On necessary and sufficient conditions for
$L^2$-well-posedness of mixed
problems for hyperbolic equations II

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

We consider hyperbolic mixed problems $(P, B_j)$ in a quadrant $R_+ \times R^n_+$:

- $Pu = f$ in $t > 0, x_n > 0, x' \in R^{n-1}$,
- $B_j u = 0$ (j = 1, ..., l) in $t > 0, x_n = 0, x' \in R^{n-1}$,
- $\partial^*_t u = u_k$ (k = 0, 1, ..., m - 1) in $t = 0, x_n > 0, x' \in R^{n-1}$

where $R^n_+ = \{x = (x', x_n) \in R^n ; x_n > 0\}$, $\partial_t = \frac{\partial}{\partial t}$, $D_{x_j} = (-i)\frac{\partial}{\partial x_j}$, $P = P(t, x; \partial_t, D_x)$ is a strictly t-hyperbolic operator of order $m$ and $B_j = B_j(t, x'; \partial_t, D_x)$ are boundary operators of order $m_j < m$. Furthermore $P$ is assumed non-characteristic with respect to the hyperplane $x_n = 0$.

This paper consists of two parts. The first part (§ 2 and § 3) is concerned with constant coefficient problems for homogeneous operators $P$ and $B_j$. In previous paper [2] we assume that $\{B_j\}$ is normal; that is, $m_j \neq m_k$ if $j \neq k$ and the hyperplane $x_n = 0$ is non-characteristic for $B_j$. However, when $\{B_j\}$ is not always normal, it will be suitable for our purpose to use the following

DEFINITION. The mixed problem $(P, B_j)$ with homogeneous initial condition is $L^2$-well-posed with decreasing order $\nu$ ($\nu \geq 0$, an integer) if and only if there exist positive constants a and C such that for any $f$ with $e^{-at} f \in H^{\nu+1}_0(R^n_+ \times R^n_+)$ the problem has a unique solution $u$ with $e^{-at} u \in H^m(R_+ \times R^n_+)$ satisfying

$$\sum_{j+|\alpha| \leq m-1} \int_0^\infty \int_{R^n_+} e^{-2at} |(\partial^*_t D^\alpha x) u(t, x)|^2 dt dx \leq C \sum_{j+|\alpha| \leq m-1} \int_0^\infty \int_{R^n_+} e^{-2at} |(\partial^*_t D^\alpha f) (t, x)|^2 dt dx.$$  

In § 2 a necessary and sufficient condition for $L^2$-well-posedness with decreasing order $\nu$ is given by the terms of compensating function (cf.
Theorem 4.1 in [2]). In § 3 we show that under the $L^2$-well-posedness with decreasing order $\nu$ Lopatinskii’s determinant $R(\tau_0, \sigma_0) \neq 0$ for $Re \tau_0 > 0$ and $\sigma_0 \in R^{n-1}$ is equivalent to the fact that $B_j(\tau_0, \sigma_0, \lambda)$ are linearly independent as polynomials in $\lambda$.

The second part is concerned with variable coefficient problems. In this part we always assume that $\{B_j\}$ is normal. For initial data $U(t_0) = (u_0, u_1, \cdots, u_{m-1})$ we set up a convenient function space $\mathscr{H}(t_0)$; that is, $U(t_0) \in \mathscr{H}(t_0)$ if and only if $u_k \in H^{m-k}(R_+^n)$ and satisfy the compatibility condition

$$\sum_{k=0}^{m} b_{j,k}(t, x') D_x u_k = 0 \quad \text{on} \quad t = t_0, x_n = 0$$

where

$$B_j(t, x'; \partial_t, D_x) = \sum_{k=0}^{m} b_{j,k}(t, x'; D_x) \partial_t^k.$$ 

**Definition.** The mixed problem $(P, B_j)$ is strongly $L^2$-well-posed if and only if there exist positive constants $T$ and $C$ such that for an arbitrarily fixed time $t_0 \in [0, T)$ the problem $(P, B_j)$ with $f \in H_0^1((t_0, T) \times R_+^n)$ and $U(t_0) \in \mathscr{H}(t_0)$ has a unique solution $u \in \mathcal{E}^{0}((t_0, T), H^{m}(R_{+}^{n})) \cap \cdots \cap \mathcal{E}^{m}((t_0, T), H^{0}(R_{+}^{n}))$ which satisfies energy inequalities

$$(1.2) \quad \|u(t, \cdot)\|_{m-1}^2 \leq C(\|U(t_0)\|_{m-1}^2 + \int_{t_0}^{t} \|f(s, \cdot)\|_0^2 ds),$$

$$(1.3) \quad \|u(t, \cdot)\|_{m}^2 \leq C(\|U(t_0)\|_{m}^2 + \int_{t_0}^{t} \|f(s, \cdot)\|_1^2 ds)$$

for any $t \in [t_0, T]$, where

$$\|u(t, \cdot)\|_k^2 = \sum_{j=0}^{k} \|\partial_t^j u(t, \cdot)\|_{k-j}^2,$$

$$\|U(t_0)\|_k^2 = \sum_{j=0}^{k} \|u_j(\cdot)\|_{k-j}^2,$$

$$\|u(\cdot)\|_k^2 = \sum_{|\alpha| \leq k} \int_{R_{+}^{n}} |D_x^\alpha u(x)|^2 dx.$$

In § 4 we show the following: If a variable coefficient problem $(P, B_j)$ is strongly $L^2$-well-posed, then each constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point is $L^2$-well-posed (with decreasing order $\nu=0$), provided that the corresponding Lopatinskii’s determinant $R(t, x'; 1, 0) \neq 0$ on the boundary. Combining this and results in [1] we obtain a certain characterization of strongly $L^2$-well-posed problems with real boundary condition for the case of second order.

This note is the supplement of our previous papers [1] and [2].
§ 2. Necessary and sufficient condition

In this section and the following we consider constant coefficient problems \((P, B_j)\) with homogeneous initial condition. Here \(P\) and \(B_j\) are homogeneous operators.

