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Local solution of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes

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Introduction

The Cauchy problem for nonlinear hyperbolic equations in Gevrey classes was studies by Leray-Ohya [7] (c. f. [8]). They assume that the characteristics are of constant multiplicity or smooth. In this paper we shall remove this restriction.

We consider the following equations for the unknowns $u(x)=(u_1(x), \cdots, u_N(x))$, $x=(x_0, x_1, \cdots, x_n)=(x_0, x')\in R^{n+1}$,

$$F_i\left(x, D^{M_i}u(x)\right)=0 \text{ in } \Omega, \qquad i=1, \cdots, N, \tag{0. 1}$$

where Ω is a neighborhood of 0 in R^{n+1} and

$$D^{M_i}u(x)=\left\{D^{M_{i1}}u_1(x), \cdots, D^{M_{iN}}u_N(x)\right\}$$

$$D^{M_{ij}}u_j(x)=\left\{(\partial/\partial x_0)^{\alpha_0}(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}u_j(x)\ ;\ \alpha=(\alpha_0, \alpha_1, \cdots, \alpha_n)\in M_{ij}\right\}$$

and M_{ij} is a finite set of non negative multi indices.

We assume that $\{F_i\}$ is a Leray-Volevich system of order m, that is, there exist non negative integers $n_1, \cdots, n_N$ such that for $\alpha\in M_{ij}$,

$$|\alpha|=\alpha_0+\alpha_1+\cdots+\alpha_n\leqslant m+n_j-n_i, \qquad i, j=1, \cdots, N. \tag{0. 2}$$

Then we can prescribe the following Cauchy data to the equations (0. 1),

$$(\partial/\partial x_0)^j u_i(0, x')=\varphi_{ji}(x'), \qquad j=0, \cdots, m-1,\ i=1, \cdots, N. \tag{0. 3}$$

We introduce coordinate variables

$$y_{ij}=(y_\alpha\ ;\ \alpha\in M_{ij}) \text{ in } R^{r_{ij}}, \qquad i, j=1, \cdots, N,$$

$$y_i=(y_{ij}\ ;\ j=1, \cdots, N) \text{ in } R^{r_i}, \qquad i=1, \cdots N,$$

$$y=(y_i\ ;\ i=1, \cdots, N) \text{ in } R^r,$$

where r_{ij} is the number of the elements of M_{ij}, $r_i=r_{i1}+\cdots+r_{iN}$ and $r=r_1+\cdots+r_N$.

We assume that $F_i(x, y)$, $i=1, \cdots, N$ are in Gevrey class s in x and analytic in $y \in V$ (V is a compact set in R^r), that is, there are positive numbers C and A such that

$$|D_x^\alpha D_y^\beta F_i(x, y)| \leqslant CA^{|\alpha|+|\beta|} |\alpha|!^s |\beta|!, \ x \text{ in } \Omega, \qquad y \text{ in } V, \tag{0. 4}$$

for $\alpha \in N^{n+1}$, $\beta \in N^r$.

We define the characteristic matrix for $\{F_i\}$ as follows

$$p_{ij}(x, y, \xi) = \sum_{\substack{|\alpha|=m+n_j-n_i \\ \alpha \in M_{ij}}} (\partial/\partial y_\alpha) F_i(x, y) \xi^\alpha, \qquad i, j = 1, \cdots, N,$$

which is a polynomial in ξ of degree $m+n_j-n_i$. We call the determinant of $\{p_{ij}(x, y, \xi)\}$ characteristic polynomial and denote it by $p(x, y, \xi)$. We say that a system $\{F_i\}$ is hyperbolic in $\Omega \times V$ with respect to ξ_0, if the characteristic polynomial $p(x, y, \xi_0 - \sqrt{-1}\ \lambda, \xi')$ does not vanish for (x, y, ξ, λ) in $\Omega \times V \times R^{n+1} \times R^1 \backslash 0$ and $p(x, y, 1, 0) = 1$. Then $p(x, y, \xi_0, \xi')$ has only real roots $\lambda_j(x, y, \xi')$ $(j=1, \cdots, mN)$ with respect to ξ_0 for (x, y, ξ') in $G \times V \times R^n \backslash 0$. Assume that the multiplicities of the roots $\lambda_j(x, y, \xi')$ do not exceed ν for any $(x, y, \xi') \in G \times V \times R^n \backslash 0$.

For K a closed set in R^n, we denote by $\gamma_A^{(s)}(K)$ the class of all functions $u(x')$ satisfying

$$|D^\alpha u(x')| \leqslant CA^{|\alpha|} |\alpha|!^s,$$

for $x' \in K$ and $\alpha \in N^n$, and by $\gamma_A^{k(s)}([0, T] \times K)$ the class of all k times continuously differentiable function of x_0 in $\gamma_A^{(s)}(K)$. We denote $\gamma^{k(s)} = \bigcup_{A>0} \gamma_A^{k(s)}$ and $\gamma^{\infty(s)} = \bigcap_{k \geq 0} \gamma^{k(s)}$.

We consider the Cauchy problem (0, 1) with intial data (0. 3). Then we obtain following theorems,

THEOREM 0. 1. *Assume that $\{F_i(x, y)\}$ is a hyperbolic Leray-Volevich system of order m, satisfying* (0. 4), *and that the multiplicities of the roots of it's characteristic polynomial do not exceed ν. If $1 < s < \nu(\nu-1)^{-1}$, then for any initial data $\{\varphi_{ji}(x')\}$ in $\gamma^{(s)}(K)$, there exist $T>0$ and $K_0 \subset K$ such that the Cauchy problem* (0. 1) *and* (0. 3) *has a solution $\{u_i(x)\}$ in $\gamma^{\infty(s)}([0, T] \times K_0)$.*

THEOREM 0. 2. *Assume that the conditions of Theorem* 0. 1 *are valid. Let $\{\varphi_{ji}(x')\}$ and $\{\varphi_{ji}(x')\}$ be in $\gamma^{(s)}(K)$ such that $\varphi_{ji}(x') = \psi_{ji}(x')$ for $x' \in K_0 \subset K$. Let $\{u_i\}$ and $\{v_i\}$ be a solution of* (0. 1) *with initial data $\{\varphi_{ji}\}$ and $\{\psi_{ji}\}$ respectively. Then there exists a positive number λ_0 such that*

$$\{u_i(x)\} = \{v_i(x)\} \quad \textit{for} \quad x \in \{x \in R^{n+1};\ 0 \leqslant x_0 \leqslant \lambda_0 |x' - y'|,\ y' \in K_0\}.$$

Recently, in the linear case, the above theorems have proved by Bronshtein [2]. Kajitani [4] and Nishitani [6] have given another proof. To prove Theorem 0.1 and 0.2 we shall derive an energy estimate for the linearized equations of (0.1) by methods in [4] and apply Schauder's fixed point theorem.

§ 1. Preliminaries

We introduce the following notation; $x=(x_0, x_1, \cdots, x_n)=(x_0, x')$ in R^{n+1} and $\xi=(\xi_0, \xi')$ is dual variable with a inner product $x\xi=\sum_i x_i\xi_i$, $D=(D_0, D_1, \cdots, D_n)$, $D_j=-\sqrt{-1}\ \partial/\partial x_j$ $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_n)=(\alpha_0, \alpha')$ in N^{n+1}, $|\alpha|=\Sigma\alpha_i$, $\langle\xi\rangle_h^2=h^2+\xi_0^2+\xi_1^2+\cdots+\xi_n^2$, $\langle\xi'\rangle_h^2=h^2+\xi_1^2+\cdots, \xi_n^2$, $\langle D\rangle_h^2=h^2-\Delta_x$, $\langle D'\rangle_h^2=h^2-\Delta_{x'}$, $(h>0)$. We denote by $W_{l,q,h}$, where l, q in R^1 and $h>0$, the class of functions $u(x)$ such that $\langle D\rangle_h^l\langle D'\rangle_h^q u$ in $L^2(R^{n+1})$. We put $\Lambda=\Lambda(x_0, T, h, \kappa, D')=(T-x_0)(h+\langle D'\rangle_h^\kappa)$, where $T\in R^1$, $h>0$ and $\kappa=s^{-1}$. We define $e^\Lambda u(x)$,

$$e^\Lambda u(x)=e^{\Lambda(x_0,D')}u(x)=\int e^{[ix'\xi'+\Lambda(x_0,T,h,\kappa,\xi')]}\tilde{u}(x_0, \xi')\, d\!\!{}^-\xi' ,$$

where $d\!\!{}^-\xi'=(2\pi)^{-n}d\xi'$ and $\tilde{u}(x_0, \xi')$ stands for a Fourier transform of u with respect to x'. Denote by $W_{l,q,h}^{(\Lambda)}$ all functions $u(x)$ such that $e^\Lambda u$ in $W_{l,q,h}$. When l is a non negative integer, for $\Omega_{T_0}=[0, T_0]\times R^n$, $(T_0>0)$, we define $W_{l,q,h}(\Omega_{T_0})$, all functions $u(x)$ such that $D_0^j\langle D'\rangle_h^{l-j+q}u(j=0, \cdots, l)$, in $L^2(\Omega_{T_0})$ and also define $W_{l,q,h}^{(\Lambda)}(\Omega_{T_0})$ analogously.

LEMMA 1.1. *Let f and g be in $W_{l,q,h}^{(\Lambda)}$, where $\Lambda=(T-x_0)(h+\langle D'\rangle_h^\kappa)$, $T\geq 0$ and $0<\kappa<1$. Then if $(n+2)$, $q\geq 0$, and supp $f\subset\left\{x_0\leq\frac{T}{2}\right\}$, then the product $f\cdot g$ is also in $W_{l,q,h}^{(\Lambda)}$ and satisfies*

$$\|f\cdot g\|_{W_{l,q,h}^{(\Lambda)}}\leq C_l\|f\|_{W_{l,q,h}^{(\Lambda)}}\|g\|_{W_{l,q,h}^{(\Lambda)}}, \tag{1.1}$$

where C_l is independent of h.

PROOF. We have

$$\begin{aligned}\|fg\|_{W_{l,q,h}^{(\Lambda)}}^2&=\sum_{|\beta|+j\leq l}\|e^\Lambda h^{l-j}D^\beta(fg)\|_{W_{0,q,h}}^2\\ &\leq C\sum_{\substack{|\beta|+j\leq l\\ \beta'+\beta''=\beta}}\|e^\Lambda h^{l-j}D^{\beta'}fD^{\beta''}g\|_{W_{0,q,h}}^2\\ &\leq Ch^{2l}\sum_{\beta'+\beta''=\beta}\|e^{\Lambda(\xi')}\langle\xi'\rangle_h^q\int\tilde{f}_{\beta'}(x_0, \xi'-\eta')\,\tilde{g}_{\beta''}(x_0, \eta')\,d\eta'\|_{L^2}^2\\ &\leq Ch^{2l}\sum\|K(x_0, \xi', h)\int e^{\Lambda(\xi'-\eta')}\langle\xi'-\eta'\rangle_h^q\tilde{f}_{\beta'}(x_0, \xi'-\eta')\\ &\quad\times e^{\Lambda(\eta')}\langle\eta'\rangle_h^q\tilde{g}_{\beta''}(x_0, \eta')\,d\eta\|_{L^2}^2\end{aligned}$$

where $\tilde{f}_{\beta'}(x_0, \xi')$ and $\tilde{g}_{\beta''}$ are Fourier transforms of $D^{\beta'}f$ and $D^{\beta''}g$ with respect to x' and

$$K(x_0, \xi', h) = \inf_{\eta' \in R^n} e^{\Lambda(\xi') - \Lambda(\xi' - \eta') - \Lambda(\eta')} \left\{ \langle \xi' \rangle_h \langle \xi' - \eta' \rangle_h^{-1} \langle \eta' \rangle_h^{-1} \right\}^q .$$

Noting that $\Lambda(\xi') - \Lambda(\xi' - \eta') - \Lambda(\eta') = (T - x_0)(\langle \xi' \rangle_h^{\kappa} - \langle \xi' - \eta' \rangle_h^{\kappa} - \langle \eta' \rangle_h^{\kappa}) \leqslant -hT/2$ for $x_0 \leqslant T/2$ and $(\langle \xi' \rangle_h \langle \xi' - \eta' \rangle_h^{-1} \langle \eta' \rangle_h^{-1})^q \leqslant 1$ for $q \geqslant 0$, we have

$$K(x_0, \xi', h) \leqslant e^{-hT/2}, \qquad x_0 \leqslant T/2 .$$

Hence we obtain by virtue of Hausdorff-Yong's inequality,

$$\begin{aligned} \|fg\|^2_{W^{(\Lambda)}_{l,q,h}} &\leqslant C e^{-Th} h^2 \Big\{ \sum_{|\beta'| \geqslant \frac{l}{2}} \|e^{\Lambda(\xi')} \langle \xi' \rangle_h^q \tilde{f}_{\beta'}\|^2_{L^2} \|e^{\Lambda(\xi')} \langle \xi' \rangle_h^q \tilde{g}_{\beta''}\|^2_{L^1} \\ &\quad + \sum_{|\beta'| < \frac{l}{2}} \|e^{\Lambda(\xi')} \langle \xi' \rangle_h^q \tilde{f}_{\beta'}\|^2_{L^1} \|e^{\Lambda(\xi')} \langle \xi' \rangle_h^q \tilde{g}_{\beta''}\|^2_{L^2} \Big\} \\ &\leqslant C \Big\{ \|f\|_{W^{(\Lambda)}_{l,q,h}} \|g\|_{W^{(\Lambda)}_{l/2+[n/2]+1,q,h}} \\ &\quad + \|f\|_{W^{(\Lambda)}_{l/2+[n/2]+1,q,h}} \|g\|_{W^{(\Lambda)}_{l,q,h}} \Big\} \\ &\leqslant C \|f\|_{W^{(\Lambda)}_{l,q,h}} \|g\|_{W^{(\Lambda)}_{l,q,h}}, \qquad \text{for} \quad l \geqslant n+2 . \end{aligned}$$

LEMMA 1.2. *Let f be in $W^{(\Lambda)}_{l,q,h}$, g in $W^{(\Lambda)}_{m,q,h}$ and supp $f \subset \{x_0 \leqslant T\}$. Then if l is sufficiently large and $q \geqslant 0$, the product $f \cdot g$ is in $W^{(\Lambda)}_{m,q,h}$ and satisfies*

$$\|f \cdot g\|_{W^{(\Lambda)}_{m,q,h}} \leqslant C \|f\|_{W^{(\Lambda)}_{l,q,h}} \|g\|_{W^{(\Lambda)}_{m,q,h}},$$

where $\Lambda = (T - x_0)(h + \langle D' \rangle_h^{\kappa})$, $T \geqslant 0$, $0 < \kappa < 1$, and C is independent of h.

