IMMERSIONS OF TOPOLOGICAL MANIFOLDS

BY

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1. Introduction, Definitions and Notation.

Let M and Q be topological manifolds, with $\dim M \leq \dim Q$. In [2] Gauld proved the classification theorem of M' immersions for M, when topological manifold M' is given such as M a locally flat submanifold of the interior of M' and $\dim M' = \dim Q$. An M' immersion means an immersion of V in Q, where V is a neighbourhood of M in M' and two such are identified if they agree over a common neighbourhood of M. cf. Lashof [6], Less [7] and Kirby [5].

Considering an induced neighbourhood of an immersion $I^r \times M \rightarrow I^r \times Q$ with uniformity, we obtain the classification theorem for immersions of topological manifolds instead of M' immersion, where topological manifold is paracompact Hausdorff. Similar result was proven by Hirsh [4] in differentiable case, and by Haefliger and Poenaru [3] in PL case.

Theorem 1. *Let M^m, Q^q be manifolds with $\dim M < \dim Q$. Let N be a closed subset of M. Suppose*

$$\theta : U \longrightarrow Q$$

be an immersion of a neighbourhood of N in M.
Then the correspondance which assigns to an immersion

$$f : M \longrightarrow Q$$

its differential

$$df : TM \longrightarrow TQ$$

induces a bijection between the regular homotopy classes relative to N of immersions of M in Q and the homotopy classes relative to N of representations from TM to TQ.

We define two (complete) semi-simplicial complexes $\mathfrak{J}_\theta(M,Q)$ and $\mathfrak{R}_\theta(M, Q)$.

Definition of $\mathfrak{J}_\theta(M, Q)$. A typical k-simplex of $\mathfrak{J}_\theta(M, Q)$ is a map

$$f : \Delta^k \times M \longrightarrow \Delta^k \times Q$$

which has the uniform immersion property with respect to Δ^k, i.e. for each $(t, x) \in \Delta^k \times M$, there exists a neighbourhood U of t in Δ^k and local charts

$$g : U\times R^m \longrightarrow U\times M$$
$$h : U\times R^q \longrightarrow U\times Q$$

such that g and h commute with the projection on U. Furthermore f must agree with $id_{\Delta^k}\times\theta$ on some neighbourhood of N in M.

Definition of $\mathfrak{R}_\theta(M, Q)$. This is an s. s. complex of representative germs of TM in TQ, a k-simplex of $\mathfrak{R}_\theta(M, Q)$ is represented by a pair (Φ, φ) such that the following diagram commutes.

$$\begin{array}{ccc}
\Delta_k\times M & \xrightarrow{\quad\varphi\quad} & \Delta^k\times Q \\
\Big\downarrow{\scriptstyle id\times\Delta} & & \Big\downarrow{\scriptstyle id\times\Delta} \\
\Delta^k\times V & \xrightarrow{\quad\Phi\quad} & \Delta^k\times Q\times Q \\
\Big\downarrow{\scriptstyle id\times pr_1} & & \Big\uparrow{\scriptstyle id\times pr_1} \\
\Delta^k\times M & \xrightarrow{\quad\varphi\quad} & \Delta^k\times Q
\end{array}$$

where $\Delta: M\to M\times M$ is the diagonal map, and V is a neighbourhood of $\Delta(M)$ in $M\times M$. φ and Φ must satisfy the following properties. They commute with the projection on the Δ^k, φ agrees with $id\times\theta$ on $\Delta^k\times$ (a neighbourhood of N in M), Φ agrees with $id\times\theta\times\theta$ on $\Delta^k\times$ (a neighbourhood of $\Delta(N)$ in $M\times M$), and the map: $\Delta^k\times V\to\Delta^k\times M\times Q$ given by $(t, x, x')\to(t, x, pr_3\Phi(t, x, x'))$ has the uniform immersion property with respect to Δ^k.

Face operations are defined by restrictions to a paticular face. Each of the above s. s. complexes is a Kan complex without degeneracies. cf. Rourke and Sanderson [10].

A 0-simplex of $\mathfrak{J}_\theta(M, Q)$ is called an immersion. Two immersions $f, f' : M\to Q$ are regularly homotopic relative to N if and only if they determine vertices of the same 1-simples of $\mathfrak{J}_\theta(M, Q)$. Two representations

$$(\Phi, \varphi),\ (\Phi', \varphi') : TM \longrightarrow TQ$$

are homotopic relative to N if and only if they determine vertices of the same 1-simplex of $\mathfrak{R}_\theta(M, Q)$. When $N=\phi$, we omit the subscript θ.