We take Laplace transform in \(t\) and Fourier transform in \(x\) and \(\hat{u}(\tau, \sigma, \lambda)\) and \(\hat{u}(\tau, \sigma, x_n)\) denote the Fourier-Laplace image of \(u(t, x', x_n)\) with respect to \((t, x', x_n)\) and \((t, x')\) respectively. By the assumption on \(P\) the number \(l(m-l)\) of roots \(\lambda_j^{+}(\tau, \sigma)(\lambda_k^{-}(\tau, \sigma))\) of \(P(\tau, \sigma, \lambda)=0\) in \(\lambda\), which have positive (negative) imaginary part, is independent of \((\tau, \sigma)\in \mathbb{C}_+ \times \mathbb{R}^{n-1}\), where \(\mathbb{C}_+ = \{\tau \in \mathbb{C}; \text{Re} \tau > 0\}\).

Taking now Fourier-Laplace transform the problem \((P, B_j)\) becomes formally to the boundary value problem of ordinary differential equations depending on parameters \((\tau, \sigma)\in \mathbb{C}_+ \times \mathbb{R}^{n-1}\):

\[
P(\tau, \sigma, D_{x_n}) \hat{u}(\tau, \sigma, x_n) = \hat{f}(\tau, \sigma, x_n) \quad \text{in} \; x_n > 0,
\]

\[
B_j(\tau, \sigma, D_{x_n}) \hat{u}(\tau, \sigma, x_n) = 0 \quad (j=1, \cdots, l) \quad \text{on} \; x_n = 0.
\]

Let \(R(\tau, \sigma)\) be Lopatinskii's determinant; that is,

\[
R(\tau, \sigma) = \det \left( B_j(\tau, \sigma, \lambda_j^{+}(\tau, \sigma)) \right) / \prod_{j>k} (\lambda_j^{+}(\tau, \sigma) - \lambda_k^{+}(\tau, \sigma))
\]

and \(R_j(\tau, \sigma, x_n)\) be the determinant replacing \(j\)-column in \(R(\tau, \sigma)\) by the transposed vector \((\exp(ix_n \lambda_1^{+}(\tau, \sigma)), \cdots, \exp(ix_n \lambda_l^{+}(\tau, \sigma)))\). Then \(R(\tau, \sigma)\) and \(R_j(\tau, \sigma, x_n)(j=1, \cdots, l)\) are analytic in \((\tau, \sigma)\in \mathbb{C}_+ \times \mathbb{R}^{n-1}\). If \(R(\tau, \sigma) \neq 0\) for some \((\tau, \sigma)\in \mathbb{C}_+ \times \mathbb{R}^{n-1}\), then it is well known that for any \(f \in \mathbb{C}_0^\infty(\mathbb{R}_+)\) the problem (2.1) has a unique bounded solution \(u \in \mathbb{C}^\infty(\mathbb{R}_+)\) which is written by the form

\[
\hat{u}(\tau, \sigma, x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_n \lambda}(\tau, \sigma)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \int_0^{\infty} G(x_n, s, \tau, \sigma) f(s) ds
\]

where

\[
G(x_n, s, \tau, \sigma) = \sum_{j=1}^l \frac{R_j(\tau, \sigma, x_n)}{R(\tau, \sigma)} \int_{\Gamma} \frac{B_j(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} e^{-ist} d\lambda
\]

and \(\Gamma = \Gamma'(\tau, \sigma)\) denotes a simple closed curve in the lower half \(\lambda\)-plane enclosing all the roots \(\lambda_j(\tau, \sigma)\).

Let \(\Sigma_+\) be the set \{\((\tau', \sigma') \in \mathbb{C}_+ \times \mathbb{R}^{n-1}; \; |\tau'|^2 + |\sigma'|^2 = 1\}\) and \(\Sigma_+\) be its closure. Furthermore let \(V\) be the zeros of \(R(\tau, \sigma)\) in \(\mathbb{C}_+ \times \mathbb{R}^{n-1}\) and \(V'\) be \(V \cap \Sigma_+\). \(V^c_+\) and \(V^c\) denote the complement of \(V'\) and \(V\) in \(\Sigma_+\) and \(\mathbb{C}_+ \times \mathbb{R}^{n-1}\) respectively. Then we obtain the following
THEOREM 2.1. Suppose that \( R(\tau, \sigma) \) is identically not zero. Then the mixed problem \( (P, B_j) \) is \( L^2 \)-well-posed with decreasing order \( \nu \) if and only if the following condition is satisfied:

For every \( (\tau'_0, \sigma'_0) \in (\Sigma_+ - \Sigma_+) \cup V' \) there exist a constant \( C(\tau'_0, \sigma'_0) \) and a neighborhood \( U(\tau'_0, \sigma'_0) \) such that

\[
\| (D_{x_n}^k G)(x_n, s, \tau', \sigma') \|_{L^2(x_n > 0)} \leq C(\tau'_0, \sigma'_0) (Re \tau')^{-\nu-1}
\]

for any \( (\tau', \sigma') \in U(\tau'_0, \sigma'_0) \cap \Sigma_+ \cap V' \) and \( k = 0, 1, \cdots, m-1 \), where

\[
\| \cdot \|_{L^2(x_n > 0)}
\]

denotes the operator norm from \( H^m(\tau > 0) \) to \( L^2(x_n > 0) \).

Since the proof of the theorem is accomplished by the almost same considerations as those in Theorem 4.1 [2], we show only different points.

(I) SUFFICIENCY (EXISTENCE OF SOLUTIONS). Let \( S \) be the set \( \{ \sigma \in \mathbb{R}^{n-1} ; R(\tau, \sigma) \) is identically zero in \( \tau \} \). Then \( S \) is a null set with respect to Lebesgue measure in \( \mathbb{R}^{n-1} \) because \( V \) is so in \( C_+ \times \mathbb{R}^{n-1} \). In what follows we assume \( \sigma \in S \) and for \( (\tau, \sigma) \in C_+ \times \mathbb{R}^{n-1} \) denotes \( (\rho^{-1} \tau, \rho^{-1} \sigma) \) where \( \rho = (|\tau|^2 + |\sigma|^2)^{\frac{1}{2}} \).