This lemma follows from Proposition 2.7 in § 2.

LEMMA 1.3. *Let $u(x)$ be in $\gamma^{l(s)}_{A_0}$ and of a compact support in $\{x \in R^{n+1};\ x_0 \geqslant 0\}$, where l a positive integer. Then there exists $T > 0$ such that for any $q \in R^1$ and $h > 0$, $u(x)$ is in $W^{(\Lambda)}_{l,q,h}$, where $\Lambda = (T - x_0)(h + \langle D' \rangle_h^{\kappa})$, $\kappa = s^{-1}$.*

PROOF. Since u is in $\gamma^{l(s)}_{A_0}$, u satisfies

$$|D^{\alpha} u(x)| \leqslant C A_0^{|\alpha'|} |\alpha'| !^s$$

for $\alpha = (\alpha_0, \alpha')$ in N^{n+1}, $\alpha_0 \leqslant l$. Hence

$$\begin{aligned} |\xi^{\alpha'} D_0^{\alpha_0} \tilde{u}(x_0, \xi')| &= \left| \int e^{-ix' \xi'} D^{\alpha} u(x)\, dx' \right| \\ &\leqslant C A_0^{|\alpha'|} |\alpha'| !^s , \end{aligned}$$

for $\alpha = (\alpha_0, \alpha')$ in N^{n+1}, $\alpha_0 \leqslant l$, which implies

$$|D_0^{\alpha_0} \tilde{u}(x_0, \xi')| \leqslant C_l (A_0 \langle \xi' \rangle^{-1})^M M!^s , \qquad \alpha_0 \leqslant l ;$$

for any positive integer M, where $\langle\xi'\rangle=(1+|\xi'|^2)^{1/2}$. Therefore,

$$|D_0^{\alpha_0}u(x_0,\xi')|\leqslant\inf_{M\in N}C(A_0\langle\xi'\rangle^{-1})^M M!^s\leqslant C_1e^{-\{(A_0^{-1}\langle\xi'\rangle)^{1/s}\}/2}$$

and

$$e^{(T-x_0)(h+\langle\xi'\rangle_h^{\kappa})}\langle\xi'\rangle_h^{l+q-\alpha_0}|D_0^{\alpha_0}\tilde{u}(x_0,\xi')|$$

$$\leqslant C_2(T,h)\,e^{(T-x_0)\langle\xi'\rangle}\langle\xi'\rangle^{l+q-\alpha_0}|D_0^{\alpha_0}\tilde{u}(x_0,\xi')|$$

$$\leqslant C_3(T,h)\,e^{-\{(A_0^{-1}\langle\xi'\rangle)^{1/s}\}/4},$$

if $T-x_0\leqslant A_0^{-1/s}/4$ and $x_0\geqslant 0$, which implies our conclusion.

Next we mention the properties of hyperbolic polynomials.

LEMMA 1.4. *Let $p(z,\xi)=\sum_{|\alpha|=m}a_\alpha(z)\xi^\alpha$ be a hyperbolic polynomial with respect to ξ_0, which coefficients $a_\alpha(z)$ are constant in the compliment of a compact set in R^d and which multiplicity of roots is at most ν. Then the following properties hold,*

(i) $|p(z,\xi_A)^{-1}|\leqslant C\langle\xi\rangle_h^{-m}\langle\xi'\rangle_h^{\tau}$, $\quad(z,\xi)\in R^{d+n+1}$,

(ii) $|p^{(\alpha)}(z,\xi_A)\,p(z,\xi_A)^{-1}|\leqslant C\langle\xi'\rangle_h^{-\kappa}$, $\quad(z,\xi)\in R^{d+n+1}$, $|\alpha|=1$,

(iii) $|p_{(\alpha)}(z,\xi_A)\,p(z,\xi_A)^{-1}|\leqslant C\langle\xi'\rangle_h^{1-\kappa}$, $\quad(z,\xi)\in R^{d+n+1}$, $|\alpha|=1$,

where C is independent of h and $\xi_A=(\xi_0-i(h+\langle\xi'\rangle_h^{\kappa}),\xi')$, $\tau=\nu(1-\kappa)$.

The proof of (ii) and (iii) follows from Bronshtein [1].

§ 3. Pseudo-differential operators in Gevrey class

We introduce a class of pseudo-differential operators in Gevrey classes.

DEFINITION 2.1. *Denote by $S_{\rho,\delta,h}^{(p)}$, a set of all functions $a(x',\xi')$ in $C^\infty(R^{2n})$ such that*

$$|a|_l^{(p)}=\sup_{\substack{(x',\xi')\in R^{2n}\\|\alpha+\beta|\leq l}}\frac{|a^{(\alpha)}{}_{(\beta)}(x',\xi')|}{\langle\xi'\rangle_h^{p-\rho|\alpha|+\delta|\beta|}}<\infty$$

for each $l\in N$, where $p\in R^1$, and $0\leqslant\delta\leq\rho\leq 1$.

DEFINITION 2.2. *Denote by $S_{\rho,\delta,h}^{(m,p)}$ a set of all functions $a(x,\xi)\in C^\infty(R^{2(n+1)})$ such that*

$$|a|_l^{(m,p)}=\sup_{\substack{(x,\xi)\in R^{2(n+1)}\\\alpha,\beta\in N^{n+1}\\|\alpha+\beta|\leq l}}\frac{|a_{(\beta)}^{(\alpha)}(x,\xi)|}{\langle\xi\rangle_h^{m-\alpha_0}\langle\xi'\rangle_h^{p-\rho|\alpha'|+\delta|\beta|}}<\infty$$

for each $l \in N$, *where* $m, q \in R^1$, $0 \leqslant \delta \leqslant \rho \leq 1$.

We define $\mathscr{A}^{(p)}_{s,A,h}$ *all functions* $a(x', \xi')$ *satisfying*

$$|a|^{(p)}_{s,A,l} = \sup_{\substack{\gamma \in N^n \\ |\alpha+\beta| \leq l \\ (x',\xi') \in R^{2n}}} \frac{|a^{(\alpha)}_{(\beta+\gamma)}(x', \xi')|}{A^{|\gamma|} |\gamma| !^s \langle \xi' \rangle_h^{p-|\alpha|}} < \infty ,$$

for each $l \in N$, *and also define by* $C_0^\infty([-T, T]; \mathscr{A}^{(p)}_{s,A,h})$ *all functions* $a(x_0, x', \xi')$ *satisfying*

$$[a]^{(p)}_{s,A,l} = \sup_{\substack{0 \leq i \leq l \\ |x_0| \leqslant T}} \left|\left(\frac{\partial}{\partial x_0}\right)^{l-i} a(x_0, \cdot)\right|^{(p)}_{s,A,i} < \infty ,$$

for $l \in N$ *and* $a = 0$ *for* $|x_0| > T$.

As usual, for a symbol $a(x, \xi)$ we define a pseudodifferential operator $a(x, D)$,

$$a(x, D)\, u(x) = \int e^{ix\xi} a(x, \xi)\, \hat{u}(\xi)\, \xi ,$$

where $\hat{u}(\xi)$ stands for a Fourier transform of $u(x)$ and $đ\xi = (2\pi)^{-(n+1)} d\xi$. We define $a_\Lambda(x, D)$ a transform of $a(x, D)$ by e^Λ as follows

$$a_\Lambda(x, D) = e^\Lambda a(x, D)\, e^{-\Lambda} .$$

Lemma 2. 3. *Let* $a(x', \xi')$ *be in* $\mathscr{A}^{(p)}_{s,A,h}$. *Then* $a_\Lambda(x', D') = e^\Lambda a(x', D')\, e^{-\Lambda}$ *where* $\Lambda = M(h + \langle D' \rangle_h^\kappa)$, $\kappa = s^{-1}$ *and* $|M| \leq (24 n^\kappa A^\kappa)^{-1}$, *is a pseudo-differential operator and its symbol* $a_\Lambda(x', \xi')$ *is in* $S^{(p)}_{1,0,h}$ *and satisfies*

$$(2.1) \qquad |a_\Lambda|^{(p)}_l \leq C_l |a|^{(p)}_{s,A}, \qquad \left[\frac{l}{1-\kappa}\right] + l + 1 ,$$

where C_l *is independent of* a *and* h *and* $[\cdot]$ *is a Gauss' symbol.*

It's proof is refered to Proposition 2. 3 in [4].

Definition 2. 4. *Denote by* $\mathscr{A}^{(p,\Lambda)}$ *all functions* $a(x', \xi')$ *in* $C^\infty(R^n \times R^n)$ *such that*

$$||a||^{(p,\Lambda)}_l = \sup_{\substack{\xi \in R^n \\ |\alpha+\beta| \leq l}} \frac{||e^\Lambda a^{(\alpha)}{}_{(\beta)}(\cdot, \xi')||_{L^2(R^n_x)}}{\langle \xi \rangle_h^{p-|\alpha|}} < \infty ,$$

for each $l \in N$, *where* $\Lambda = M(h + \langle D' \rangle_h^\kappa)$, $M \in R^1$, $0 < \kappa < 1$ *and* $p \in R^1$ $p \in R^1$.

Remark. If $a(x', \xi') \in \mathscr{A}^{(p,\Lambda)}$ and $\Lambda = M(h + \langle D' \rangle_h^\kappa)$, $M \geq 0$, then it follows from Sobolev's lemma that $a(x', \xi') \in S^{(p)}_{1,0,h}$ and $|a|^{(p)}_l \leq C ||a||^{(p,\Lambda)}_{l+n+1}$.