When M is a manifold with boundary, M is a submanifold of the interior of M', for some manifold M' with $\dim M=\dim M'$. For instance such a manifold M' is obtained from the disjoint union of M and $\partial M\times I$ by identifying $x\in\partial M$ with $(x, 0)\in\partial M\times I$. Moreover if M'' is other manifold of $\dim M$ containing M in its interior, then there exists a homeomorphism of a neighbourhood of M in M' onto a neighbourhood of M in M''

fixing M. By an immersion of a manifold M with boundary in Q, we mean an immersion

$$f: V \longrightarrow Q$$

where V is a neighbourhood of M in M', and such two immersions are identified if they agree over a common neighbourhood of M. The tangent bundle TM of M is defined by $TM'|M$. In this case $\mathfrak{J}_\theta(M, Q)$ and $\mathfrak{R}_\theta(M, Q)$ are independent on the choice of M'.

Definition. The differential

$$d: \mathfrak{J}_\theta(M, Q) \longrightarrow \mathfrak{R}_\theta(M, Q)$$

is an s.s. map as follows. For k-simplex $f \in \mathfrak{J}_\theta(M, Q)$, $d(f) = \Phi$, where

$$\Phi: \Delta^k \times TM \longrightarrow \Delta^k \times TQ$$

is defined by $\Phi(t, x, x') = (t, pr_2 f(t, x), pr_2 f(t, x'.))$.

Now the main theorem is the following.

Theorem 2. *Let $\dim M < \dim Q$.*
The map

$$d: \mathfrak{J}_\theta(M, Q) \longrightarrow \mathfrak{R}_\theta(M, Q)$$

is a homotopy equivalence.

Theorem 2 implies Theorem 1 as an immediate corollary. When $\dim M = \dim Q$, the theorem is obtained in Gauld [2]. Therefore we suppose $\dim M < \dim Q$.

2. Proof of Theorem 2.

First we prove when $M - N$ is a handle body, that is, there exists a sequence of manifolds $N = M_0, M_2 \cdots, M_l = M$ $(l \leqq \infty)$ such that M_i is obtained from M_{i-1} by adding a handle. Suppose that M is obtained from M_0 by adding a k-handle and $\dim M = \dim M_0 < \dim Q$.

Lemma 3. *The map*

$$i^*: \mathfrak{J}(M, Q) \longrightarrow \mathfrak{J}(M_0, Q)$$

is a fibration.

Lemma 4. *The map*

$$i^*: \mathfrak{R}(M, Q) \longrightarrow \mathfrak{R}(M_0, Q)$$

is a fibration, where i^* is the restriction map induced by the inclusion

$$i: M_0 \subset\longrightarrow M.$$

Lemma 5. *The map*

$$d:\mathfrak{J}_\theta(B^k\times B^{m-k},Q)\longrightarrow\mathfrak{R}_\theta(B^k\times B^{m-k},Q)$$

is a homotopy equivalence, where θ is an immersion of $\partial B^k\times B^{m-k}$ in Q.

Lemma 6. *The map*

$$d:\mathfrak{J}(B^m,Q)\longrightarrow\mathfrak{R}(B^m,Q)$$

is a homotopy equivalence.

The fibres of the fibrations

$$i^*:\mathfrak{J}(M,Q)\longrightarrow\mathfrak{J}(M_0,Q)$$
$$i^*:\mathfrak{R}(M,Q)\longrightarrow\mathfrak{R}(M_0,Q)$$

are isomorphic to $\mathfrak{J}_\theta(B^k\times B^{m-k},Q)$ and $\mathfrak{R}_\theta(B^k\times B^{m-k},Q)$ respectively. By Lemma 5 of Gauld [2], a homotopy equivalence

$$d:\mathfrak{J}(M_0,Q)\longrightarrow\mathfrak{R}(M_0,Q)$$

implies the homotopy equivalence

$$d:\mathfrak{J}(M,Q)\longrightarrow\mathfrak{R}(M,Q)$$

By induction, we obtain theorem 2 when $M-N$ is a finite handle body. When $M-N$ is an infinite handle body, consider the projective system of homotopy equivalences given by the above. Taking the limite, the required result is obtained. cf. Philips [9].
In the general case the proof is similar to Gauld [2].