LEMMA A. There exists an analytic extension \( \tilde{G}(\tau, \sigma) \) in \( \tau \in C_+ \) of \( G(x_n, s, \tau, \sigma) \) as an operator from \( H^m(\tau > 0) \) to \( L^2(x_n > 0) \) such that \( \tilde{G}(\tau, \sigma) f \in H^{m-1}(x_n > 0) \) for \( f \in H^m(\tau > 0) \) and

\[
\| D_{x_n}^k \tilde{G}(\tau, \sigma) f \|_{L^2(x_n > 0)} \leq C(\tau'_0, \sigma'_0) (Re \tau')^{-\nu-1} \rho^{k-m+1} \rho^{-2\nu} \int_0^\infty |(D_s^\nu f)(0)\|_{L^2(x_n > 0)} \]

for any \( (\tau'_0, \sigma'_0) \) with \( R(\tau'_0, \sigma'_0) = 0 \), \( (\tau, \sigma) \) with \( (\tau', \sigma') \in U(\tau'_0, \sigma'_0) \cap (C_+ \times \mathbb{R}^{n-1}) \) and \( k = 0, 1, \cdots, m-1 \). Here \( C(\tau'_0, \sigma'_0) \) and \( U(\tau'_0, \sigma'_0) \) are the same ones in Theorem 2.1.

PROOF. If \( \sigma_0 \in S \) and \( R(\tau_0, \sigma_0) = 0 \), then \( \tau_0 \) is an isolated point in \( C_+ \). In virtue of (2.3) and the relations \( \langle D_{x_n}^k G(x_n, s, \tau, \sigma) \rangle = \rho^{k-m+1} \rho^{-2\nu} \int_0^\infty |(D_s^\nu f)(\rho^{-1}s)|_{L^2(x_n > 0)} \]

(2.5) \[
\leq \rho^{k-m-1} \rho^{-2\nu} \int_0^\infty |g(x_n)\|_{L^2(x_n > 0)} \int_0^\infty |(D_s^\nu f)(\rho^{-1}s)|_{L^2(x_n > 0)} \]

\[
\leq \rho^{k-m-\frac{1}{2}} C(\tau'_0, \sigma'_0) (Re \tau')^{-\nu-1} \rho^{k-m+1} \rho^{-2\nu} \int_0^\infty |(D_s^\nu f)(\rho^{-1}s)|_{L^2(x_n > 0)} \]
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$$= C(\tau_0, \sigma_0)(Re\tau)^{-1} \rho^{-m+k+1} \|g\|_{L^2(x_n>0)} \left( \sum_{\mu=0}^\nu \rho^{-2\mu} \int_0^\infty |(D_\epsilon^\mu f)(s)|^2 ds \right)^{1/2}$$

$$= C(\tau_0, \sigma_0)(Re\tau)^{-1} \rho^{k-m+1} \|g\|_{L^2(x_n>0)} \left( \sum_{\mu=0}^\nu \rho 2^{-2\mu} \int_0^\infty |(D_\epsilon^\mu f)(s)|^2 ds \right)^{1/2}$$

where $(\tau, \sigma)$ with $(\tau', \sigma') \in (\tau_0, \sigma_0) \cap V^t \ \Sigma_+$. In particular, it follows from above that, with some $C(\tau_0, \sigma_0)$,

$$|\left(\int_0^\infty (D_{x_n}^k G)(x_n, s, \tau, \sigma_0)f(s)ds, g(x_n)\right)_{L^2(x_n>0)}| \leq C(\tau_0, \sigma_0) \|f\|_{H_0^\nu}(s>0) \|g\|_{L^2(x_n>0)}$$

in a small neighborhood of $\tau_0$. Hence $(\int_0^\infty (D_{x_n}^k G)(x_n, s, \tau, \sigma_0)f(s)ds, g(x_n))_{L^2(x_n>0)}$ has an analytic extension in $C_+$. By Riesz theorem and (2.6) there exist operators $\tilde{G}_k(\tau, \sigma_0)$ ($k=0, 1, \ldots, m-1$) from $H_0^\nu(s>0)$ to $L^2(x_n>0)$ such that $\tilde{G}_k(\tau, \sigma_0)f, g)_{L^2(x_n>0)}$ is analytic in $C_+$ and

$$(\tilde{G}_k(\tau, \sigma_0)f, g)_{L^2(x_n>0)} = (\int_0^\infty (D_{x_n}^k G)(x_n, s, \tau, \sigma_0)f(s)ds, g(x_n))_{L^2(x_n>0)}$$

for $\tau \neq \tau_0$. In virtue of Banach-Steinhaus theorem there exist uniform derivatives

$$\left(\frac{d}{d\tau} \tilde{G}_k(\tau, \sigma_0)f, g\right)_{L^2(x_n>0)} = \left(\frac{d}{d\tau} \tilde{G}_k(\tau, \sigma_0)f, g\right)_{L^2(x_n>0)}$$

such that

$$\frac{d}{d\tau} (\tilde{G}_k(\tau, \sigma_0)f, g)_{L^2(x_n>0)} = \left(\frac{d}{d\tau} \tilde{G}_k(\tau, \sigma_0)f, g\right)_{L^2(x_n>0)}.$$