Proposition 2. 5. *Let* $a(x', \xi')$ *be in* $\mathscr{A}^{(p,\Lambda)}$, *where* $\Lambda = M(h + \langle D' \rangle_h^\kappa)$, $0 < \kappa < 1$, $M \geqslant 0$ *and* $p \in R^1$. *Then* $a_\Lambda(x', D') = e^\Lambda a(x', D')\, e^{-\Lambda}$ *is a pseudo dif-*

ferential operators and it's symbol $a_\Lambda(x',\xi')$ *belongs to* $S^{(p)}_{1,0,h}$ *and satisfies*

(2.2) $$|a|^{(p)}_{l}\leq C_l\,||a||^{(p,\Lambda)}_{[\frac{l}{1-\kappa}]+n+1+l}\,.$$

for each $l\in N$, *where* C_l *is independent of* h *and* $[\cdot]$ *is a Gauss' symbol.*

PROOF. It follows from [5] that the symbol $a_\Lambda(x,\xi)$ is given by

$$a_\Lambda(x,\xi)=os-\iint_{R^{2n}} e^{-iy\eta+\Lambda(\xi+\eta)-\Lambda(\xi)}\,a(x+y,\xi)\,dy\,d\!\!{}^-\eta$$

where $x,\ \xi\in R^n$ and $os-\iint$ means an oscillatory integral. Hence we have by Taylor expansion,

$$a_\Lambda(x,\xi)=\sum_{|\alpha|<N}a_{(\alpha)}(x,\xi)\,\lambda_\alpha(\xi)+r_N(x,\xi)\equiv r(x,\xi)+r_N(x,\xi)$$

where $\lambda_\alpha(\xi)=\alpha!^{-1}(D^\alpha_\eta e^{\Lambda(\xi+\eta)-\Lambda(\xi)})_{\eta=0}$ and

$$r_N(x,\xi)\equiv\sum_{|\alpha|=N}\frac{1}{\alpha!}os-\iint e^{-iy\eta+\Lambda(\xi+\eta)-\Lambda(\xi)}\,y^\alpha\int_0^1 a_{(\alpha)}(x+\theta y,\xi)\,d\theta\,dy\,d\!\!{}^-\eta\,.$$

Noting that $|D^\beta_\xi\lambda_\alpha(\xi)|\leq C_{\alpha\beta}\langle\xi\rangle_h^{-(1-\kappa)|\alpha|-|\beta|}$,

(2.3) $$\begin{aligned}|r^{(\alpha)}{}_{(\beta)}(x,\xi)|&\leq\sum_{|\gamma|<N}\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}|a^{(\alpha')}_{(\gamma+\beta)}(x,\xi)|\,|\lambda_\gamma{}^{(\alpha'')}(\xi)|\\&\leq C_\alpha|a|^{(p,\Lambda)}_{|\alpha+\beta|+N+n}\langle\xi\rangle_h^{p-|\alpha|}\,.\end{aligned}$$

On the other hand, we have

$$r_N(x,\xi)=\sum_{|\gamma|=N}\frac{1}{\gamma!}os-\iint e^{-iy\eta+\Lambda(\xi+\eta)-\Lambda(\xi)}\,\mu_\gamma(\xi+\eta)\,a_{(\gamma)}(x+\theta y,\xi)\,d\theta\,dy\,d\!\!{}^-\eta$$

where

$$\mu_\gamma(\xi+\eta)=e^{-\Lambda(\xi+\eta)}D^\gamma_\eta e^{\Lambda(\xi+\eta)}\,.$$

Hence

$$\begin{aligned}r^{(\alpha)}_{N(\beta)}(x,\xi)&=\sum_{|\gamma|=N}\frac{1}{\gamma!}\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\iiint_0^1 e^{-iy\eta}\mu^{(\alpha')}_\gamma(\xi+\eta)\,a^{(\alpha'')}_{(\gamma+\beta)}(x+\theta y,\xi)\,d\theta\,dy\,d\!\!{}^-\eta\\&=\sum_{|\gamma|=N}\frac{1}{\gamma!}\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\int e^{\Lambda(\xi+\eta)-\Lambda(\xi)}\mu^{(\alpha')}_\gamma(\xi+\eta)\,F^{(\alpha'')}_{(\gamma+\beta)}(x,\xi,\eta,)\,d\!\!{}^-\eta\,,\end{aligned}$$

where

$$\begin{aligned}F^{(\alpha'')}_{(\gamma+\beta)}(x,\xi,\eta)&=\int_0^1\int e^{-iy\eta}a^{(\alpha'')}_{(\gamma+\beta)}(x+\theta y,\xi)\,dy\,d\theta\\&=\int_0^1\int e^{-i(y-x)\frac{\eta}{\theta}}a^{(\alpha'')}_{(\gamma+\beta)}(y,\xi)\frac{dy}{\theta^n}\,d\theta\\&=\int_0^1 e^{ix(\frac{\eta}{\theta})}\theta^{-n}\,\hat{a}^{(\alpha'')}_{(\gamma+\beta)}\left(\frac{\eta}{\theta},\xi\right)d\theta\,,\end{aligned}$$

here $\hat{a}(\eta, \xi)$ stands for a Fourier transform of $a(x, \xi)$ with respect to x. Noting that $\Lambda(\xi+\eta)-\Lambda(\xi)-\Lambda(\eta)\leq 0$ and

$$|\mu_{\gamma}^{(\alpha')}(\xi+\eta)|\leq C\langle\xi+\eta\rangle_h^{-(1-\kappa)(|\gamma|+|\alpha'|)},$$

$$|r_{N(\beta)}^{(\alpha)}(x, \xi)|\leq C\sum\int\langle\xi+\eta\rangle^{-(1-\kappa)N}e^{\Lambda(\eta)}|F_{(\gamma+\beta)}^{(\alpha'')}(x, \xi, \eta)|\,d\eta$$

$$\leqslant C\langle\xi\rangle_h^{-(1-\kappa)N}\sum\int\langle\eta\rangle_h^{+(1-\kappa)N}e^{\Lambda(\eta)}|F_{(\gamma+\beta)}^{(\alpha'')}|\,d\eta$$

$$\leqslant C\langle\xi\rangle_h^{-(1-\kappa)N}\iint_0^1\langle\eta\rangle_h^{+(1-\kappa)N}e^{\Lambda(\eta)}\theta^{-n}\left|\hat{a}_{(\gamma+\beta)}^{(\alpha'')}\left(\frac{\eta}{\theta}, \xi\right)\right|\,d\theta d\eta$$

$$\leqslant C\langle\xi\rangle_h^{-(1-\kappa)N}\sum\int\langle\eta\rangle_h^{+(1-\kappa)N}e^{\Lambda(\eta)}|\hat{a}_{(\gamma+\beta)}^{(\alpha'')}(\eta, \xi)|\,d\eta$$

$$\leqslant C\langle\xi\rangle_h^{-(1-\kappa)N+p}|a|_{[(1-\kappa)^{-1}N+(n+1)/2]+1+N'+|\alpha+\beta|}^{(p,\Lambda)},$$

where we used $\langle\theta\eta\rangle_h\leqslant\langle\eta\rangle_h$ and $\Lambda(\theta\eta)\leqslant\Lambda(\eta)$ for $0\leqslant\theta\leqslant 1$. If we take N such that $(1-\kappa)N\geqslant|\alpha|$, we obtain

$$|r_{N}^{(\alpha)}{}_{(\beta)}(x, \xi)|\leqslant C\langle\xi\rangle_h^{p-|\alpha|}||a||_{[|\alpha|+(n+1)/2]+N+|\alpha+\beta|+1}^{(p,\Lambda)}$$

which implies (2. 2) with (2. 3).

Next we indicate the boundedness of pseudodifferential operators in $W_{l,q,h}$ and $W_{l,q,h}^{(\Lambda)}$ which proof is refers to [5].

PROPOSITION 2. 6. *Let $a(x, \xi)$ be in $S_{\rho,\delta,h}^{(m,p)}$, where $0\leqslant\delta<\rho\leqslant 1$ and $\delta<1/2$. Then there exists a positive number $C_{l,q}$ independent of h such that*

$$||au||_{W_{l,q,h}}\leqslant C_{l,q}|a|_{M_0(l,q)}^{(m,p)}||u||_{W_{l+m,p+q,h}},$$

for any u in $W_{l+m,p+q,h}$, where

$$M_0(l, q)=[(1-\delta/2)^{-1}(|l|+|q|+l_0+n+2)]+l_0 \tag{2. 4}$$

$$l_0=[\delta(n+2)(\rho-\delta)^{-1}]+1\,.$$

We denote by $C_0^k([-T, T]:\mathscr{A}^{(p,\Lambda)})$, where $\Lambda=(T-x_0)(h+\langle D'\rangle_h^{\kappa})$ all functions $a(x, \xi')$ such that $x_0\in R^1\rightarrow a(x_0, x', \xi')$ in $\mathscr{A}^{(p,\Lambda)}$ is k times continuously differentiable and $a(x, \xi')=0$ for $|x_0|\geq T$, and we put

$$[a]_k^{(p,\Lambda)}=\sup_{\substack{0\leqslant j\leqslant k\\|x_0|\leqslant T}}|(\partial/\partial x_0)^j a(x_0, \cdot, \cdot)|_{k-j}^{(p,\Lambda)}\,.$$

Analogously we define $C_0^k([-T, T]\,;\,\gamma_A^{(s)}(R^n))$, all functions $a(x)$ such that $x_0\rightarrow a(x_0, x')$ in $\gamma_A^{(s)}(R^n)$ is k times continuously differentiable and $a=0$ for $|x_0|\geq T$, and denote

$$[a]_{s,A,k} = \sup_{\substack{0\leqslant j\leqslant k\\ |x_0|\leqslant T}} |(\partial/\partial x_0)^j a(x_0, \cdot)|_{s,A,k-j}\,.$$

Then Proposition 2.5 and 2.6 imply directly

PROPOSITION 2.7. *Let* $a(x,\xi')$ *be in* $C_0^k([-T,T]\,;\,\mathscr{A}^{(p,\Lambda)}$, *where* $\Lambda=(T-x_0)(h+\langle D'\rangle_h^\kappa)$. *Then we have* $C_{l,q}$ *independent of* h *such that*

$$||au||_{W_{l,q,h}^{(\Lambda)}} \leqslant C_{l,q}[a]_{M_\Lambda(l,q)}^{(p,\Lambda)}||u||_{W_{l,q+p,h}^{(\Lambda)}}$$

for u *in* $W_{l,q+p,h}^{(\Lambda)}$ *and* $l,\ q\in R^1$ *such that* $M_\Lambda(l,q)\leqslant k$, *where*

$$M_\Lambda(l,q)=[M_0(l,q)(1-\kappa)^{-1}]+1+n+M_0(l,q)\,, \tag{2.5}$$

and $M_0(l,q)$ *is given by* (2.4).

PROPOSITION 2.8. *Let* $a(x)$ *be in* $C_0^k([-T,T]\,;\,\gamma_{A_0}^{(s)}(R^n))$. *Then if* $0\leq T\leqslant(24n^\kappa A_0^\kappa)^{-1}$ *and* $\kappa=s^{-1}$, *we have*

$$||au||_{W_{l,q,h}^{(\Lambda)}} \leqslant C[a]_{s,A_0,M_\Lambda(l,q)}||u||_{W_{l,q,h}^{(\Lambda)}}\,,$$

for any u *in* $W_{l,q,h}^{(\Lambda)}$, $M_\Lambda(l,q)\leqslant k$, *where* $M_\Lambda(l,q)$ *given by* (2.5) *and* $\Lambda=(T-x_0)(h+\langle D'\rangle_h^\kappa)$.

Applying the above Propositions, we calculate a norm of composite functions in $W_{l,q,h}^{(\Lambda)}$.

PROPOSITION 2.9. *Let* $a(x,y)$ *be in* $C^k(R^{n+1}\times B_{R_0})$, *where* $B_{R_0}=\{y\in R^r\,;\ |y|\leqslant R_0\}$ *and satisfies*

$$a(x,0)=0$$

$$a(x,y)=a_0(y)\,, \qquad |x_0|>T\,,$$

$$|D_x^\alpha D_y^\beta a(x,y)|\leqslant C_{\alpha_0}A_{\alpha_0}^{|\alpha'|+|\beta|}|\alpha'|\,!^s|\beta|\,!\,,$$

for any $\alpha=(\alpha_0,\alpha')\in N^{n+1}$, $\alpha_0\leq k$, $\beta\in N^r$ and $(x,y)\in R^{n+1}\times B_{R_0}$. *Assume that* $0\leqslant T\leqslant(24n^\kappa A_0^\kappa)^{-1}$ *and* $v=(v_1,\cdots,v_r)$ *is in* $(W_{l,q,h}^{(\Lambda)})^r$, *where* $\Lambda=(T-x_0)(h+\langle D'\rangle_h^\kappa)$, $\kappa=s^{-1}$ *and* $q\in R^1$, $M_\Lambda(l,q)\leq k$ *and satisfies*

$$||v||_{W_{l,q,h}^{(\Lambda)}}=\left\{\sum_{i=1}^r||v_i||_{W_{l,q,h}^{(\Lambda)}}^2\right\}^{1/2}\leqslant(2rA_0)^{-1}\,,$$

then $a\circ v(x)=a(x,v(x))$ *is in* $W_{l,q,h}^{(\Lambda)}$ *and satisfies*

$$||a\circ v||_{W_{l,q,h}^{(\Lambda)}}\leqslant C(A_0)\,,$$

where $C(A_0)$ *is independent of* h.

PROOF. By Taylor's expansion, we have

$$a(x,y)=\sum_{\beta\neq 0} a_{(\beta)}(x)\, y^\beta/\beta\,!$$

where $a_{(\beta)}(x)=(\partial/\partial y)^\beta a(x,0)$. Then it follows from (2.6) that $a_{(\beta)}(x)$ is in $C_0^\infty([-T,T]\,;\,\gamma_{A_0}^{(s)}$ and satisfies

(2.7) $\qquad [a_{(\beta)}]_{s,A_0,l}<C_l A_0^{|\beta|}\,|\beta|\,!\,.$

Hence we have by Lemma 1.1, Proposition 2.7 and (2.7)

$$\|a\circ v\|_{W_{l,q,h}^{(A)}}\leqslant C_l\sum_{\beta\neq 0}\frac{1}{\beta\,!}[a_{(\beta)}]_{s,A_0,M_A(l,q)}\left(\|v\|_{W_{l,q,h}^{(A)}}\right)^{|\beta|}$$

$$\leqslant C_l\sum_{j=1}^{\infty}\left(rA_0\|v\|_{W_{l,q,h}^{(A)}}\right)^j\leqslant C_l\,,$$

where C_l is independent of h.