Proofs of Lemma 4–6 are similar to those of Gauld [2]. We need only to prove Lemma 3.

3. Induced neighbourhood by an immersion.

Suppose a map

$$F:I^k\times M\longrightarrow I^k\times Q$$

has an uniform immersion property.

Definition. A neighbourhood of M induced by f is a triple (M',i,Φ) such that M' is a manifold with $\dim M'=\dim Q$, $i:M\to M'$ is a locally flat embedding into the interior of M', a map

$$\Phi:I^k\times M'\longrightarrow I^k\times Q$$

has an uniform immersion property with respect to I^k, and the following diagram commutes.

$$\begin{array}{ccc} I^k\times M & \xrightarrow{\quad id\times i\quad} & I^k\times M' \\ & \searrow f \qquad \Phi \swarrow & \\ & I^r\times Q & \end{array}$$

In this section we prove the existance and the uniqueness of an induced neighbourhood by f, when M is compact.

Lemma 7 and 8 are due to Haefliger and Poenaru [3].

Lemma 7. *Suppose*

$$F: I^r\times M \longrightarrow I^r\times Q$$

han an uniform immersion property, and X is compact. Then for any $t\in I^r$ there exist a cubic neighbourhood I_t of t in I^r and an induced neighbourhood (M', i, Φ) by $F|I_t\times M$.

Proof. Fix $t\in I^r$. For any $x\in M$, there are a neighbourhood $U(x)$ of t in I^r and $V'(x)$ in M such that $F|U(x)\times V'(x)$ is an embedding of $U(x)\times V'(x)$ in $I^r\times Q$. Define the metric on M.

For the open covering $\{V'(x)\}_{x\in M}$ of M, there is $p>0$ such that any open ball $V_p(x)$ with the radius p and the center x is contained in some $V'(x')$. We can choose a finite covering $\{V_{p/3}(x_1), \cdots, V_{p/3}(x_l)\}$ of M. Let the covering $\{V_1, \cdots, V_l\}$ of M be a refinement of $\{V_{p/3}(x_1), \cdots, V_{p/3}(x_l)\}$ such that $\overline{V}_i\subset V_{p/3}(x_i)$ and $V_i\cap V_j=\phi$ implies $\overline{V}_i\cap\overline{V}_j=\phi$.

There are a neighbourhood U_i of t in I^r and an open set $N_i\subset Q$ such that $U_i\subset U(x_i)$, $F(U_i\times V_i)\subset U_i\times N_i$, and if $\overline{V}_i\cap\overline{V}_j=\phi$ and $\overline{V}_i\cup\overline{V}_j\subset V'(x)$ for some $x\in M$ then $N_i\cap N_j=\phi$. Let $I_t\subset\bigcap_i U_i$ be a cubic neighbourhood of t in I^r. Then the open coverings $\{V_i\}$, $\{N_i\}$ satisfy

1. $F|(I_t\times V_i)$ is an embedding of $I_t\times V_i$ into $I_t\times N_i$.
2. If $V_i\cap V_j=\phi$ and $V_i\cap V_l\neq\phi$, $V_j\cap V_l\neq\phi$ for some V_l, then $N_i\cap N_j=\phi$.

Now we construct a q-manifold M' as follows. Consider the disjoint sum $\bigcup_i\{i\}\times N_i$. The relation which identifies (j, x) and (k, y) if $x=y$ and $V_j\cap V_k\neq\phi$, is an equivalence relation. The quotient space M' of $\bigcup_i\{i\}\times N_i$ by this relation is q-manifold. The point represented by (j, x) is denoted by $[j, x]$.

The map

$$\varphi: M' \longrightarrow Q$$

where $\varphi([j, x]=x$ is an immersion.