This shows that $\tilde{G}_k(\tau, \sigma_0)$ is an analytic extension in $C_+$ of $(D_{x_n}^k G)(x_n, s, \tau, \sigma_0)$ as an operator from $H_0^\nu(s>0)$ to $L^2(x_n>0)$ and $\|\tilde{G}_k(\tau, \sigma)f\|_{L^2(x_n>0)}$ is the same bound as (2.4). Since

$$(-1)^k \left(\tilde{G}(\tau_0, \sigma_0)f, D_{x_n}^k g\right)_{L^2(x_n>0)}$$

$$= \lim_{\tau \to \tau_0} (-1)^k \left(\int_0^\infty G(x_n, s, \tau, \sigma_0)f(s)ds, (D_{x_n}^k g)(x_n)\right)_{L^2(x_n>0)}$$

$$= \lim_{\tau \to \tau_0} \left(\int_0^\infty (D_{x_n}^k G)(x_n, s, \tau, \sigma_0)f(s)ds, g(x_n)\right)_{L^2(x_n>0)}$$

$$= \left(\tilde{G}_k(\tau_0, \sigma_0)f, g\right)_{L^2(x_n>0)}$$

where $\tilde{G}(\tau, \sigma_0) = \tilde{G}_0(\tau, \sigma_0)$ and $g \in C_0^\infty(R)$. $D_{x_n}^k (\tilde{G}(\tau_0, \sigma_0)f)$ is in $L^2(x_n>0)$ and equal to $\tilde{G}_k(\tau_0, \sigma_0)f$. This finishes the proof.

Let us set
\( \hat{u}(\tau, \sigma, x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\sigma x_n}}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \tilde{G}(\tau, \sigma) \hat{f}(\tau, \sigma, s) \)

where \( f \in C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^n_+) \). Then by Lemma A and the method of the proof of Theorem 4.1 in [2] we have the following

**Lemma B.** \( \hat{u}(\tau, \sigma, x_n) (\sigma \in S) \) is a solution in \( H^n(x_n > 0) \) of (2.1) and satisfies

\[
\|\hat{u}(\tau, \cdot, \cdot)\|_{m-1}^2 \leq C(Re\tau)^{-2(v+1)1} \|f(\tau, \cdot, \cdot)\|_{\nu}^2,
\]

\[
\|\hat{u}(\tau, \cdot, \cdot)\|_m^2 \leq C(Re\tau)^{-2(v+1)} \|f(\tau, \cdot, \cdot)\|_{\nu+1}^2
\]

where

\[
\|\hat{u}(\tau, \cdot, \cdot)\|_k^2 = \sum_{j=0}^{k} \int_{\mathbb{R}^{n-1}} \rho^{2(k-j)} d\sigma \int_0^\infty |(D^{i}x_n \vee n)(\tau, \sigma, x_n)|^2 dx_n.
\]

Let us set

\[
u(t, x) = \frac{1}{(2\pi)^n} \int_{a-i\infty}^{a+i\infty} \int_{\iota^{n-1}} \hat{u}(\tau, \sigma, x_n) d\sigma d\tau
\]

where \( f \in C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^n_+) \) and \( \hat{u}(\tau, \sigma, x_n) \) is defined by (2.7). Then by Lemma B \( u(t, x) \) is a solution of the problem \( (P, B_j) \) which satisfies \( e^{-at}u \in H^n(\mathbb{R}_+ \times \mathbb{R}^n_+) \), (1.1) and

\[
a^{2(v+1)} \sum_{j+|\alpha| \leq m} \int_{\mathbb{R}_+^n} e^{-2at} |(D^j \partial^\alpha_t u)(t, x)|^2 dt dx
\]

\[
\leq C \sum_{j+|\alpha| \leq \nu+1} \int_{\mathbb{R}_+^n} e^{-2at} |(\partial^\alpha_t D^j f)(t, x)|^2 dt dx.
\]

\( C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^n_+) \) is dense in \( H^{\nu+1}_0(\mathbb{R}_+ \times \mathbb{R}^n_+) \). By (1.1), (2.9) and the limit process we obtain a solution \( u(t, x) \) for \( f \in H^{\nu+1}_0(\mathbb{R}_+ \times \mathbb{R}^n_+) \).

(II) Necessity and uniquenness of solutions.

**Lemma C.** Suppose that for fixed \( \tau_0 \in C_+ \{B_j(\tau_0, \sigma, D_s x_n)\} \) is normal in a bounded open set \( D \subset \mathbb{R}^{n-1} \) and \( R(\tau_0, \sigma) \neq 0 \) for \( \sigma \in D \). Let \( \hat{u}(\sigma, x_n) \) be a function whose distribution derivatives in \( x_n \) up to \( m \) belong to \( L^2(D \times \mathbb{R}_+) \). Then the problem (2.1) has the following uniqueness property; that is, if for any \( \hat{\varphi}(\sigma, x_n) \in C^\infty_0(D \times \mathbb{R}_+) \) whose support in \( \sigma \) is contained in \( D \)

\[
(P(\tau_0, \sigma, D_{x_n}) \hat{u}, \hat{\varphi})_{L^2(D \times \mathbb{R}_+)} = 0,
\]

\[
(B_j(\tau_0, \sigma, D_{x_n}) \hat{u})_{x_n=0}, \hat{\varphi}_{x_n=0})_{L^2(D)} = 0 \quad (j=1, \ldots, l),
\]

then \( \hat{u} = 0 \) in \( L^2(D \times \mathbb{R}_+) \).

To prove Lemma C we may use the dual problem.
**Lemma D.** Suppose that \( \{B_j(\tau_0, \sigma_0, D_{x_n})\} \) is not normal and \( R(\tau_0, \sigma_0) \neq 0 \). Then there exist a point \((\tilde{\tau}_0, \tilde{\sigma}_0)\) sufficiently close to \((\tau_0, \sigma_0)\) and a normal set \( \{B'_j(\tilde{\tau}_0, \sigma, D_{x_n})\} \) in a sufficiently small neighborhood \( D(\tilde{\sigma}_0) \) such that Lopatinskii's determinant for \( (P(\tilde{\tau}_0, \sigma, D_{x_n}), B'_j(\tilde{\tau}_0, \sigma, D_{x_n})) \) does not vanish in \( D(\tilde{\sigma}_0) \) and \( (P(\tilde{\tau}_0, \sigma, D_{x_n}), B_j(\tilde{\tau}_0, \sigma, D_{x_n})) \) and \( (P(\tilde{\tau}_0, \sigma, D_{x_n}), B'_j(\tilde{\tau}_0, \sigma, D_{x_n})) \) in \( D(\tilde{\sigma}_0) \) have the same solutions.