Let Q be a cube consisting of x in R^{n+1} with each $|x_j|<1$ $j=1,\cdots,n+1$, and $Q_k(k=1,\cdots)$ an enumeration of tranlates of Q centered at the integer lattice point $g^{(k)}$ of R^{n+1}. Let $\theta(x)$ in $C_0^\infty(R^{n+1})$ be supported in Q, and $\varphi_j(x)=\theta(x-g^{(j)})\times\left[\sum_{k=1}^{\infty}\theta(x-g^{(k)})^2\right]^{-1/2}$ and $\chi(x)\in C_0^\infty R^{n+1})$ such that $\chi=1$ on $\mathscr{B}_{R_1}=\{x\in R^1,\ |x|\leq R_1\}$. We define $\varphi_{jk}(x,\xi')$, $j,\ k=0,1,\cdots$,

$$\varphi_{00}(x,\xi')=\left(1-\chi(x)\right)\left(\chi^2+(1-\chi)^2\right)^{-1/2}$$

$$\varphi_{0j}=\varphi_{k0}=0\,,\qquad j,\ k=1,2,\cdots$$

$$\varphi_{jk}(x,\xi')=\varphi_j\left((x_A-g^{(k)})\langle\xi'\rangle_h^\sigma\right)\varphi_k(x)\,\chi(x)\left(\chi^2+(1-\chi)^2\right)^{-1/2}$$

$$j,\ k=1,2,\cdots,$$

where $x_A=(x_0,x'+\mathrm{grad}_{\xi'}\langle\xi'\rangle_h^\kappa)$ (if there is no confusion we write also $x_A=e^A x\, e^{-A}$) and $0<\sigma<1/2$. Then we have

LEMMA 2.10. *$\{\varphi_{jk}(x,\xi')\}_{j,k=0,1,\cdots}$ satisfies*

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\varphi_{jk}(x,\xi')^2=1\,,\qquad on\ R^{n+1}\times R^n\,,$$

(2.8) $\qquad \left|\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\varphi_{jk(\beta)}^{(\alpha)}(x,\xi')\right|\leqslant C_{\alpha\beta}\langle\xi'\rangle_h^{\sigma|\beta|-(1-\sigma)|\alpha|}$

on $R^{n+1}\times R^n$, for $(\alpha,\beta)\in N^{n+1}\times N^n$.

It follows from (2.8) that $\varphi_{jk(\beta)}^{(\alpha)}(x,\xi')$ is in $S_{1-\sigma,\sigma,h}^{(0,\sigma|\beta|-(1-\sigma)|\alpha|)}$ uniformly in (j,k), if we take $0<\sigma<1/2$. Therefore we obtain the following proposition by Lemma 2.1 and Proposition 2.6.

PROPOSITION 2.11. *Let $\{\varphi_{jk}(x,D)\}$ be pseudo-differential operators with symbols $\{\varphi_{jk}(x,\xi')\}$. and $0<\sigma<1/2$. Then there are positive numbers C and h_0 such that*

$$C^{-1}\|u\|^2_{W_{l,q,h}} \leqslant \sum_{j,k} \|\varphi_{jk}u\|^2_{W_{l,q,h}} \leqslant C\|u\|^2_{W_{l,q,h}},$$

for any u *in* $W_{l,q,h}$ *and* $h \geqslant h_0$.

We put

$$x_{jk} = x_{jk}(D') = g^{(k)} + g^{(j)}\langle D'\rangle_h^{-\sigma}, \qquad j,\ k = 1, 2, \cdots. \tag{2.9}$$

If there is no confusion, we write also $x_{jk} = g^{(k)} + g^{(j)}\langle\xi'\rangle_h^{-\sigma}$. Since the support of $\varphi_{jk}(x, \xi')$ is contained in $\{(x, \xi') \in R^{n+1} \times R^n\,;\ |x_\Lambda - x_{jk}| \leqslant \langle\xi'\rangle_h^{-\sigma}\}$, $(x_\Lambda - x_{jk})^\gamma \varphi_{jk}(x, \xi')$ is in $S^{(0,-|\gamma|\sigma)}_{1-\sigma,\sigma,h}$ uniformly in (j, k). Hence we have

Proposition 2.12. (c.f. [5]). *Assume that*

$$a_{jk}(x, \xi) = \sum_{|\gamma|=d} (x_\Lambda - x_{jk})^\gamma b_{jk\gamma}(x, \xi)$$

where $b_{jk}(x, \xi)$ *is in* $S^{(m,p)}_{1,0,h}$ *uniformly in* (j, k). *Then we have a positive number* $C_{l,q}$ *independent of* h *such that*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}\varphi_{jk}u\|^2_{W_{l,q,h}} \leqslant C[\sup_{j,k} |b_{jk}|^{(m,p)}_{M_1(l,q)}]^2 \|u\|^2_{W_{l+m,p+q-d\sigma,h}}, \tag{2.10}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|[a_{jk}, \varphi_{jk}]\, u\|^2_{W_{l,q,h}} \leqslant C[\sup_{j,k} |b_{jk}|^{(m,p)}_{M_1(l,q)}]^2 \|u\|^2_{W_{l+m,p+q-d\sigma+\sigma-1,h}}, \tag{2.11}$$

where

$$M_1(l, q) = M_0(l, q) + 2[n2^{-1}+1] + 2\left[\frac{|m|+|q|+n+M_0(l, q)}{2(1-\sigma)} + 1\right] \tag{2.11$'$}$$

and $M_0(l, q)$ *is given by* (2.4).

Lemma 2.13. *Let* $a(x, \xi')$ *be in* $C_0^\infty([-T, T]\,;\ \mathscr{A}^{(p,\Lambda)})$ *or in* $C_0^\infty([-T, T]\,;\ \mathscr{A}^{(p)}_{s,A_0,h})$, *where* $\Lambda = (T - x_0)(h + \langle D'\rangle_h^\varepsilon)$. *Then*

$$a(x, D') - a(x_{jk}, D') = \sum_{|\alpha|=1} a_{(\alpha)}(x_{jk}, D')(x - x_{jk})^\alpha$$
$$+ \sum_{|\alpha|=2} Q_{jk\alpha}(a)(x, D')(x - x_{jk})^\alpha + R_{jk}(a)(x, D'),$$

where $x_{jk} = x_{jk}(D')$ *is given by* (2.9) *and* $Q_{jk\alpha}(a)(x, \xi')$ *in* $C_0^\infty([-T, T]\,;\ \mathscr{A}^{(p,\Lambda)})$ (*or* $C_0^\infty([-T, T]\,;\ \mathscr{A}^{(p)}_{s,A_0,h})$) *and* $R_{jk}(a)(x, \xi')$ *in* $C_0^\infty([-T, T]\,;\ \mathscr{A}^{(p-1,\Lambda)})$ (*or* $C_0^\infty([-T, T]\,;\ \mathscr{A}^{(p-1)}_{s,A_0,h})$) *uniformly in* (j, k) *respectively, and it holds for* $l \in N$,

$$\sup_{j,k,\alpha}\, [Q_{jk\alpha}(a)]_l^{(p,\Lambda)} \leqslant C_l [a]^{(p,\Lambda)}_{l+2} \qquad \left(or\ [a]^{(p)}_{a,A_0,l+2}\right) \tag{2.12}$$

$$\sup_{j,k}\, [R_{jk}(a)]_l^{(p-1,\Lambda)} \leqslant C_l [a]^{(p,\Lambda)}_{l+2} \qquad \left(or\ [a]^{(p)}_{s,A_0,l+2}\right) \tag{2.13}$$

where C_l *is independent of* h.

PROOF. By Taylor expansion, we have

$$a(x, \xi')-a\left(x_{jk}(\xi'), \xi'\right)=\sum_{|\alpha|=1} a_{(\alpha)}(x_{jk}, \xi')(x-x_{jk})^{\alpha}$$
$$+\sum_{|\alpha|=2}\frac{1}{\alpha!}\int_0^1 a_{(\alpha)}\left(x_{jk}+\theta(x-x_{jk}), \xi'\right)(x-x_{jk})^{\alpha}d\xi .$$

Hence we obtain

$$Q_{ik\alpha}(a)(x, \xi')=\int_0^1 a_{(\alpha)}\left(x_{jk}+\theta(x-x_{jk}), \xi'\right)d\theta/\alpha!$$

and

$$R_{jk}(a)(x, \xi')=\sum_{|\alpha|=1}\partial_{\xi}^{\alpha}\left(a_{(\alpha)}\left(x_{jk}(\xi), \xi'\right)\right)$$
$$+\sum_{|\alpha|=2} os-\iint e^{-iy\eta'}Q_{jk\alpha}(a)(x, \xi'+\eta')(x+y-x_{jk})\,dyd\eta'$$
$$-\sum_{|\alpha|=2}Q_{jk\alpha}(a)(x, \xi')(x-x_{jk})^{\alpha},$$

which satisfy (2. 12) and (2. 13) respectively.

Noting that $e^{\Lambda}a(x_{jk}(D'), D')e^{-\Lambda}=a(x_{jk}, D')$, we have,

COROLLARY 2. 14. *Let $a(x, \xi')$ be in $C_0^{\infty}([-T, T]; \mathscr{A}^{(p,\Lambda)})$ or $C_0^{\infty}([-T, T]; \mathscr{A}_{s,A_0,h}^{(p)})$, where $\Lambda=(T-x_0)(h+\langle D'\rangle_h^{\kappa})$. Then we have*

$$a_{\Lambda}(x, D')=a\left(x_{jk}(D'), D'\right)+\sum_{|\alpha|=1}a_{(\alpha)}(x_{jk}, D')(x_{\Lambda}-x_{jk})^{\alpha}$$
$$+\sum_{|\alpha|=2}Q_{jk\alpha}(a)_{\Lambda}(x, D')(x_{\Lambda}-x_{jk})^{\alpha}+S_{jk}(a)_{\Lambda}(x, D'), \tag{2. 14}$$

where $Q_{jk\alpha}(a)_{\Lambda}(x, \xi')$ is in $S_{1,0,h}^{(0,p)}$ and $S_{jk}(a)_{\Lambda}(x, \xi')$ in $S_{1,0,h}^{(0,p-1)}$ uniformly in (j, k) and satisfy

$$|Q_{jk\alpha}(a)_{\Lambda}|_l^{(0,p)}\leqslant C_l[a]_{[(l+2)(1-\kappa)^{-1}]+n+l+2}^{(0,p)},$$
$$\left(or\ [a]_{s,A_0,[(l+2)(1-\kappa)^{-1}]+n+l+2}^{(p)}\right) \tag{2. 15}$$

$$|S_{jk}(a)_{\Lambda}|_l^{(0,p-1)}\leqslant C_l[a]_{[(l+2)(1-\kappa)^{-1}]+n+l+2}^{(p,\Lambda)}$$
$$\left(or\ [a]_{s,A_0,[(l+2)(1-\kappa)^{-1}]+n+l+2}^{(p)}\right). \tag{2. 16}$$

PROOF. (2. 15) and (2. 16) follow from (2. 12), (2. 13) and Proposition 2. 5.

LEMMA 2. 15. *Let $a(x, \xi')$ be in $C_0^{\infty}([-T, T]; \mathscr{A}^{(p,\Lambda)})$ or $C_0^{\infty}([-T, T]; \mathscr{A}_{s,A_0,h}^{(p)})$, where $\Lambda=(T-x_0)(h+\langle D'\rangle_h^{\kappa})$. Then we have,*

$$[a_{\Lambda}, \varphi_{jk}]=\sum_{|\alpha|=1}\left\{a_{(\alpha)}(x_{jk}, D')+\sum_{|\beta|=1}a_{(\beta)}(x_{jk}, D')(x_{jk}^{\beta})^{(\alpha)}\right\}\varphi_{jk(\alpha)}(x, D')$$
$$+T_{jk}(a)(x, D'), \tag{2. 17}$$

where $T_{jk}(a)(x,\xi')$ *is in* $S^{(0,p-1)}_{1-\sigma,\sigma,h}$ *and satisfies*

$$(2.18)\qquad \sum_{j,k}\|T_{jk}(a)u\|^2_{W_{l,q,h}}\leqslant C_{l,q}\Big\{[a]^{(p,A)}_{M_2(l,q)}\|u\|_{W_{l,q+p-1,h}}\Big\}^2$$

$$\Big(or\ \Big\{[a]^{(p)}_{s,A_0,M_1(l,q)}\|u\|_{W_{l,q+p-1,h}}\Big\}^2\Big)$$

where

$$(2.19)\qquad M_2(l,q)=\Big[\Big(M_1(l,q)+2\Big)(1-\kappa)^{-1}\Big]+n+M_1(l,q)+2\,,$$

and $M_1(l,q)$ *is given by* (2.11)$'$.