The map

$$G: I_t\times M \longrightarrow I_t\times M'$$

defined by $G(t', x)=(t', [i, pr_2F(t', x)])$ with $x\in V_i$ is well-defined and an embedding commuting with the projection on the I_t factor.
Moreover

$$(id_{I_t}\times\varphi)\circ G = F|(I_t\times M).$$

Let the map

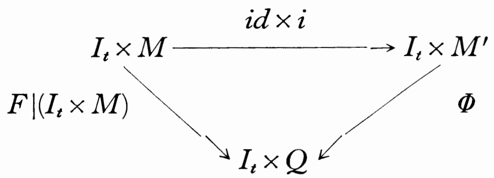
$$i: M \longrightarrow M'$$

is the embedding given by $i(x)=pr_2\circ G(t, x)$. By the Isotopy Extension Theorem of Edward and Kirby [1], there is a cube of homeomorphisms

$$H: I_t\times M' \longrightarrow I_t\times M'$$

such that $G=H\circ(id_{I_t}\times i)$.

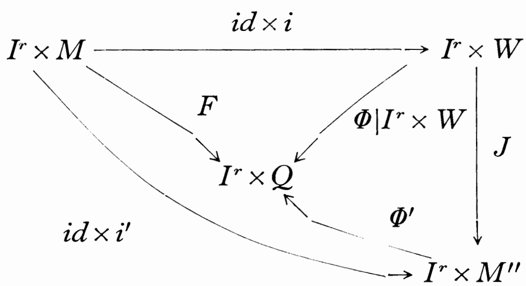
Let $\Phi=(id_{I_t}\times\varphi)\circ H$. Then Φ has an uniform immersion property with respect to I_t, and the following diagram commutes.



Lemma 8. *Suppose M is a compact manifold and a map*

$$F: I^r\times M \longrightarrow I^r\times Q$$

has an uniform immersion property with respect to I^r. If (M', i, Φ) and (M'', i', Φ') are neighbourhoods of M induced by F. Then there exist a neighbourhood W of $i(M)$ in M' and a homeomorphish J of $I^r\times W$ onto a neighbourhood of $I^r\times i'(M)$ in $I^r\times M''$ commuting with the projection on X with the following commutative diagram.



Proof. As in [3. §7. Proposition 2], there exist a open covering $\{W_i\}_{i=1\cdots,l}$ of $I^r\times i(M)$ in $I^r\times M'$ and $\{W_i'\}_{i=1\cdots,l}$ of $I^r\times i'(M)$ in $I^r\times M''$ such that the following conditions are satisfied.

1. $\Phi_i=\Phi\,|\,W_i$ and $\Phi_i'=\Phi'\,|\,W_i'$ are homeomorphisms of W_i and W_i' into $I^r\times Q$ respectively. And image $\Phi_i\subset$Image Φ_i'.

2. If $W_i\cap W_j=\Phi$, there is an open set V in $I^r\times M''$ such that $W_i'\cup W_j'\subset V$ and $\Phi\,|\,V$ is a homeomorphism.

3. $\Phi_i'^{-1}\Phi_i(t, i(x))=(t, i'(x))$ for $x\in M$.

The local homeomorphism J' of $\bigcup_i W_i$ in $I^i\times M''$ given by $J'(t, x)=\Phi_i'^{-1}\Phi_i(t, x)$ for $(t, x)\in W_i$ is well defined. J' commutes with the projection on I^r. Because $J'(t, i(x))=(t, i'(x))$, J' is injective on $I^r\times i(M)$. By Whitehead [11, §4], there is a neighbourhood W of $i(M)$ in M' such that $I^r\times W\subset\bigcup_i W_i$. $J=J'|I^r\times W$ and W satisfy the required conditions.

Lemma 9. *Suppose M is compact.*
A map

$$F: I^r\times M\longrightarrow I^r\times Q$$

has an uniform immersion property with respect to I^r.

Then there is a neighbourhood (M', i, Φ) of M induced by F.

Proof. For any $t\in I^r$, there is a cubic neighbourhood I_t of t in I^r, with a neighbourhood (M_t, i_t, Φ_t) of M induced by $F|I_t\times M$. Let n be a positive integer such that any cube $\bigtimes_{k=1}^{r}[a_k, b_k]\subset I^r$, where $|a_k-b_k|<\frac{2}{n}$, is contained in some I_t. Define the cube $I_i\subset I^r$, $i=1,\cdots,n^r$, as follows.

$$I_i=\bigtimes_{k=1}^{r}\left[\frac{j_k}{n},\ \frac{j_k+1}{n}\right]$$

where $i=1+\sum_{k=1}^{r}n^k j_k$, $j_k=0,\cdots,n-1$.