**Proof.** Let us set

\[
B_j(\tau, \sigma, D_{x_n}) = \sum_{k=0}^{m_j'} b_{j,k}(\tau, \sigma) D_{x_n}^k.
\]

Here we may assume that \( m_j' \leq m_j \) and \( b_{j,m_j'}(\tau, \sigma) \) is identically not zero in a neighborhood of \((\tau_0, \sigma_0)\). Furthermore we may assume that \( R(\tau, \sigma) \neq 0 \) in a neighborhood of \((\tau_0, \sigma_0)\). We carry out the following two processes in this neighborhood: First if \( b_{j,m_j'}(\tau_0, \sigma_0) = 0 \) then there is a point \((\tilde{\tau}_0, \tilde{\sigma}_0)\) closed to \((\tau_0, \sigma_0)\) such that \( b_{j,m_j'}(\tilde{\tau}_0, \sigma) \neq 0 \) in a neighborhood of \( \tilde{\sigma}_0 \). Thus we replace \( B_j(\tau_0, \sigma_0, D_{x_n}) \) by \( b_{j,m_j'}(\tilde{\tau}_0, \sigma)^{-1} B_j(\tilde{\tau}_0, \sigma, D_{x_n}) \). Second, after the first process, if \( m'_j = m'_k \) then we replace \( B_j \) or \( B_k \) by \( B_j - B_k \). Remark that in these processes it is invariant that Lopatinskii's determinant does not vanish. Hence, after each process, it does not occur the case that \( B_j \) and \( B_k \) \((j \neq k)\) are monomials in \( D_{x_n} \) of same degree. Therefore Lemma D is obtained by carrying out successively these processes from \( B_j \) of highest order in \( D_{x_n} \).

To prove the uniqueness let \( \hat{u}(\tau, \sigma, x_n)(Re\tau \geq a) \) be the Fourier-Laplace transform of a solution \( u \) of \( (P, B_j)(f = 0, e^{-at} u \in H^m(R_+ \times R^n)) \). Then for an arbitrarily fixed point \( \tau_0 \) with \( Re\tau_0 \geq a \) the two equations in Lemma C are valid for any \( \hat{\phi}(\sigma, x_n) \in C_0'(R^{n-1} \times R_+) \). From the proof of Lemma D we see that the set \( Q \) of all the points \((\tilde{\tau}_0, \tilde{\sigma}_0)\) with \( Re\tilde{\tau}_0 \geq a \) satisfying the conclusion in Lemma D is almost everywhere equal to \( \{\tau; Re\tau \geq a\} \times R^{n-1} \). Therefore it follows from Lemma C that \( \hat{u} = 0 \) in \( L^2(Q \times R_+) \), which implies that for some \( \alpha'(\alpha' \geq a) \) \( \hat{u}(\alpha' + i\eta, \sigma, x_n) = 0 \) almost everywhere in \( (\eta, \alpha, x_n) \).

Now we prove the necessity of Theorem 2.1. First, in the proof of theorem 4.1 in [2] pp. 142–144, a sequence \( \{(\tau'_p, \sigma'_p)\} \) may be replaced by \( \{(\tilde{\tau}'_p, \tilde{\sigma}'_p)\} \) where the conclusion of Lemma D is satisfied for each point \((\tilde{\tau}'_p, \tilde{\sigma}'_p)\). Here it may be assumed that if \( p \to \infty \),

\[
(Re\tilde{\tau}'_p)^{n+1} \left\| (D_{x_n}^k G)(x_n, s, \tilde{\tau}'_p, \tilde{\sigma}'_p) \right\| \rightarrow \infty
\]

because \( \left\| (D_{x_n}^k G)(x_n, s, \tau, \sigma) \right\| \rightarrow \infty \) is continuous in \((\tau, \sigma)\).

Second we use Lemma C in order to show the inequality in lines 7–10 in [2] p. 143. Thus we may prove our assertion as we have done in [2].
§ 3. Lopatinskii's determinant

In this section we prove the following

THEOREM 3.1. Suppose that a constant coefficient problem \((P, B_j)\) is \(L^2\)-well-posed with decreasing order \(\nu\) and \(R(\tau, \sigma)\) is identically not zero. Then \(R(\tau_0, \sigma_0)\neq 0\) for \((\tau_0, \sigma_0)\in \mathbb{C}_+ \times \mathbb{R}_{n-1}\) is equivalent to the fact that \(B_j(\tau_0, \sigma_0, \lambda)\) \((j=1, \cdots, l)\) are linearly independent as polynomials in \(\lambda\).

From Theorem 3.1 we obtain immediately

COROLLARY 3.2. Under the assumptions of Theorem 3.1. and the normality of \(\{B_j\}\), \(R(\tau, \sigma)\neq 0\) for any \((\tau, \sigma)\in \mathbb{C}_+ \times \mathbb{R}^{n-1}\).

PROOF OF THEOREM 3.1. \(R(\tau_0, \sigma_0)\neq 0\) for \((\tau_0, \sigma_0)\in \mathbb{C}_+ \times \mathbb{R}^{n-1}\) is equivalent to the fact that \(B_j(\tau_0, \sigma_0, \lambda)\) are linearly independent modulo \(\prod_{j=1}^{l}(\lambda-\lambda_j^+)(\tau_0, \sigma_0)\) as polynomials in \(\lambda\). Hence the necessity is obvious.