PROOF. We put

$$\begin{aligned} c_{jk}(x,D') &= a_A(x,D')-a(x_{jk},D')\\ &=\sum_{|\alpha|=1}a_{(\alpha)}(x_{jk},D')(x_A-x_{jk})^\alpha\\ &\quad+\sum_{|\alpha|=2}Q_{jk\alpha}(a)_A(x,D')(x_A-x_{jk})^\alpha+S_{jk}(a)_A(x,D')\,. \end{aligned}$$

Then it follows from (2.15), (2.16) and (2.11) that

$$\begin{aligned} (2.20)\qquad \sum_{j,k}\|[C_{jk},\varphi_{jk}]u\|^2_{W_{l,q,h}} &\leqslant C\Big\{\sup_{j,k,|\alpha|=1,}|a_{(\alpha)}(x_{jk},\cdot)|^{(0,p-1)}_{M_1(l,q)}\Big\}^2\|u\|^2_{W_{l,q+p-1,h}}\\ &\quad+C_{l,q}\Big\{\sup_{j,k,|\alpha|=2,}|Q_{jk\alpha}(a)_A|^{(0,p)}_{M_1(l,q)}\|u\|_{W_{l,q+p-\sigma-1,h}}\Big\}^2\\ &\quad+C_{l,q}\Big\{\sup_{j,k}|S_{jk}(a)_A|^{(0,p-1)}_{M_1(l,q)}\|u\|_{W_{l,q+p-2+\sigma,h}}\Big\}^2\\ &\leqslant C_{l,q}\Big\{[a]^{(p,A)}_{M_2(l,q)}\|u\|_{W_{l,q+p-1,h}}\Big\}^2\,. \end{aligned}$$

On the other hand,

$$\Big([a\Big(x_{jk}(D'),D'\Big),\varphi_{jk}]\Big)(x,\xi')=\sum_{|\alpha|=1}a_{jk}{}^{(\alpha)}(\xi')\varphi_{jk(\alpha)}(x,\xi')+\tilde{c}_{jk}(x,\xi')\,,$$

where $a_{jk}(\xi')=a(x_{jk}(\xi'),\xi')$ and

$$\begin{aligned} \tilde{c}_{jk}(x,\xi') &= os-\iint e^{-iy'\eta'}a_{jk}(\xi'+\eta')\varphi_{jk}(x+y,\xi')\,dy'd\eta'\\ &\quad-\sum_{|\alpha|=1}a_{jk}{}^{(\alpha)}(\xi')\varphi_{jk}(x,\xi')\\ &= os-\iint e^{-iy'\eta'}\sum_{|\alpha|=2}a^{(\alpha)}_{jk}(\xi'+\eta')\int_0^1\varphi_{jk}(x+\theta y,\xi')\,d\theta dy'd\eta'\,. \end{aligned}$$

Noting that $a_{jk}(\xi')$ is in $S^{(0,p)}_{1,0,h)}$ uniformly in (j,k), and $\varphi_{jk}(x,\xi')$ satisfies (2.8), we obtain by virtue of Proposition 2.6,

$$\begin{aligned} \sum_{j,k}\|\tilde{c}_{jk}u\|^2_{W_{l,q,h}} &\leqslant C_{l,q}\Big\{\sup|a_{jk}|^{(0,p)}_{M_1(l,q)}\|u\|_{W_{l,q+p-2(1-\sigma),h}}\Big\}^2\\ &\leqslant C_{l,q}\Big\{[a]^{(p,A)}_{M_2(l,q)}\|u\|_{W_{l,q+p-2(1-\sigma),h}}\Big\}^2\,, \end{aligned}$$

which implies (2.18) with (2.20).

Thus summarizing up, we obtain,

PROPOSITION 2.16. *Let* $a(x,\xi)=\sum_{i=0}^{m} a_i(x,\xi')\,\xi_0^{m-i}$ *be polynomial in* ξ_0 *of which coefficients* $a_i(x,\xi')=b_i(x,\xi')+c_i(x,\xi')+d_i(\xi')$, *where* b_i *is in* $C_0^\infty([-T,T];\ \mathscr{A}^{(i,A)})$, c_i *in* $C_0^\infty([-T,T];\ \mathscr{A}^{(i)}_{s,A_0,h})$ *and* d_i *in* $S^{(i)}_{1,0,h}$ *respectively, and* $\Lambda=(T-x_0)(h+\langle D'\rangle_h^\kappa)$, $\kappa=s^{-1}$ *and* $0\leqslant T\leqslant(24n^\kappa A_0^\kappa)^{-1}$. *Then* $a_\Lambda(x,\xi)$ *is in* $S^{(m,0)}_{1,0,h}$ *and satisfies,*

$$\text{(2.21)}\qquad \left\{a(x_{jk}(D'),D_\Lambda)-a_\Lambda(x,D)\right\}\varphi_{jk}(x,D')+[a_\Lambda,\varphi_{jk}]$$

$$=\sum_{|\alpha|=1} a_{(\alpha)}(x_{jk},D_\Lambda)\,(x_\Lambda-x_{jk})^\alpha\varphi_{jk}(x,D')$$

$$+\sum_{|\alpha|=1}\left\{a^{(\alpha)}(x_{jk},D_\Lambda)+\sum_{|\beta|=1} a_{(\beta)}(x_{jk},D_\Lambda)\,x_{jk}^{\beta(\alpha)}(D)\right\}\varphi_{jk(\alpha)}(x,D)$$

$$+R_{jk}(a)(x,D)\,,$$

where $R_{jk}(a)(x,\xi)$ *is in* $S^{(m,-1)}_{1-\sigma,\sigma,h}$ *and*

$$\text{(2.22)}\qquad \sum_{j,k}||R_{jk}(a)\,u||^2_{W_{l,q,h}}\leqslant C_{l,q}\Big\{\sup_i\,[b_i]^{(i,A)}_{M_2(l,q)}+\sup_i\,[c_i]^{(i)}_{M_2(l,q)}$$

$$+\sup_i\,|d_i|^{(i)}_{M_2(l,q)}\Big\}^2||u||^2_{W_{l+m,q-1,h}}$$

where $M_2(l,q)$ *is given by* (2.19).

PROOF. We have

$$a_\Lambda(x,D)=\sum_{i=0}^{m} e^\Lambda a_i(x,D')\,D_0^{m-i}e^{-\Lambda}$$

$$=\sum_{i=0}^{m} a_{i\Lambda}(x,D')\,(D_{0\Lambda})^{m-i}\,,$$

Since $a_{i\Lambda}(x,\xi')$ is in $S^{(0,i)}_{1,0,h}$ and $\xi_{0\Lambda}^{m-i}=(\xi_0-\sqrt{-1}\ \langle\xi'\rangle_h^\kappa)^{m-i}$ in $S^{(m-i,0)}_{1,0,h}$, it is clearly that $a_\Lambda(x,\xi)$ is in $S^{(m,0)}_{1,0,h}$. Moreover for $a_{i\Lambda}(x,D')$, (2.14) and (2.17) are valid. Noting that $e^\Lambda a(x_{jk}(D'),D)\,e^{-\Lambda}=a(x_{jk},D_\Lambda)$, we have

$$\text{(2.23)}\qquad a_\Lambda(x,D)-a(x_{jk},D_\Lambda)=\sum_{i=0}^{m}\Big(b_{i\Lambda}(x,D')-b_i(x_{jk},D')\Big)\,D_{0\Lambda}^{m-i}$$

$$+\sum_{i=1}^{m}\Big(c_{i\Lambda}(x,D')-c_i(x_{jk},D')\Big)\,D_{0\Lambda}^{m-i}$$

$$=\sum_{|\alpha|=1} a_{(\alpha)}(x_{jk},D_\Lambda)\,(x_\Lambda-x_{jk})^\alpha$$

$$+\sum_{i=0}^{m}\Big\{\sum_{|\alpha|=2} Q_{jk\alpha}(b_i+c_i)_\Lambda(x,D')\,(x_\Lambda-x_{jk})^\alpha+S_{jk}(b_i+c_i)_\Lambda\Big\}\,(D_{0\Lambda})^{m-i}$$

$$=\sum_{|\alpha|=1} a_{(\alpha)}(x_{jk},D_\Lambda)\,(x_\Lambda-x_{jk})^\alpha+\tilde{S}_{jk}(x,D)\,.$$

It follows from Corollary 2.14 that $\tilde{S}_{jk}(x, D)\varphi_{jk}(x, D')$ satisfies (2.22). On the other hand we have

$$[a_A, \varphi_{jk}]=\left[a_A-a(x_{jk}, D_A), \varphi_{jk}\right]+\left[a(x_{jk}, D_A), \varphi_{jk}\right].$$

It follows from that Proposition 2.12 that $[a_A-a(x_{jk}, D_A), \varphi_{jk}]$ satisfies (2.22). The last term is

$$\left[a(x_{jk}, D_A), \varphi_{jk}\right]=\sum_{i=0}^{m}\left\{a_i(x_{jk}, D')\,[D_{0A}^{m-i}, \varphi_{jk}] + \left[a_i(x_{jk}, D'), \varphi_{jk}\right]D_{0A}^{m-i}\right\}. \tag{2.24}$$

Noting that $[D_{0A}^{m-i}, \varphi_{jk}]=(m-i)(D_{0A})^{m-i}(D_0\varphi_{jk})(x, D')+\varphi_{jki}(x, D')$, where $\varphi_{jki}(x, \xi)$ is in $S_{1-\sigma,\sigma,h}^{(m-i-2,2\sigma)}$ uniformly in (j, k) and satisfies from (2.8),

$$\sum_{j,k}|\varphi_{jki}{}_{(\beta)}^{(\alpha)}(x, \xi')|\leqslant C_{\alpha\beta}\langle\xi\rangle_h^{m-i-2-\alpha_0}\langle\xi'\rangle_h^{2\sigma-(1-\sigma)|\alpha'|+\sigma|\beta|}.$$

Hence we have from (2.24) and (2.17),

$$\left[a(x_{jk}, D_A), \varphi_{jk}\right]=\sum_{|\alpha|=1}\left\{a^{(\alpha)}(x_{jk}, D_A)+\sum_{|\beta|=1}a_{(\beta)}(x_{jk}, D_A)\,x_{jk}^{\beta(\alpha)}\right\} \times\varphi_{jk(\alpha)}(x, D')+\tilde{T}_{jk}(a)(x, D),$$

where

$$\tilde{T}_{jk}(a)(x, D)=\sum_{i=0}^{m}\left\{a_i(x_{jk}, D')\,\varphi_{jki}(x, D)+T_{jk}(a_i)\right\}D_{0A}^{m-i} + \sum_{i=0}^{m}\sum_{|\beta|=1}a_{(\beta)}(x_{jk}, D_A)\,[x_{jk}^{\beta(\alpha)}\varphi_{jk(\alpha)}, D_{0A}^{m-i}].$$

Hence

$$\sum_{j,k}\|\tilde{T}_{jk}(a)\,u\|_{W_{l,q,h}}^2\leqslant C_{l,q}\sum_{j,k}\sum_{i=0}^{m}\left\{\left(|a_i|_0^{(i)}\,\|\varphi_{jki}u\|_{W_{l+i,q,h}}\right)^2 + \sum_{|\beta|=1}\left(|a_i|_1^{(i)}\,\|[x_{jk}^{\beta(\alpha)}\varphi_{jk(\alpha)}, D_{0A}^{m-i}]\,u\|_{W_{l+i,q,h}}\right)^2 + \|T_{jk}(a_i)\,D_{0A}^{m-i}u\|_{W_{l,q,h}}^2\right\}.$$

Since $x_{jk}^{\beta(\alpha)}(\xi')$ is in $S_{1,0,h}^{(0,-|\alpha|)}$ uniformly in (j, k), it follows from Proposition 2.11 that

$$\sum_{j,k}\|[x_{jk}^{\beta(\alpha)}\varphi_{jk(\alpha)}, D_{0A}^{m-i}]\,u\|_{W_{l+i,q,h}}^2\leqslant C_{l,q}\|u\|_{W_{l+m,q+2\sigma-2,h}}^2.$$

Noting that $T_{jk}(a_i)=T_{jk}(b_i)+T_{jk}(c_i)+T_{jk}(d_i)$, we have by Lemma 2.15

$$\sum_{j,k}\|T_{jk}(a_i)\,D_{0A}^{m-i}u\|_{W_{l,q,h}}^2\leqslant C_{l,q}\left\{[b_i]_{M_2(l,q)}^{(i,A)}+[c_i]_{M_2(l,q)}^{(i)}+|d_i|_{M_2(l,q)}^{(i)}\right\}^2 \times\|u\|_{W_{l+m,q-1,h}}^2.$$

Thus we obtain

$$R_{jk}(a)=\tilde{S}_{jk}(x, D)\,\varphi_{jk}(x, D')+\tilde{T}_{jk}(a)\,(x, D)\,,$$

which satisfies (2. 22).