Let

$$X_l=\bigcup_{i\le l}I_i\,.$$

Inductively we will construct a neighbourhood X_l' of X_l in I^r and a neighbourhood $(\widetilde{M}_l, \tilde{i}_l, \tilde{\Phi}_l)$ of M induced by $F|(X_l'\times M)$. When $l=n^r$, $(\widetilde{M}_l, \tilde{i}_l, \tilde{\Phi}_l)$ is the required neighbourhood.

It is clear when $l=1$.

Suppose we are given X_l' and $(\widetilde{M}_l, \tilde{i}_l, \tilde{\Phi}_l)$. X_{l+1}' and $(\widetilde{M}_{l+1}, \tilde{i}_{l+1}, \tilde{\Phi}_{l+1})$ are given as follows. Cubes $I_{l,i}$ and $I_{l,i}'$ $(1\le i\le l+1)$ are defined by

$$I'_{l,i}=\mathop{\times}_{k=1}^{r}\left[\frac{j_k}{n}-\delta(j_k)\cdot P_l,\ \frac{j_k+1}{n}+\delta'(j_k)\cdot P_l\right]$$

$$I'_{l,i}=\mathop{\times}_{k=1}^{r}\left[\frac{j_k}{n}-\frac{1}{2}\delta(j_k)\cdot P_l,\ \frac{j_k+1}{n}+\frac{1}{2}\delta'(j_k)\cdot P_l\right]$$

where $\delta(j_k)=0$ if $j_k=0$, $\delta(j_k)=1$ if $j_k\neq 0$
$\delta'(j_k)=0$ if $j_k=n-1$, $\delta'(J_k)=1$ if $j_k\neq n-1$

and P_l is a positive number such that $P_l<\frac{1}{4n}$ and, $I_{l,i}\subset X'_l$ for any $i\leq l$. $Y_l=\bigcup_{i\leq l} I_{l,i}$ and $Z_l=\bigcup_{i\leq l} I'_{l,i}$ are neighbourhoods of X_l in I^r, and contained in X'_l. For $I_{l,l+1}$, there exists $t_{l+1}\in I^r$ such that $I_{l,l+1}\subset I_{t_{l+1}}$.

Let $\varphi_l: I_{l,l+1}\to I^r$ and $\varphi': I^r\to\left[0,\frac{1}{2}\right]\times I^{r-1}$ be homeomorphisms such that

$$\varphi_l(Y_l\cap I_{l,l+1})=\left[0,\frac{1}{2}\right]\times I^{r-1},\ \varphi_l(Z_l\cap I_{l,l+1})=\left[0,\frac{1}{4}\right]\times I^{r-1}$$

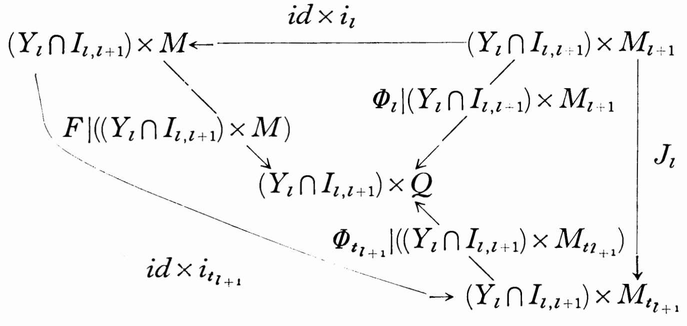
$$\varphi/\left(\left[0,\frac{1}{4}\right]\times I^{r-1}\right)=id.$$

The homeomorphism defined by

$$\varphi=\varphi_l^{-1}\circ\varphi'\circ\varphi_l: I_{l,l+1}\longrightarrow Y_l\cap I_{l,l+1}$$

satisfies $\varphi|(Z_l\cap I_{l,l+1})=id$.

