$$xf'(x)=(s+r-1)f(x)-(s+r)f(x-1).$$

Hence the function $\dfrac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)}f(x)$ satisfies all the conditions (i)—(iv) with $s+r$ in place of s, and, since these conditions uniquely determine the function $f_{s+r}(x)$, it follows that

$$f_{s+r}(x)=\frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)}f(x),$$

which is the required result.

Remark. The substance of Lemma 2 is a particular case of a slightly more general result. Thus, let $h(x)=h(x;c)$ be a function defined by the following conditions, $c=(c_0, c_1, \cdots, c_n)$ being an $(n+1)$-tuple of constants ($n\geqq 0$ fixed):

(i) $h(x)=0$ for $x<0$,

(ii) $h(x)$ is continuous for $x>0$,

(iii) $\lim\limits_{x\to+0} xh(x)=0$,

(iv) $xh'(x)=(c_0-1)h(x)+\sum\limits_{j=1}^{n} c_j h(x-j)$ for all $x>0$, $x\neq m$ $(1\leqq m\leqq n)$.

(Obviously the above conditions for $h(x)=h(x;c)$ imply that $c_0>0$ and $h(x)=Bx^{c_0-1}$ for $0<x\leqq 1$ with B a constant and, for $x>1$, $h(x)$ is uniquely determined once B is fixed. We shall be concerned with those functions $h(x)$ only which are not identically zero.) Then we have

$$h(x;a+b)=K\int_0^x h(y;a)h(x-y;b)\,dy \qquad (x\neq 0),$$

where K is a constant and where we set

$$a+b=(a_0+b_0, a_1+b_1, \cdots, a_n+b_n)$$

if

$$a=(a_0, a_1, \cdots, a_n) \quad \text{and} \quad b=(b_0, b_1, \cdots, b_n).$$

2. The Explicit Formula. We now consider the Laplace transform of $f_s(x)$,

$$F_s(\xi)=\int_0^{\infty} e^{-\xi x} f_s(x)\,dx\,,$$

where ξ is a complex variable. The integral defining $F_s(\xi)$ is absolutely convergent on the line $\operatorname{Re}\xi=0$ (cf. §3 below). Also, $F_s(\xi)$ is, as a function of s, continuous for $s>0$, ξ ($\operatorname{Re}\xi \geqq 0$) being fixed.

In view of Lemma 2 we have

$$F_{s+r}(\xi)=\frac{\Gamma(s+r)}{\Gamma(s)\,\Gamma(r)}\,F_s(\xi)F_r(\xi) \qquad (s>0,\ r>0)$$

or

$$\frac{F_{s+r}(\xi)}{\Gamma(s+r)}=\frac{F_s(\xi)}{\Gamma(s)}\,\frac{F_r(\xi)}{\Gamma(r)} \qquad (s>0,\ r>0).$$

Hence, by applying Lemma 1 to $\phi(s)=F_s(\xi)/\Gamma(s)$, we get for $s>0$

$$\frac{F_s(\xi)}{\Gamma(s)}=e^{2\pi iAs}\left(\frac{F_1(\xi)}{\Gamma(1)}\right)^s$$

or

$$F_s(\xi)=\Gamma(s)\left(F_1(\xi)\right)^s,$$

the constant A being necessarily zero since for any $s>0$ $F_s(\xi)$ has a positive real value for real $\xi \geqq 0$.

We see from the explicit formula for $f_1(x)$ ([2; §1]) that

$$F_1(\xi)=e^C \exp\left(\int_0^{-\xi}\frac{e^z-1}{z}\,dz\right).$$

Therefore,

$$F_s(\xi)=\Gamma(s)e^{Cs}\exp\left(s\int_0^{-\xi}\frac{e^z-1}{z}\,dz\right) \qquad (s>0). \tag{3}$$

By a standard inversion formula for the Laplace transform, we thus obtain the following result.

Theorem. *We have for* $s>0$

$$f_s(x)=\lim_{T\to\infty}\frac{\Gamma(s)e^{Cs}}{2\pi i}\int_{-iT}^{iT}\exp\left(-xt+s\int_0^t\frac{e^z-1}{z}dz\right)dt \qquad (x\neq 0).$$

3. Notes. 1) We note that for $s=1$ the right-hand side of the equality in the theorem is equal to $\dfrac{1}{2}$ at $x=0$ and for $s>1$ it is equal to 0 at $x=0$. Also, if $s=1$ then we have

$$f_1(x) = \frac{e^C}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left(-xt + \int_0^t \frac{e^z - 1}{z}\, dz\right) dt$$

for all $x \neq 0$, and if $s > 1$ then

$$f_s(x) = \frac{\Gamma(s) e^{Cs}}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left(-xt + s\int_0^t \frac{e^z - 1}{z}\, dz\right) dt$$

for all x, $-\infty < x < \infty$, $f_s(0)$ being defined to be equal to 0.

2) de Bruijn and van Lint [3; I] have also considered the function $g_s(x)$ $(s \geqq 0)$ defined by the conditions:

(i) $g_s(x) = 0$ for $x < 0$,

(ii) $g_s(x)$ is continuous for $x \geqq 0$,

(iii) $g_s(x) = x^s$ for $0 \leqq x \leqq 1$,

(iv) $x g_s'(x) = s g_s(x) - s g_s(x-1)$ for $x > 1$.

As is noted in [3; II, §2], we have for $s > 0$

$$g_s(x) = s\int_0^x f_s(y)\, dy\,.$$

They showed in [3; I, §2] that if $s = 0$ then $g_s(x) = g_0(x) = 1$ for all $x > 0$ and if $s > 0$ then $g_s(x)$ is a positive, monotone increasing function of x for $x > 0$.

It is also proved there that we have

$$\lim_{x\to\infty} g_s(x) = \Gamma(s+1) e^{Cs} \qquad (s > 0)$$

and this implies at once that

$$\int_0^\infty f_s(x)\, dx = \Gamma(s) e^{Cs} \qquad (s > 0),$$

which clearly agrees with (3). And in the course of its proof they found a formula which is essentially the same as (3). (In fact, using the relation between $f_s(x)$ and $g_s(s)$, we can show that $F_s(\xi)$ $(s > 0)$ satisfies as a function of ξ the differential equation

$$\xi F_s'(\xi) = s(e^{-\xi} - 1) F_s(\xi)\,,$$

and, by integrating this equation, we get the formula (3).) Thus, our main interest of this note is in deriving the explicit formula for $f_s(x)$ from a somewhat different point of view, that is, on the basis of Lemma 2 which shows an interesting interrelation existent among the functions $f_s(x)$ $(s > 0)$.

References

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