§ 3. Quasilinear case

We at first consider a following quasilinear equation of unknown $u(x)$,

$$\begin{cases} p(x, D^{m-1}u, D)\,u(x)=b(x, D^{m-1}u)\,, \\ (\partial/\partial x_0)^j u(0, x')=0\,, \qquad j=0, 1, \cdots, m-1\,, \end{cases} \tag{3.1}$$

where $p(x, y, \xi)$ is a hyperbolic polynomial with respect to ξ_0. Here we assume that the coefficients $a_\alpha(x, y)$ of $p(x, y, \xi)$ satisfy (0. 4) for (x, y) in $G\times V$. Let $\theta(x)=\theta_0(x), \cdots, \theta_n(x))$ be a mapping of R^{n+1} to G such that

$$\theta(x)=\begin{cases} x & \text{for } |x|\leqslant R_1/2\,, \\ 0 & \text{for } |x|\geqslant R_1\,, \end{cases}$$

and each $\theta_i(x)$ $(i=0, \cdots, n)$ be in $\gamma_{A_0}^{(s)}(R^{n+1})$, and $\rho(x)$ be a function in $C_0^\infty(R^{n+1})$ such that $\int\rho dx=1$ and $\chi(x)\in\gamma_{A_0}^{(s)}(R^{n+1})$ such that $\chi(x)=1$ on $|x|\leqslant R_1/2$, $\chi(x)=0$ for $|x|\geq R_1$. We put

$$\begin{aligned} p_\varepsilon(x, D^{m-1}u, D) &= p\Big(\theta(x), \rho_\varepsilon*(\chi D^{m-1}u), D\Big) \\ &= \sum_{|\alpha|=m} a_\alpha\Big(\theta(x), \rho_\varepsilon*(\chi D^{m-1}u)\Big)\, D^\alpha\,, \end{aligned} \tag{3.2}$$

where $\rho_\varepsilon(x)=\varepsilon^{-(n+1)}\rho(\varepsilon^{-1}x)$. Then, if u is in $W_{\hat{l},\hat{q},h}^{(\Lambda)}$, where $\Lambda=(T-x_0)\,(h+\langle D'\rangle_h^\kappa)$, $\kappa=s^{-1}$, $0\leq T\leq\min\,((24n^\kappa A_0^\kappa)^{-1}, R_1)$, then $\rho_\varepsilon*(\chi D^{m-1}u)$ in $W_{l-m+1,q,h}^{(\Lambda)}$ for any l, q in R^1 and $\varepsilon>0$, and satisfies for $\hat{l}$ and $\hat{q}$,

$$\|\rho_\varepsilon*\chi D^{m-1}u\|_{W_{\hat{l}-m+1,\hat{q},h}^{(\Lambda)}}\leqslant C_0\|u\|_{W_{\hat{l},\hat{q},h}^{(\Lambda)}}\,.$$

where C_0 is independent of h and ε. Therefore it follows from Proposition 2. 9 that if $l\geqslant n+3$, $\hat{q}\geq 0$ and u is in $W_{\hat{l},\hat{q},h}^{(\Lambda)}$ satisfies

$$\|u\|_{W_{\hat{l},\hat{q},h}^{(\Lambda)}}\leqslant(2rA_0C_0)^{-1}\,, \tag{3.3}$$

then $a_\alpha(\theta(x), \rho_\varepsilon*(\chi D^{m-1}u)-a_\alpha(\theta(x), 0)$ is in $W_{\hat{l}-m+1,\hat{q},h}^{(\Lambda)}$ and

$$\|a_\alpha(\theta, \rho_\varepsilon*\chi D^{m-1}u)-a_\alpha(\theta, 0)\|_{W_{\hat{l}-m+1,\hat{q},h}^{(\Lambda)}}\leqslant C(A_0)\,, \tag{3.4}$$

where $\Lambda=(T-x_0)\,(h+\langle D'\rangle_h^\kappa)$, $\kappa=s^{-1}$ and

$$0\leqslant T\leqslant\min\Big((24n^\kappa A_0^\kappa)^{-1}, R_1\Big)\,. \tag{3.5}$$

Moreover for $l>\hat{l}$ or $q>\hat{q}$, we have

$$(3.4)_\varepsilon \qquad ||a_\alpha(\theta, \rho_\varepsilon * \chi D^{m-1}u) - a_\alpha(\theta, 0)||_{W^{(A)}_{\hat{l}-m+1, \hat{q}, h}} \leqslant C(A_0, \varepsilon),$$

for $\varepsilon>0$, where $C(A, \varepsilon)$ depends on ε but does not on h.

Now, instead of (3.1) we consider the Cauchy problem for $p_\varepsilon(x, D^{m-1}u, D)$ as follows,

$$(3.6) \qquad \begin{cases} p_\varepsilon(x, D^{m-1}u, D)\, v(x) = f(x), & x \in [0, T] \times R^n, \\ (\partial/\partial x_0)^j u(0, x') = 0, & j = 0, \cdots, m-1. \end{cases}$$

If $u(x)$ is given, then (3.6) is an equation for $v(x)$. Assume that $u(x)$ in $W^{(A)}_{\hat{l}, \hat{q}, h}$ satisfies (3.3), where $A=(\mathrm{T}-\mathrm{x}_0)(h+\langle D'\rangle_h^\kappa)$ satisfies (3.5), and $\hat{l}$ is a large integer and $\hat{q}\geq 0$, fixed, and that the coefficients $a_\alpha(x, y)$ satisfy (0.4) and $p(x, y, \xi)$ is hyperbolic with respect to ξ_0. Then $a_\varepsilon(x, \xi)=p_\varepsilon(x, D^{m-1}u, \xi)$ is also hyperbolic with respect to ξ_0, and it follows from Lemma 1.4 that there exists a positive constant $C=C(||u||_{W^{(A)}_{\hat{l}, \hat{q}, h}})$ such that

$$(3.7) \qquad |a_\varepsilon(x, \xi_A)^{-1}| \leqslant C\langle \xi \rangle_h^{-m} \langle \xi' \rangle_h^{\tau},$$

$$(3.8) \qquad \left| \frac{a_\varepsilon^{(\alpha)}(x, \xi_A)}{a_\varepsilon(x, \xi_A)} \right| \leqslant C\langle \xi' \rangle_h^{-\kappa}$$

$$(3.9) \qquad \left| \frac{a_{\varepsilon(\alpha)}(x, \xi_A)}{a_\varepsilon(x, \xi_A)} \right| \leqslant C\langle \xi' \rangle_h^{1-\kappa},$$

for $|\alpha|=1$, $x\in R^{n+1}$, $\xi_A=(\xi_0-i(h+\langle \xi' \rangle_h^\kappa), \xi')$, $\xi\in R^{n+1}$, where $\kappa=s^{-1}$, $\tau=\nu(1-\kappa)$ and ν is the maximal multiplicity of roots of a_ε.

THEOREM 3.1. *Assume that the above conditions for p are valid and $\hat{l}$ is a large integer and $\hat{q}\geq\tau$. Then for any l, q in R^1 and $\varepsilon>0$ there exist positive numbers $h=h(l, q, \varepsilon)$ and $C=C(l, q, \varepsilon)$ such that for u in $W^{(A)}_{\hat{l}, \hat{q}, h}$ satisfying* (3.3) *and any v in $W^{(A)}_{l, q, h}$*

$$(3.10) \qquad ||v||_{W^{(A)}_{l, q, h}} \leqslant C(l, q, \varepsilon)\, ||p_\varepsilon(\cdot, D^{m-1}u, D)\, v||_{W^{(A)}_{l-m, q+\tau, h}}$$

where $A=(T-x_0)(h+\langle D'\rangle_h^\kappa)$, $\kappa=s^{-1}$, $0<s<\nu(\nu-1)^{-1}$ and T satisfies (3.5). *In paticular if $0\leq l\leq \hat{l}+1$ and $0\leq q\leq \hat{q}-\tau$, then $h(l, q, \varepsilon)$ and $C(l, q, \varepsilon)$ do not depend on ε.*

For the adjoint operator $p_\varepsilon^{(*)}$ of p_ε, we obtain analogously,

THEOREM 3.2. *Assume that the conditions of Theorem* 3.1 *are valid. Then for any l, q in R^1 and $\varepsilon>0$ there exist $h=h(l, q, \varepsilon)$ and $C=C(l, q, \varepsilon)$ such that for u in $W^{(A)}_{\hat{l}, \hat{q}, h}$ satisfying* (3.3) *and for any w in $W^{(-A)}_{l, q, h}$,*

$$||w||_{W^{(-A)}_{l, q, h}} \leq C(l, q, \varepsilon)\, ||p_\varepsilon^{(*)}(\cdot, D^{m-1}u, D)\, w||_{W^{(-A)}_{l-m, q+\tau, h}}.$$

Therefore it follows from that Theorem 3.1 and Theorem 3.2 that by virtue of Riesz' theorem we obtain an existence theorem (c. f. [4]) as follows,

THEOREM 3.3. *Assume that the conditions of Theorem* 3.1 *are valid. Then if* $0\leqslant l\leq\hat{l}+1$ *and* $0\leq q<\hat{q}-\tau$, *there exist* $h=h(l, q)$ *and* $C=C(l, q)$ *such that for any* u *in* $W^{\{A\}}_{\hat{l},\hat{q},h}$ *satisfying* (3.3) *and for any* f *in* $W^{\{A\}}_{l-m,q+\tau,h}$, *we have a unique solution* v *in* $W^{\{A\}}_{l,q,h}$ *of the following equation,*

$$p\Big(\theta(x), \chi D^{m-1}u(x), D\Big)\, v(x)=f(x), \qquad x \text{ in } R^{n+1}, \tag{3.11}$$

and v *satisfies*

$$\|v\|_{W^{\{A\}}_{l,q,h}}\leqslant C(l, q)\,\|f\|_{W^{\{A\}}_{l-m,\,q+\tau,h}}\,.$$

Since the conditions of Theorem 3.3 are invariant under Holmgen's transforms, the solution of (3.11) with zero initial data has a finite propagation speed. Therefore we obtain the following (c. f. Theorem 4.5 in [4]),

THEOREM 3.4. *Assume that the conditions of Theorem* 3.1 *are valid. If the support of* f *is compact in* $[0, \infty)\times R^n$, *then the solution* v *of* (3.11) *has a compact support in* $[0, T_0]\times R^n=\Omega_{T_0}$ *for* $T_0>0$ *and satisfies*

$$\|v\|_{W^{\{A\}}_{l+1,q-\tau,h(\Omega_{T_0})}}\leqslant C(l, q)\,\|f\|_{W^{\{A\}}_{l-m+1,q,h(\Omega_{T_0})}}, \tag{3.12}$$

for $2m+(n+1)/2<l\leqslant\hat{l}$ *and* $\tau\leqslant q\leqslant\hat{q}$.

PROOF OF THEOREM 3.1. We put

$$a_\epsilon(x, D)=p_\epsilon(x, D^{m-1}u, D)$$

$$g=e^{-\Lambda}a_\epsilon(x, D)\,v=a_{\epsilon\Lambda}(x, D)\,w\,, \qquad w=e^{\Lambda}v\,,$$

$$a_{\epsilon\Lambda}(x, D)=e^{\Lambda}a_\epsilon e^{-\Lambda}\,, \qquad a_{\epsilon jk}(D)=a_\epsilon\Big(x_{jk}(D), D_\Lambda\Big)$$

$$D_\Lambda=e^{\Lambda}De^{-\Lambda}=\Big(D_0-i(h+\langle D'\rangle_h^{\kappa}), D'\Big)$$

where $x_{jk}(D)$ is given by (2.9). Then applying $\varphi_{jk}(x, D')$ to g and noting that $e^{\Lambda}a_\epsilon(x_{jk}, D)\,e^{-\Lambda}=a_\epsilon(x_{jk}, D_\Lambda)=a_{\epsilon jk}(D)$,

$$a_{\epsilon jk}(D)\,\varphi_{jk}(x, D')\,w=\Big(a_{\epsilon jk}(D)-a_{\epsilon\Lambda}(x, D)\Big)\,\varphi_{jk}w+[a_{\epsilon\Lambda}, \varphi_{jk}]\,w$$
$$+\varphi_{jk}(x, D')\,g(x)\,,$$

where [,] stands for a commutator. Since $a_{\epsilon jk}(\xi)=a_\epsilon(x_{jk}(\xi'), \xi_\Lambda)$ satisfies (3.7), there exists the inverse $a_{\epsilon jk}(D)^{-1}$ of $a_{\epsilon jk}(D)$ in $W_{l,q,h}$. Hence from (2.21) we have

$$(3.13)\qquad \varphi_{jk}w = a_{\varepsilon jk}(D)^{-1}\Big[\sum_{|\alpha|=1}\Big\{a_{\varepsilon(\alpha)}(x_{jk}, D_A)(x_A - x_{jk})^{\alpha}\varphi_{jk}w + \sum_{|\alpha|=1}\Big\{a_\varepsilon^{(\alpha)}(x_{jk}, D_A) + \sum_{|\beta|=1} a_{\varepsilon(\beta)}(x_{jk}, D_A)\, x_{jk}^{\beta(\alpha)}\Big\}\varphi_{jk}w + R_{jk}(a_\varepsilon)(x, D)\,w + \varphi_{jk}(x, D')\,g(x)\Big]$$