For neighbourhoods $(\widetilde{M}_l,\ \tilde{\imath}_l,\ \widetilde{\Phi}_l|((Y_l\cap I_{l,l+1})\times\widetilde{M}_l))$ and $(M_{t_{l+1}},\ i_{t_{l+1}},\ \Phi_{t_{l+1}}|((Y_l\cap I_{l,l+1})\times M_{t_{l+1}}))$ of M induced by $F|((Y_l\cap I_{l,l+1})\times M))$, there is a neighbourhood $\widetilde{M}_{l+1}$ of $\tilde{\imath}_l(M)$ in $\widetilde{M}_l$ and a homeomorphism J_l of $(Y_l\cap I_{l,l+1})\times M_{l+1}$ into $(Y_l\cap I_{l,l+1})\times M_{t_{l+1}}$ commuting with the projection on $Y_l\cap I_{l,l+1}$, with the following commutative diagram;



Let $X'_{l+1}=\mathrm{Int}\,Z_l\cup\mathrm{Int}\,I_{l,l+1}$ and $\tilde{\imath}_{l+1}=\tilde{\imath}_l$, where $\mathrm{Int}\,Z_l$ and $\mathrm{Int}\,I_{l,l+1}$ are interiors of Z_l and $I_{l,l+1}$ in I^r respectively.

The map

$$\tilde{\Phi}_{l+1} : X'_{l+1} \times \widetilde{M}_{l+1} \longrightarrow X_{l+1} \times Q$$

given by

$$\tilde{\Phi}_{l+1}(t, x) = \begin{cases} \tilde{\Phi}_l(t, x) & \text{if} \quad (t, x) \in (\mathrm{Int}\, Z_l) \times \widetilde{M}_{l+1} \\ \Phi_{t_{l+1}} \circ (\varphi^{-1} \times id) \circ J_l \circ (\varphi \times id)(t, x) & \\ & \text{if} \quad (t, x) \in (\mathrm{Int}\, I_{l,l+1}) \times \widetilde{M}_{l+1} \end{cases}$$

is well defined because J_l commutes with the projection on the first factor, and $\varphi|(Z_l \cap I_{l,l+1}) = id$. Clearly $\tilde{\Phi}_{l+1}$ has the uniform immersion property, and commutes with the projection on X'_{l+1}.

$J_{l,t} \circ \tilde{i}_l(x) = i_{t_{l+1}}(x)$ for $(t, x) \in (Y_l \cap I_{l,l+1}) \times M$ where $J_{l,t}(x) = pr_2 \circ J_l(t, x)$. Then for $(\mathrm{t}, x) \in (\mathrm{Int}\, I_{l,l+1}) \times M$,

$$\begin{aligned} (\tilde{\Phi}_{l+1} \circ (id \times \tilde{i}_{l+1})(t, x) &= \Phi_{t_{l+1}}\Big((t, J_{l,\varphi(t)} \circ \tilde{i}_l(x)\Big) \\ &= \Phi_{t_{l+1}}\Big(t, i_{t_{l+1}}(x)\Big) \\ &= F(t, x)\,. \end{aligned}$$

Therefore $(\widetilde{M}_{l+1}, \tilde{i}_{l+1}, \tilde{\Phi}_{l+1})$ satisfies the required conditions.

4. Proof of the fibration lemma for immersions.

Now we will prove Lemma 3 when $M - M_0$ has a k-handle when $\dim M < \dim Q$. It suffices to prove the following lifting problem.

$$\begin{array}{ccc} I^r \times 0 & \xrightarrow{\quad H_0 \quad} & (B^k \times B^{m-k}, Q) \\ \cap \big\downarrow & \overset{H}{\nearrow} & \big\downarrow i^* \\ I^r \times I & \xrightarrow{\quad h \quad} & \left(\left(B^k - \frac{3}{4} B_k\right) \times B^{m-k}, Q\right) \end{array}$$

We may assume H_0, h as follows. cf. Gauld [2. Lemma 3]

$$h : I^r \times I \times \left(2B^k - \frac{1}{2}\mathring{B}^k\right) \times 2B^{m-k} \longrightarrow I^r \times I \times Q$$

$$H_0 : I^r \times 0 \times 2B^k \times 2B^{m-k} \longrightarrow I^r \times 0 \times Q$$

such that

$$h\Big|\left(I^r \times 0 \times \left(2B^k - \frac{1}{2}\mathring{B}^k\right) \times 2B^{m-k}\right) = H_0\Big|\left(I^r \times 0 \times \left(2B^k - \frac{1}{2}\mathring{B}^k\right) \times 2B^{m-k}\right)$$