Let us set \(B_j(\sigma, \tau, \lambda)=b_{j,k}(\tau, \sigma)\lambda^k\). Since \(B_j(\tau_0, \sigma_0, \lambda)\) are linearly independent, the matrix \((b_{j,k}(\tau, \sigma))\) has rank \(l\); that is, there exist \((k_1, \cdots, k_l)\) and a neighborhood \(U(\tau_0, \sigma_0)\) in \(\mathbb{C}_+ \times \mathbb{R}^{n-1}\) such that

\[\det(b_{j,k_h}(\tau, \sigma);_{h\rightarrow}^{j\downarrow}1, \cdots, l)\neq 0 \quad \text{in} \quad U(\tau_0, \sigma_0).\]

Using (3.1) we can construct a function \(v\) satisfying

\[B_j(\tau, \sigma, D_{x_n})v|_{x_n=0} = g_j \quad (j=1, \cdots, l)\]

for any \(g_j \in \mathcal{C}\) and \((\tau, \sigma)\in U(\tau_0, \sigma_0)\). In fact, if \(v(\tau, \sigma, x_n)=\sum_{h=1}^{l}v_{k_h}(\tau, \sigma)x_n^{k_h}(k_h!)^{-1}\) then (3.2) becomes

\[\sum_{h=1}^{l}b_{j,k_h}(\tau, \sigma)v_{k_h}(\tau, \sigma) = g_j \quad (j=1, \cdots, l).\]

In the rest of this section \((\tau, \sigma)\) are considered as parameters and belong to \(U(\tau_0, \sigma_0) \cap V^c\), where \(U(\tau_0, \sigma_0)\) is assumed, if necessary, sufficiently small.

Let \(u_1\) be a solution of the Cauchy problem:

\[P(\tau, \sigma, D_{x_n})u_1 = P(\tau, \sigma, D_{x_n})(\varphi v) \quad x_n>0,\]
\[D_{x_n}^ku_1 = 0 \quad (k=0, \cdots, m-1) \quad x_n=0\]

where \(\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+)\) with \(\varphi = 1 \ (0 \leq x_n \leq 2^{-1})\) and \(\varphi = 0 \ (x_n \geq 1)\). We consider the problem:

\[P(\tau, \sigma, D_{x_n})u = P(\tau, \sigma, D_{x_n})(\varphi v - \varphi u_1) \quad x_n>0,\]
\[B_j(\tau, \sigma, D_{x_n})u = 0 \quad (j=1, \cdots, l) \quad x_n=0.\]

Since \(f=P(\tau, \sigma, D_{x_n})(\varphi(v-u_1))\in \mathcal{C}_0^\infty(x_n>0)\), the problem has a unique solution:
On necessary and sufficient conditions for $L^2$-well-posedness of mixed problems

\[ u(\tau, \sigma, x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_n \lambda} \hat{f}(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \int_{0}^{\infty} G(x_n, \tau, \sigma) f(\tau, \sigma, s) ds \]

\[ = \tilde{u}_1(\tau, \sigma, x_n) + \tilde{u}_2(\tau, \sigma, x_n). \]

Since $|P(\tau, \sigma, \lambda)|^2 \geq C(Re\tau)^2 (|\tau|^2 + |\sigma|^2 + |\lambda|^2)^{m-1}$, we obtain for some constant $C(\tau_0, \sigma_0) > 0$

\[ ||\tilde{u}_1(\tau, \sigma, x_n)||_{B^{m-1}(x_n > 0)} \leq C(\tau_0, \sigma_0) ||f(\tau, \sigma, x_n)||_{L^2(x_n > 0)}. \]

Furthermore it follows from the assumption and Theorem 2.1 that for some constant $C(\tau_0, \sigma_0) > 0$

\[ ||\tilde{u}_2(\tau, \sigma, x_n)||_{L^2(x_n > 0)} \leq C(\tau_0, \sigma_0) ||f(\tau, \sigma, x_n)||_{L^2(x_n > 0)} \]

From (3.3) and (3.4) we have

\[ ||u(\tau, \sigma, x_n)||_{L^2(x_n > 0)} \leq C(\tau_0, \sigma_0) ||f(\tau, \sigma, x_n)||_{L^2(x_n > 0)}. \]

If we put $w = \varphi v - \varphi u_1 - u$, $w$ is an $L^2$-solution of

\[ P(\tau, \sigma, D_{x_n}) w = 0 \quad x_n > 0, \]

\[ B_j(\tau, \sigma, D_{x_n}) w = g_j \quad (j = 1, \cdots, l) \quad x_n = 0. \]

Furthermore, by (3.5) and the construction of $v$ and $u_1$, we see that for some $C(\tau_0, \sigma_0) > 0$

\[ ||w(\tau, \sigma, x_n)||_{L^2(x_n > 0)} \leq C(\tau_0, \sigma_0) \sum_{j=1}^{l} |g_j|^2. \]

On the other hand, if $R(\tau, \sigma) \neq 0$ then the problem (3.6) has a unique solution in $L^2(x_n > 0)$ which is written by the form

\[ w(\tau, \sigma, x_n) = \sum_{j=1}^{l} \frac{R_j(\tau, \sigma, x_n)}{R(\tau, \sigma)} g_j. \]

Now we arrange the roots $\lambda_+^k(\tau, \sigma)$ into $q$-groups $\{\lambda_+^{k,h}(\tau, \sigma) \ h = 1, 2, \cdots, k'\}$ ($k=1, \cdots, q$) in a sufficiently small neighborhood $U(\tau_0, \sigma_0)$ such that $\lambda_+^{k,1}(\tau, \sigma) = \cdots = \lambda_+^{k,k'}(\tau_0, \sigma_0)$. Let us set

\[ \gamma_{k,1}(\tau, \sigma, x_n) = \exp(ix_n \lambda_+^{k,1}(\tau, \sigma)), \]

\[ \gamma_{k,h}(\tau, \sigma, x_n) = (ix_n)^{h-1} \int_{0}^{1} d\theta_1 \cdots d\theta_{h-2} \int_{0}^{1} \theta_1^{h-2} \cdots \theta_{h-2} \exp(ix_n g_{k,h}(\tau, \sigma, \theta)) d\theta_{h-1}, \]

\[ B_j^{k,h}(\tau, \sigma) = \int_{0}^{1} d\theta_1 \cdots d\theta_{h-2} \int_{0}^{1} \theta_1^{h-2} \cdots \theta_{h-2} (\partial_{\lambda}^{h-1} B_j)(\tau, \sigma, g_{k,h}(\tau, \sigma, \theta)) d\theta_{h-1}, \]