Since for $|\alpha|=1$ $a_\varepsilon(x_{jk}(\xi'), \xi_A)^{-1}a_\varepsilon^{(\alpha)}(x_{jk}(\xi'), \xi_A)$ and $a_\varepsilon(x_{jk}(\xi'), \xi_A)^{-1}a_{\varepsilon(\alpha)}(x_{jk}(\xi'), \xi_A)$ satisfy (3.8) and (3.9) respectively, we obtain

$$\|\varphi_{jk}w\|^2_{W_{l,q,h}} \leqslant C\Big[\sum_{|\alpha|=1}\|(x_A - x_{jk})^{\alpha}\varphi_{jk}w\|^2_{W_{l,q+1-\kappa,h}} + \sum_{|\alpha|=1}\Big\{\|\varphi_{jk(\alpha)}w\|^2_{W_{l,q-\kappa,h}} + \sum_{|\beta|=1}\|x_{jk}^{\beta(\alpha)}\varphi_{jk(\alpha)}w\|^2_{W_{l,q+1-\kappa,h}}\Big\} + \|R_{jk}(a_\varepsilon)\,w\|^2_{W_{l-m,q+\tau,h}} + \|\varphi_{jk}g\|^2_{W_{l-m,q+\tau,h}}\Big]$$

where C is independent of h. Noting that $(x_{jk}^{\beta})^{(\alpha)} = (\partial/\partial\xi)^{\alpha}(x_{jk}(\xi')^{\beta})$ is in $S_{1,0,h}^{(0,-|\alpha|)}$ uniformly in (j, k), we obtain from Proposition 2.12,

$$(3.14)\qquad \sum_{j,k}\|\varphi_{jk}w\|^2_{W_{l,q,h}} \leqslant C\Big[\|w\|^2_{W_{l,q+1-\kappa-\sigma,h}} + \|w\|^2_{W_{l,q-\kappa+\sigma,h}} + \|w\|^2_{W_{l,q-\kappa,h}} + \sum_{j,k}\|R_{jk}(a_\varepsilon)\,w\|^2_{W_{l-m,q+\tau,h}} + \|g\|^2_{W_{l-m,q+\tau,h}}\Big],$$

where C is independent of ε and h. We decompose

$$\begin{aligned} a_\varepsilon(x, D) &= p(\theta(x), \rho_\varepsilon * \chi D^{m-1}u, D) \\ &= p(\theta(x), \rho_\varepsilon * \chi D^{m-1}u, D) - p(\theta(x), 0, D) \\ &\quad + p(\theta(x), 0, D) - p(0, 0, D) + p(0, 0, D) \\ &= b_\varepsilon(x, D) + c(x, D) + d(D). \end{aligned}$$

Then we have

$$b_\varepsilon(x, D) = \sum_{i=0}^{m} b_{\varepsilon i}(x, D')\, D_0^{m-i} = \sum_{|\alpha|=m} b_{\varepsilon\alpha}(x)\, D^{\alpha},$$

where $b_{\varepsilon\alpha}(x) = a_\alpha(\theta(x), \rho_\varepsilon * \chi D^{m-1}u) - a_\alpha(\theta(x), 0)$ satisfies (3.4) and $(3.4)_\varepsilon$. Hence $b_{\varepsilon i}(x, \xi')$ is in $C_0^\infty([-T, T]; \mathscr{A}^{(i,A)})$ and satisfies

$$[b_{\varepsilon i}]_l^{(i,A)} \leq C \sup_\alpha \|b_{\varepsilon\alpha}\|_{W_{l,0,h}^{(A)}}.$$

Noting that the coefficients of $c(x, D)$ and $d(D)$ do not depend on ε and u, and that the coefficients $c_i(x, \xi')$ of ξ_0^{m-i} in $c(x, \xi)$ is in $C_0^k([-T, T]; \mathscr{A}_{s,A_0,h}^{(i)})$, we obtain by virtue of (2.22) in Proposition 2.16,

$$(3.15)\qquad \sum_{j,k}\|R_{jk}(a_\varepsilon)\, w\|^2_{W_{l-m,q+\tau,h}} \leq C\Big[\sup_{0\leq i\leq m}\Big\{[b_{\varepsilon i}]^{(i)}_{M_2(l-m,q+\tau)}+|c_i|^{(i)}_{s,A_0,M_2(l-m,q+\tau)} + |d_i|^{(i)}_{M_1(l-m,q+\tau)}\Big\}\Big]^2\|w\|^2_{W_{l,q+\tau-1,h}}\,.$$

Thus if we take σ $(0<\sigma<1/2)$ such that

$$\rho=\inf\{\kappa+\sigma-1,\ \kappa-\sigma,\ \kappa,\ 1-\tau\}>0\,,$$

we have from (3.14), (3.15) and (3.4)

$$\|w\|^2_{W_{l,q,h}}\leqslant C_{l,q}\Big(1+C(\varepsilon,A_0)\Big)^2h^{-2\rho}\|w\|^2_{W_{l,q,h}}+C_{l,q}\|g\|^2_{W_{l-m,q+\tau,h}}\,.$$

Hence if we choose h such that

$$C_{l,q}\Big(1+C(\varepsilon,A_0)\Big)\,h^{-2\rho}\leqslant 1/2\,,$$

we obtain the estimate (3.10).

Next we shall prove (3.10) for a positive integer l. Applying D^α $(|\alpha|=l)$ to g, we have

$$a_{\varepsilon A}(x,D)\,D^\alpha w=\Big[a_{\varepsilon A}(x,D),D^\alpha\Big]\,w+D^\alpha g=g_\alpha\,.$$

Then by virtue of (3.14) and (3.15) we have

$$(3.16)\qquad \|D^\alpha w\|^2_{W_{0,q,h}}\leqslant C_q\Big(1+\sup_\beta\|b_{\varepsilon\beta}\|_{W^{(A)}_{M_2(-m,q),0,h}}\Big)^2h^{-2\rho}\|D^\alpha w\|^2_{W_{0,q,h}} + C_q\|g_\alpha\|^2_{W_{-m,q+\tau,h}}\,,$$

where C_q is independent of ε and h. For g_α we have

$$\|g_\alpha\|^2_{W_{-m,q+\tau,h}}\leqslant\|g\|^2_{W_{l-m,q+\tau,h}}+\|[a_{\varepsilon A},D^\alpha]\ w\|^2_{W_{-m,q+\tau,h}}\,,$$

and

$$(3.17)\qquad \|[a_{\varepsilon A},D^\alpha]\ w\|_{W_{-m,q+\tau,h}}\leqslant\|[b_{\varepsilon A},D^\alpha]\ w\|_{W_{-m,q+\tau,h}}+\|[c_A,D^\alpha]\ w\|_{W_{-m,q+\tau,h}}\,.$$

Since $(c_A)(x,\xi)$ is in $S^{(m,0)}_{1,0,h}$, we have from Proposition 2.6,

$$(3.18)\qquad \|[c_A,D^\alpha]\ w\|_{W_{-m,q+\tau,h}}\leqslant C\|w\|_{W_{l-1,q+\tau,h}}\leqslant Ch^{-\rho}\|w\|_{W_{l,q,h}}\,,$$

where C is independent of ε and h.

Finally we shall estimate $[b_{\varepsilon A},D^\alpha]\ w$. We have

$$[b_{\varepsilon A}, D^\alpha]\, w = -\sum_{0\neq\gamma\leqslant\alpha}\binom{\alpha}{\gamma} b_{\varepsilon A(\gamma)}(x, D)\, D^{\alpha-\gamma} w$$

$$= -\sum_{\gamma}\binom{\alpha}{\gamma}\sum_{|\beta|=m} b_{\varepsilon\beta A(\gamma)}(x)\, D_A^\beta D^{\alpha-\gamma} w$$

$$= -\sum_{|\beta|=m}\sum_{0<\gamma\leqslant\alpha}\binom{\alpha}{\gamma} e^A(D_{-A}^\gamma b_{\varepsilon\beta})\, D^\beta D_{-A}^{\alpha-\gamma} v + D^\beta v D_{-A}^\alpha b_{\varepsilon\beta},$$

where $D_{-A}=e^{-A}De^A$. Hence we have

$$||[b_{\varepsilon A}, D^\alpha]\, w||_{W_{-m,q+\tau,h}} \leqslant C \sum_{0<\gamma\leqslant\alpha, |\beta|=m} ||(D_{-A}^\gamma b_{\varepsilon\beta})\,(D^\beta D_{-A}^{\alpha-\gamma} v)||_{W_{-m,q+\tau,h}^{(A)}}.$$

For $0<|\gamma|\leqslant l_0$ (determined later on), we obtain by virtue of Proposition 2.7,

$$||(D_{-A}^\gamma b_{\varepsilon\beta})\,(D^\beta D_{-A}^{\alpha-\gamma} v)||_{W_{-m,q+\tau,h}^{(A)}}$$
$$\leqslant C||D_{-A}^\gamma b_{\varepsilon\beta}||_{W_{M_A(-m,q+\tau),q+\tau,h}^{(A)}}||v||_{W_{l-1,q+\tau,h}^{(A)}}$$
$$\leqslant C||b_{\varepsilon\beta}||_{W_{M_A(-m,q+\tau)+l_0,q+\cdot,h}^{(A)}}||v||_{W_{l-1,q+\tau,h}^{(A)}}.$$

For $|\gamma|>l_0$, we have analogously,

$$||(D_{-A}^\gamma b_{\varepsilon\beta})\,(D^\beta D_{-A}^{\alpha-\gamma} v)||_{W_{-m,q+\tau,h}^{(A)}}$$
$$\leqslant C||D^\beta D_{-A}^{\alpha-\gamma} v||_{W_{M_A(-m,q+\tau),q+\tau,h}^{(A)}}||b_{\varepsilon\beta}||_{W_{l-m,q+\tau,h}^{(A)}}$$
$$\leqslant C||v||_{W_{M_A(-m,q+\ )+m+l-l_0,q+\tau,h}^{(A)}}||b_{\varepsilon\beta}||_{W_{l-m,q+\tau,h}^{(A)}}.$$

Therefore if $\hat{l}\geqslant 2M_A(-m, \hat{q})+m$, we can choose l_0 such that

$$M_A(-m, \hat{q})+l_0\leqslant \hat{l}-m+1\,,$$
$$M_A(-m, \hat{q})+m-l_0\leqslant -1\,,$$

and we have from (3.4) for $|\alpha|=l\leqslant\hat{l}+1$ and $q+\tau\leqslant\hat{q}$,

$$(3.19)\qquad ||[b_{\varepsilon A}, D^\alpha]\, w||_{W_{-m,q+\tau,h}} \leqslant C||v||_{W_{l-1,q+\tau,h}^{(A)}} \leqslant Ch^{-\rho}||w||_{W_{l,q,h}^{(A)}},$$

where C is independent of h and ε. Thus we obtain from (3 16,) (3.17), (3.18) and (3.19),

$$(3.20)\qquad ||w||_{W_{l,q,h}} \leqslant Ch^{-\rho}\Big\{1+\sup ||b_{\varepsilon\beta}||_{W_{M_2(-m,q),0,h}^{(A)}}\Big\}\,||w||_{W_{l,q,h}}$$
$$+C||g||_{W_{l-m,q+\tau,h}},$$

where C is independent of h and ε. If $\hat{l}-m+1\geqslant M_2(-m, \hat{q})$, then we have by (3.4) and (3.20)

$$||v||_{W_{l,q,h}^{(A)}} \leqslant C||f||_{W_{l-m,q+\tau,h}^{(A)}},$$

for any $h \geq h_0$, where C and h_0 are independent of ε. Thus we have proved Theorem 3.1.

Now we return to the equation (3.1). By Taylor's expantion, we put

$$\tilde{u}(x) = u(x) - \sum_{j=0}^{M} (\partial/\partial x_0)^j u(0, x') x_0^j (j!)^{-1}.$$

Then we can regard (3.1) the equation of $\tilde{u}(x)$ and if we take M sufficiently large, we may assume without loss of generality

$$b(x, D^{m-1}u) = x_0^{\hat{l}-m+1} \tilde{b}(x, D^{m-1}u). \tag{3.21}$$

Let $\chi(x)$ be in $\gamma_{A_0}^{(s)}(R^{n+1})$ such that $\chi=1$ for $|x| \leqslant R_1/2$ and $\chi=0$ for $|x| \geq R_1$. Then if u is in $W_{\hat{l},\hat{q},h}^{\{A\}}$ satisfying (3.3), it follows from Lemma 1.3 and Proposition 2.9 that $\chi\tilde{b}(x, D^{m-1}u)$ is in $W_{\hat{l}-m+1,\hat{q},h}^{\{A\}}$. We denote by $B(T_0, R)$ all functions $u(x)$ in $W_{\hat{l},\hat{q},h}^{\{A\}}$ satisfying

$$||u||_{W_{\hat{l},\hat{q},h}^{\{A\}}} \leqslant (2rA_0C_0)^{-1}, \tag{3.22}$$

and

$$\{\mathrm{supp} u\} \cap \{x_0 \leqslant T_0\} \subset [0, T_0] \times \left\{|x'| \leqslant R\right\}. \tag{3.23}$$

We define an operator K of $B(T_0, R)$ to $B(T_0, R)$ such that for $K(u)$ is the solution of the following equation,

$$\begin{cases} p\left(\theta(x), D^{m-1}u, D\right) v(x) = \chi(x)\, b(x, D_u^{m-1}), & x_0 > 0, \\ D_0^j v(0, x') = 0, \quad j = 0, \cdots, m-1. \end{cases} \tag{3.24}$$

Then it follows from (3.12) in Theorem 3.4 that

$$\begin{aligned} ||K(u)||_{W_{\hat{l},\hat{q},h(\Omega_{T_0})}^{\{A\}}} &\leqslant C||xb||_{W_{\hat{l}-m,\hat{q}+\tau,h(\Omega_{T_0})}^{\{A\}}} \\ &\leqslant C_1||x_0^{-m+1}\tilde{b}||_{W_{\hat{l}-m,\hat{q}+\tau,h(\Omega_{T_0})}^{\{A\}}} \\ &\leqslant C_2 T_0 ||\tilde{b}||_{W_{\hat{l}-m,\hat{q}+\tau,h(\Omega_{T_0})}^{\{A\}}} \\ &\leqslant C_3 T_0 \leq (2rA_0C_0)^{-1}, \end{aligned}$$

if we choose T_0 sufficiently small, and that also $K(u)$ is in $W_{\hat{l}+1,\hat{q}-\tau,h}^{\{A\}}(\Omega_{T_0})$ and $\{K(u); u \in B(T_0, R)\}$ is a bounded set in $W_{\hat{l}+1,\hat{q}-\tau,h}^{\{A\}}(\Omega_{T_0})$. Moreover if we take R suitably, it is clearly that $K(u)$ has the property (3.23). Since $\tau < 1$, it follows from Relleich's theorem that the image $\{K(u); u \in B(T_0 R)\}$ is compact in $W_{\hat{l},\hat{q},h}^{\{A\}}(\Omega_{T_0})$. Therefore by applying Schauder's fixed point theorem, we know that K has a fixed point in $B(T_0, R)$. Thus we have, THEOREM 3.5. *Assume that the condition of Theorem* 3.1 *are valid. Then for any large integer* l $(l>M_2(-m, \tau))$ *there exists a positive number* h *such that there is a function* $u(x)$ *in* $W^{(4)}_{l,\tau,h}$ *satisfying* (3.1) *in a neighborhood of* 0 *in* R^{n+1}.