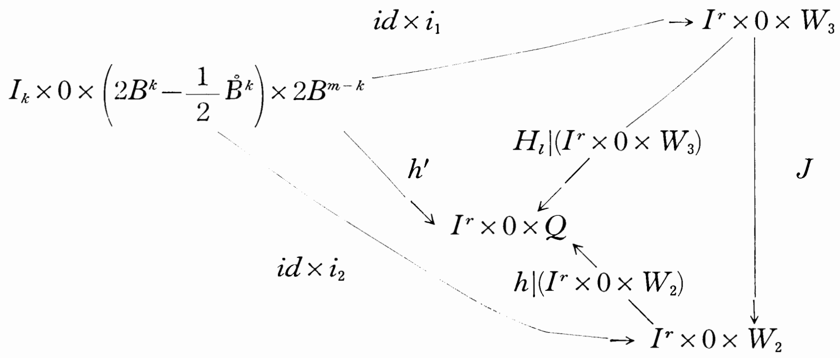
and h, H_0 have the uniform immersion property.
Our task is to extend h and H_0 to

$$H : I^r \times I \times 2B^k \times 2B^{m-k} \longrightarrow I^r \times I \times Q$$

By Lemma 9, there are a neighbourhood $(W_1, i_1, \tilde{H}_0)$ of $2B^k \times 2B^{m-k}$ induced by H_0 and a neighbourhood $(W_2, i_2, \tilde{h})$ of $\left(2B^k - \frac{1}{2}\mathring{B}^k\right) \times 2B^{m-k}$ induced by h. We may assume that

$$W_1 = 3B^k \times 4B^{n-k}$$
$$i_1 : 2B^k \times 2B^{m-k} \hookrightarrow 3B^k \times 4B^{n-k}$$

By Lemma 8, there exist a neighbourhood W_3 of $i_1\left(\left(2B^k - \frac{1}{2}\mathring{B}^k\right) \times 2B^{m-k}\right)$ in $3B^k \times 4B^{n-k}$ and a homeomorphism J of $I^r \times 0 \times W_3$ in $I^r \times 0 \times W_2$ with the following commutative diagram;



where

$$h' = h\left|\left(I^r \times 0 \times \left(2B^k - \frac{1}{2}\ddot{B}^k\right) \times 2B^{m-k}\right) = H_0\right|\left(I^r \times 0 \times \left(2B^k - \frac{1}{2}\mathring{B}^k\right) \times 2B^{m-k}\right).$$

We may assume $W_1 = \left(3B^k - \frac{1}{4}\ddot{B}^k\right) \times 4B^{n-k}$.

Let $J_t(x) = pr_3 \circ J(t, 0, x)$. The map

$$\tilde{h}' : I^r \times I \times \left(3B^k - \frac{1}{4}\mathring{B}^k\right) \times 4B^{n-k} \longrightarrow I^r \times I \times Q$$

given by $\tilde{h}'(t, t', x) = \tilde{h}\left(t, t', J_t(x)\right)$ has the uniform immersion property, and

$$\begin{aligned}\tilde{h}'\left(t, t', i_1(x)\right) &= \tilde{h}\left(t, t' J_t \circ i_1(x)\right)\\ &= \tilde{h}\left(t, t', i_2(x)\right)\\ &= h(t, t', x)\end{aligned}$$

$$\text{for}\quad (t, t', x) \in I^r \times I \times \left(2B^k - \frac{1}{2}\mathring{B}^k\right) \times 2B^{m-k}.$$

Moreover

$$\widetilde{H}_0 \left| \left(I^r \times 0 \times \left(3B^k - \frac{1}{4}\mathring{B}^k\right) \times 4B^{n-k}\right) = \tilde{h}' \right| \left(I^r \times 0 \times \left(3\mathring{B}^k - \frac{1}{4}\mathring{B}^k\right)\right) \times 4B^{m-k}\right).$$

By Gauld [2], there exists an extension of $\widetilde{H}_0$ and $\tilde{h}' \left| \left(I^r \times I \times \left(2B^k - \frac{2}{3}\mathring{B}^k\right) \times 2B^{n-k}\right)\right.$ with the uniform immersion property;

$$\widetilde{H} : I^r \times I \times 3B^k \times 4B^{n-k} \longrightarrow I^r \times I \times Q.$$

The composition

$$H = \widetilde{H}_0 \circ (id \times i_1) : I^r \times I \times 2B^k \times 2B^{m-k} \longrightarrow I^r \times I \times Q$$

is a required extemsion of H_0 and $h \left| \left(I^r \times I \times \left(2B^k - \frac{2}{3}\mathring{B}^k\right) \times 2B^{m-k}\right)\right.$.

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