\[ g_{k,h}(\tau, \sigma, \theta) = \lambda_+^{k,1}(\tau, \sigma) + (\lambda_+^{k,2}(\tau, \sigma) - \lambda_+^{k,1}(\tau, \sigma)) \theta_1 + \cdots + (\lambda_+^{k,h}(\tau, \sigma) - \lambda_+^{k,h-1}(\tau, \sigma)) \theta_1 \cdots \theta_{h-1} \quad (h \geq 2). \]
Then we have

$$R(\tau, \sigma) = \frac{B_{1}^{k,1}(\tau, \sigma) \cdots B_{1}^{k,1}(\tau, \sigma) \cdots}{\Delta(\tau, \sigma)}$$

$$= R'(\tau, \sigma)/\Delta(\tau, \sigma),$$

$$R_{j}(\tau, \sigma, x_{n}) = \frac{B_{1}^{k,1}(\tau, \sigma) \cdots T_{k,1}(\tau, \sigma, x_{n}) \cdots B_{l}^{k,1}(\tau, \sigma) \cdots}{\Delta(\tau, \sigma)}$$

$$= R_{j}'(\tau, \sigma, x_{n})/\Delta(\tau, \sigma)$$

where $\Delta(\tau, \sigma) \neq 0$ in $U(\tau_{0}, \sigma_{0})$. If follows from (3.8) that

$$w(\tau, \sigma, x_{n}) = \sum_{k,h}(\sum_{j=1}^{l} \frac{A_{j}^{k,h}(\tau, \sigma)}{R'(\tau, \sigma)}g_{j})\gamma_{k,h}(\tau, \sigma, x_{n})$$

where $A_{j}^{k,h}$ is a cofactor of $R'(\tau, \sigma)$ with respect to $B_{j}^{k,h}(\tau, \sigma)$. Since $T_{k,1}(\tau, \sigma, x_{n})$ are linearly independent, it follows from (3.7) and the same method in [2] p. 146 that for some $C(\tau_{0}, \sigma_{0}) > 0$

$$(3.9)$$

$$\left| \frac{A_{j}^{k,h}(\tau, \sigma)}{R'(\tau, \sigma)} \right| < C(\tau_{0}, \sigma_{0})$$

$$(j=1, \cdots, l, \ k=1, \cdots, q \ \text{and} \ h=1, \cdots, k').$$

By the definition of $A_{j}^{k,h}(\tau, \sigma)$ we have

$$R'(\tau, \sigma)^{-1} = \det\left(\frac{A_{j}^{k,h}(\tau, \sigma)}{R'(\tau, \sigma)}\right).$$

Hence it follows from this and (3.9) that for some $C(\tau_{0}, \sigma_{0}) > 0$

$$R'(\tau, \sigma) > C(\tau_{0}, \sigma_{0}).$$

In virtue of the continuity of $R(\tau, \sigma)$ we conclude that $R(\tau_{0}, \sigma_{0}) \neq 0$.

§ 4. Necessary condition for $L^{2}$-well-posedness
(The case of variable coefficients)

In this section we consider variable coefficient problems $(P, B_{j})$. Here coefficients are smooth and constant except a compact set in $R^{n+1}$. 

Let $(P^0, B^0_j)_{(t, x')}$ be a constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point $(t, x', 0)$ and $R(t, x'; \tau, \sigma)$ be Lopatinskii's determinant for the problem $(P^0, B^0_j)_{(t, x')}$. Then we have the following

**Theorem 4.1.** Suppose that a variable coefficient problem $(P, B_j)$ is strongly $L^2$-well-posed and $R(t, x'; 1, 0) \neq 0$ for any boundary point $(t, x', 0)$. Then each constant coefficient problem $(P^0, B^0_j)_{(t,x')}$ is $L^2$-well-posed (with $\nu = 0$).

**Proof.** First we shall show that for an arbitrarily fixed boundary point $(t_0, x_0', 0) (0 \leq t_0 < T)$ the problem $(P^0, B^0_j)_{(t_0, x_0')}$ with $f = 0$ and initial data $U(t_1) = (u_0, u_1, \cdots, u_{m-1}) (0 \leq t_1 < T)$ is strongly $L^2$-well-posed. Here $u_k \in C^\infty_0 (R^n_+)$ for $k = 0, 1, \cdots, m - 1$. By the assumption there exists a unique solution $v_r \in C^0((t_0, T), H^m(R^n_+)) \cap \cdots \cap C^m((t_0, T), H^0(R^n_+))$ of the problem $(P, B_j)$ with $f = 0$ and initial data $V_r(t_0) = (v_{0, r}, v_{1, r}, \cdots, v_{m-1, r})$ which satisfy energy inequalities (1.2) and (1.3). Here $v_{k, r}(x) = \epsilon^{-k} u_k((x' - x_0') \epsilon^{-1}, x_n \epsilon^{-1})$. Let us set $u_r(s, y) = v_r(t_0 + \epsilon(s - t_1), x_0' + \epsilon y', \epsilon y_n)$. Then $u_r(s, y)$ becomes a solution of the equations:

for $s \in [t_1, t_1 + \epsilon^{-1}(T - t_0))$

\[
P(t_0 + \epsilon(s - t_1), x_0' + \epsilon y', \epsilon y_n; \epsilon^{-1} \partial_s, \epsilon^{-1} D_y) u = 0 \quad \text{in} \quad y \in R^n_+,
\]

\[
B_j(t_0 + \epsilon(s - t_1), x_0' + \epsilon y'; \epsilon^{-1} \partial_s, \epsilon^{-1} D_y) u = 0 \quad (j = 1, \cdots, l) \quad \text{in} \quad y_n = 0, y' \in R^{n-1},
\]

with initial data $(s = t_1)$

\[
\partial^*_s u_r = \epsilon^k v_{k, r}(x_0' + \epsilon y', \epsilon y_n) = u_k(y) \quad (k = 0, 1, \cdots, m - 1) \quad \text{in} \quad y \in R^n_+.
\]