§ 4. Proof of Theorem 0.1

We shall reduce the nonlinear equations (0.1) to the quasilinear ones following Dionne [3].

We put

$$p_{jk}(x, y, D) = \sum_{\alpha \in M_{jk}} F_{jy_\alpha}(x, y) D^\alpha ,$$

and

$$\begin{aligned} &U=(U_0, U_1, \cdots, U_{n+1}), \qquad U_i=(U_{1i}, \cdots, U_{Ni}),\ i=0, 1, \cdots, n+1, \\ (4.1)\qquad &U_{ji}=\partial/\partial x_i U_j, \qquad j=1, \cdots, N,\ i=0, 1, \cdots, n, \\ &U_{jn+1}=u_j, \qquad j=1, \cdots, N. \end{aligned}$$

Differentiating (0.1) by x_i,

$$(4.2)\qquad \sum_{k=1}^{N} p_{jk}(x, D^{M_j}u, D)\, U_{ki}+F_{x_i}(x, D^{M_j}u)=0, \quad j=1, \cdots, N,\ i=0, \cdots, n.$$

By Taylor's expansion,

$$(4.3)\qquad F_j(x, D^{M_j}u)-F(x, 0)=\sum_{k=1}^{N} p_{jk}(x, D^{M_j}u, D)\, U_{kn+1} + G_j(x, D^{M_j}u), \qquad j=1, \cdots, N,$$

where $G_j(x, y)=\int_0^1 \sum_{i=1}^{N} \sum_{\alpha\in M_{ji}} F_{iy_\alpha}(x, \theta y)\, y_\alpha d\theta$.

Here we can assume without loss of generality that

$$(4.4)\qquad m+n_k-n_j \geq 1, \qquad j,\ k=1, \cdots, N.$$

In fact, if it is not so, we choose a positive integer m' such that $m+m'+n_k-n_j \geqslant 1$ for all j, k. Then we operate a strictly hyperbolic operator $p_j(D)$ of order m' to $F_j=0$,

$$(4.5)\qquad p_j(D)\, F_j(x, D^{M_j}u)=\tilde{F}_j(x, D^{\tilde{M}_j}u)=0, \qquad j=1, \cdots, N.$$

Then it is clearly that

$$|\alpha| \leqslant m+m'+n_k-n_j \quad \text{for} \quad \alpha\in \tilde{M}_{jk}.$$

and the characteristic polynomial $\tilde{p}(x, y, \xi)$ for $\{\tilde{F}_j\}$ is

$$\tilde{p}(x, y, \xi) = \prod_{j=1}^{N} p_j(\xi)\, p(x, y, \xi)\,,$$

of which maximal multiplicity is same to one of $p(x, y, \xi)$, if we choose $p_j(\xi)$ suitably. It is evident that the equations (0. 1) are equivalent to (4. 5).

Let $e_l = (0, \cdots, 0, 0, \overset{l}{\hat{1}}, 0, \cdots, 0)$ be a unit vector in R^{n+1}. We put

$$l(\alpha) = \inf(l\,;\ \alpha_l \neq 0) \qquad \text{for} \quad \alpha \neq 0\,,$$

$$\gamma(\alpha) = e_{l(\alpha)} \qquad \text{for} \quad \alpha \neq 0\,.$$

For $\alpha \in M_{jk}$, we define

$$\beta(\alpha) = \alpha - \gamma(\alpha),\ \alpha \in M_{jk},\ |\alpha| = m + n_k - n_j\,.$$

Then we can rewrite

$$D^{M_{jk}} u_k = \{D^{L^i_{jk}} U_{ki},\ i = 0, \cdots, n+1\}\,, \qquad j,\ k = 1, \cdots, N\,,$$

where

$$L^i_{jk} = \begin{cases} \{\beta \in M_{jk}\,;\ ^{\exists}\alpha \in M_{jk},\ |\alpha| = m + n_k - n_j,\ s.t. \\ \qquad \beta = \beta(\alpha),\ i = l(\alpha)\}\,, \qquad i = 0, \cdots, n\,, \\ \{\beta \in M_{jk}\,;\ |\beta| < m + n_k - n_j\}\,, \qquad i = n+1\,. \end{cases}$$

Then we have evidently

(4. 6) $\quad |\beta| \leqslant m - 1 + n_k - n_j\,, \qquad \beta \in L^i_{jk}$, for $i = 0, 1, \cdots, n+1,\ j,\ k = 1, \cdots, N$.

We define

$$D^{L_j} U = \{D^{L^i_{jk}} U_{ki}\,;\ i = 0, \cdots, n+1,\ k = 1, \cdots, N\}\,.$$

Now we obtain the following equations,

$$\text{(4. 7)} \qquad \begin{cases} \sum_{k=1}^{N} p_{jk}(x, D^{L_j} U, D)\, U_{ki} = B_{ji}(x, D^{L_j} U), \\ \qquad\qquad j = 1, \cdots, N,\ i = 0, 1, \cdots, n+1\,. \\ D_0^t U_{ki}(0, x') = 0\,, \qquad t = 0, \cdots, m-1,\ i = 0, \cdots, n+1\,, \end{cases}$$

where we may assume that the initial data are zero and B_{ji} satisfies

$$\text{(4. 8)} \qquad B_{ji}(x, y) = x_0^{\hat{l} - m + n_j} \tilde{B}_{ji}(x, y)\,.$$

Moreover we assume that the coefficients of $p_{jk}(x, y, \xi)$ and $\tilde{B}_{ji}(x, y)$ satisfy (0. 4). Then we consider the linearized equations of (4. 7) as follows

$$\text{(4. 9)} \qquad \sum_{k=1}^{N} p_{jk}\Big(\theta(x), \chi D^{L_j} U, D\Big)\, V_{ki} = \chi B_{ji}(x, D^{L_j} U)\,,$$

where $\theta(x)$ and $\chi(x)$ are given in § 3.

THEOREM 4.1. *Assume that $\hat{l}$ is a large integer, $\hat{q}\geq\tau$, and $U_{ji}(x)$ is in $W^{(A)}_{\hat{l},\hat{q},h}$ satisfying*

$$(4.10)\qquad \Big\{\sum_{j,i}\|U_{ji}\|^2_{W^{(A)}_{\hat{l}+n_j,\hat{q},h}}\Big\}^{1/2}\leqslant(2rC_0A_0)^{-1}.$$

Then the following energy estimates hold

$$(4.11)\qquad \sum_{j,i}\|V_{ji}\|^2_{W^{(A)}_{l+n_j,q,h}}\leqslant C_{l,q}\sum_{j,i}\|\sum_{k=1}^N p_{jk}V_{ki}\|^2_{W^{(A)}_{l-m+n_j,q+\tau,h}}$$

for $2mN+(n+1)/2+1<l\leqslant\hat{l}+1,\ 0<q<\hat{q}-\tau.$

PROOF. We put

$$f_{ji}=\sum_{k=1}^N p_{jk}V_{ki}.$$

Then we have for $|\alpha|=l$,

$$(4.12)\qquad \sum_{k=1}^N p_{jk}(D^\alpha V_{ki})=-\sum_k[p_{jk},D^\alpha]\,V_{ki}+D^\alpha f_{ji}=\tilde{f}_{ji}.$$

It follows from (4.10) and Proposition 2.9 that the coefficients of $(p_{jk}(x, D^{L_j}U, D)-p_{jk}(x,0,D))$ are in $W^{(A)}_{\hat{l}-m+1-n_j,\hat{q},h}$. Hence we have by Lemma 1.2,

$$(4.13)\qquad \|\sum_k[p_{jk},D^\alpha]\,V_{ki}\|^2_{W^{(A)}_{n_j-m,q+\tau,h}}\leqslant C\sum_k\|V_{ki}\|^2_{W^{(A)}_{l+n_k-1,q+\tau,h}}.$$

We put

$$D^\alpha V_{ki}=\sum_{t=1}^N H_{jt}(x,D^LU,D)\,W_t,$$

where $\{H_{kt}(x,y,\xi)\}$ is the cofactor matrix of $\{p_{jk}(x,y,\xi)\}$. Hence

$$(4.14)\qquad \sum_k p_{jk}(x,D^{L_j}u,D)\,H_{kt}(x,D^LU,D)=\delta_{jt}p(x,D^LU,D)+r_{jt}(x,D^LU,D),$$

where $p(x,y,\xi)=\det\{p_{jk}(x,y,\xi)\}$ is hyperbolic in ξ_0 of order mN and

$$(4.15)\qquad \text{order } H_{jk}\leqslant mN-n_k+n_j-m,$$

$$(4.16)\qquad \text{order } r_{jk}\leqslant mN-n_k+n_j-1.$$

Hence by (4.12) and (4.13)

$$p\big(\theta(x),\chi D^LU,D\big)\,W_j=-\sum_k r_{jk}W_k+\tilde{f}_{ji}.$$

Therefore by virtue of Theorem 3.1, (4.13) and (4.16) we have

$$\sum_{j=1}^{N} \|W_j\|^2_{W^{\{A\}}_{mN-m+n_j,q,h}} \leqslant C_q \sum_{j=1}^{N} \|\sum_k r_{jk} W_k + \tilde{f}_{ji}\|^2_{W^{\{A\}}_{n_j-m,q+\tau,h}}$$

$$\leqslant C\Big\{\sum_k \|W_j\|^2_{W^{\{A\}}_{mN-m+n_j-1,q+\tau,h}} + \sum_k \|V_{ki}\|^2_{W^{\{A\}}_{l+n_k-1,q+\tau,h}} + \sum_j \|f_{ji}\|^2_{W^{\{A\}}_{l+n_j-m,q+\tau,h}}\Big\},$$

where C is independent of h. Since $\tau<1$, if h is large, by (4.15) we have

$$\sum_{j,k} \|V_{ji}\|^2_{W^{\{A\}}_{l+n_j,q,h}} \leqslant \sum_{j,k} \|H_{jk} W_k\|^2_{W^{\{A\}}_{l+n_j,q,h}}$$

$$\leqslant C\sum_j \|W_j\|^2_{W^{(A)}{mN-m+n_j,q,h}}$$

$$\leqslant C_{l,q}\Big\{\sum_{j,i} \|V_{ji}\|^2_{W^{\{A\}}_{l+n_j-1,q+\tau,h}} + \sum_{j,i} \|f_{ji}\|^2_{W^{\{A\}}_{l-m+n_j,q+\tau,h}}\Big\},$$

which implies (4.11).

It follows from Theorem 4.1 that the linearized equations (4.9) have a solution $V=\{V_{ji}\}$ (c.f. Theorem 4.5 in [4]).

We define an operator K as

$$K(U)=V,$$

and denote by $B(T_0, R)$ all functions $U(x)=\{U_{ji}\}$, U_{ji} in $W^{\{A\}}_{l+n_j,\hat{q},h}([0, T_0]\times R^n)$ satisfying (4.10) and

$$\operatorname{supp} U_{ji}\cap\{x_0<T_0\}\subset[0, T_0]\times\{|x'|\leqslant R\}.$$

Then K is a compact operator in $B(T_0, R)$, if T_0 and R are chosen suitably. Therefore (4.7) has a solution $U=\{U_{ji}\}$ and $\{U_{jn+1}\}$ satisfies the equations (0.4) in a neighborhood of 0 in R^{n+1}. Thus we have proved Theorem 0.1.

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