Furthermore it follows from (1.2) and (1.3) that $u_r$ satisfy

\[
\| u_r(s, \cdot) \|_{m-1, \cdot} \leq C \| U(t_1) \|_{m-1, \cdot} \quad (t_1 \leq s \leq t_1 + \epsilon^{-1}(T - t_0))
\]

\[
\| u_r(s, \cdot) \|_{m, \cdot} \leq C \| U(t_1) \|_{m, \cdot}
\]

where

\[
\| u(s, \cdot) \|_{k, \cdot} = \| u(s, \cdot) \|_k + \sum_{k=0}^{k-1} \epsilon^{k-h} | u(s, \cdot) |_h
\]

and $| u(s, \cdot) |_k$ denotes the norm obtained by replacing $| \alpha | \leq k$ in definition of $\| u(s, \cdot) \|_k$ by $| \alpha | = k$. Therefore, by (4.1) and (4.2), there exists a weak limit $u(s, \cdot)$ of a subsequence of $\{ u_r(s, \cdot) \}$ in $H^m(R^n_+)$ as $\epsilon \to 0$ such that $u(s, y)$ is a solution in $C^0((t_1, T), H^m(R^n_+)) \cap \cdots \cap C^m((t_1, T), H^0(R^n_+))$ of the problem $(P^0, B^0_j)_{(t_1, x_0')}$ and satisfies the energy inequalities.
Since $R(t_0, x'_t ; 1, 0) \neq 0$ the problem $(P^0, B^0_j)_{(t_0, x'_0)}$ has a finite propagation speed (See [6]). Hence the problem $(P^0, B^0_j)_{(t_0, x'_0)}$ has a unique solution. Using Poincaré lemma and the finiteness of propagation speed it follows from (4.3) that for any $s$ ($t_1 \leq s \leq T$)

$$|u(s, \cdot)|_{m-1} \leq C(s) |U(t_1)|_{m-1}$$

(4.4)

where $C(s)$ depends continuously on $s$ and propagation speed. Thus we can define an operator $G(s, t_1)$ from initial data $U(t_1) \in C_0^\infty(R_+^n)^m$ to the solution $u(s, y)$.

Next we shall show that the problem $(P, B_j)_{(t_0, x'_0)}$ with $f \in C_0^\infty((0, T) \times R_+^n)$ and zero initial data is $L^2$-well-posed ($\nu = 0$). Let us set

$$u(t, x) = \int_0^t G(t, s) F(s) \, ds$$

where $F(s) = (0, \ldots, 0, f(s, x))$. Then $u(t, x)$ becomes a solution of the problem $(P^0, B^0_j)_{(t_0, x'_0)}$ such that, by (4.4),

$$\|u(t, \cdot)\|_{m-1}^2 \leq C(t) \int_0^t \|f(t, \cdot)\|_2^2 \, dt \quad (0 \leq t \leq T)$$

By integrating (4.5) from 0 to $T$ we obtain for some $C(T) > 0$

$$\int_0^T \|u(t, \cdot)\|_{m-1}^2 \, dt \leq C(T) \int_0^T \|f(t, \cdot)\|_2^2 \, dt.$$  

(4.6)

Therefore the problem $(P^0, B^0_j)_{(t_0, x'_0)}$ with $f \in C_0^\infty((0, T) \times R_+^n)$ and homogeneous initial-boundary conditions has a unique solution $u \in H^m((0, T) \times R_+^n)$ satisfying (4.6). Only this fact is used in the proof of the necessity of Theorem 4.1 in [2]. Thus the proof is complete.

Finally we consider mixed problems of second order:

$$P = \partial_t^2 - 2 \sum_{j=1}^n a_j(t, x) \partial_t \partial_{x_j} - \sum_{j,k=1}^n a_{jk}(t, x) \partial_{x_j} \partial_{x_k} + \text{first order term},$$

$$B = \partial_{x_n} - \sum_{j=1}^{n-1} b_j(t, x') \partial_{x_j} - c(t, x') \partial_t + h(t, x')$$

where $\sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k > 0$ for any non zero vector $\xi \in R^n$ and all the coefficients are real valued.

Combining Theorem 4.1 with results in [1] we obtain

**Theorem 4.2.** Suppose that $R(t, x' ; 1, 0) \neq 0$ on the boundary. Then
a variable coefficient problem \((P, B)\) is strongly \(L^2\)-well-posed if and only if each constant coefficient problem \((P^0, B^0)\) is \(L^2\)-well-posed (with \(\nu=0\)).

**Remark.** Let \((P^0, B^0)\) denotes \((P^0, B^0)(t, x')\) for a fixed point \((t, x', 0)\). Then the following statements (I), (II) and (III) are equivalent:

(I) \((P^0, B^0)\) is \(L^2\)-well-posed.

(II) Lopatinskiǐ's determinant for \((P^0, B^0)\) does not vanish in \(C_+ \times \mathbf{R}^{n-1}\) and \((P, B)\) has no supersonic speeds.

(III) \(a_{nn}c + a_n \geq 0\) and the quadratic form \(H(\sigma) = (a_{nn}c + a_n)^2 (a_{nn}e - b^2) - 2(a_{nn}c + a_n)(a_{nn}a - a_n b)(a_{nn}b + b) - (a_{nn} + a_n^2)(a_{nn}b + b)^2\) is positive semi-definite, where

\[
\begin{align*}
e &= \sum_{j,k=1}^{n-1} a_{jk} \sigma_j \sigma_k, \\
b &= \sum_{j=1}^{n-1} a_{nj} \sigma_j, \\
a &= \sum_{j=1}^{n-1} a_j \sigma_j, \\
b &= \sum_{j=1}^{n-1} b_j \sigma_j, \\
\end{align*}
\]

\((\sigma \in \mathbf{R}^{n-1})\).

The equivalence (I) and (III) has been proved in [1] and other equivalences are proved by using results in § 2 of [1] and [6].

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**References**


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