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The Cauchy problem for effectively hyperbolic operators

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1. Introduction

Following the pioneering work of Oleinik [8], Ivrii & Petkov [5] and then Hörmander [2] analysed the structure of the principal symbol of a weakly hyperbolic differential operator near a double characteristic. In particular, Ivrii & Petkov conjectured that if the only multiple characteristics are double and these are effectively hyperbolic, in the sense that the Hamilton map has a non-zero real eigenvalue, then the Cauchy problem is well-posed for any lower order terms; that is, strongly well-posed in C^{∞} in the sense of Ivrii [3]. In this paper the well-posedness modulo C^{∞} of the Cauchy problem is proved using microlocal energy estimates derived for an analogously defined class of effectively hyperbolic pseudodifferential operators. More restricted results in this direction have been obtained, under additional hypotheses, by Oleinik [8], who dealt with operators in two independent variables and whose work was subsequently extended by Nishitani, by Ivrii [4] who employed a form of the principal symbol corresponding to a separation of variables and by Iwasaki [6] who assumed a slighly weaker, but still non-trivial, condition of Poisson commutation on the principal symbol see also Yoshikawa [9]. The necessity of effective hyperbolicity for the strong well-posedness in C^{∞} of the Cauchy problem for a differential operator with only double characteristics was shown by Ivrii & Petkov [5].

To define the notion of an effectively hyperbolic pseudodifferential operator, let $P\in\Psi_{cl}^{m}(X)$ be a classical pseudodifferential operator on the C^{∞} manifold X and suppose that P has real principal symbol $p\in C^{\infty}(T^{*}X\backslash 0)$. If $\bar{\rho}\in T^{*}X\backslash 0$ is a double characteristic for P:

$$p(\bar{\rho})=0\,,\qquad dp(\bar{\rho})=0$$

the Hessian of p at $\bar{\rho}$ is well-defined:

$$\mathrm{Hess}\,(p):\ T_{\bar{\rho}}M\times T_{\bar{\rho}}M\rightarrow\boldsymbol{R}\,,\qquad M=T^{*}X\,.$$

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Since M is a symplectic manifold the symplectic form $\omega = d\alpha$, where α is the canonical 1-form on M, defines an isomorphism

$$T^*_{\bar\rho}M \ni \gamma \longmapsto H_\gamma \in T_{\bar\rho}M\,, \qquad \gamma = \omega(\cdot, H_\gamma)$$

and using this the Hamilton map of p at $\bar\rho$ is defined by

$$J = J_{\bar\rho}:\ T_{\bar\rho}M \ni v \mapsto H_\gamma \in T_{\bar\rho}M\,, \qquad \gamma = \mathrm{Hess}\,(p)\,(v, \cdot)\,.$$

Now, by definition, P (or p) is effectively hyperbolic at $\bar\rho$ if $\bar\rho$ has a conic neighbourhood V in which the following three conditions are valid:

(1.1) $\Sigma_1(p) \cap V = \{\rho \in V;\ p(\rho) = 0, dp(\rho) \neq 0\}$ is dense in $\Sigma(p) \cap V = \{p \in V;\ p(\rho) = 0\}$.

(1.2) There exists $t \in C^\infty(V;\ \boldsymbol{R})$, homogeneous of degree 0, $\delta > 0$ and a Riemann metric on M such that $|H_p t| > \delta |H_p|$ on $\Sigma_1(p) \cap V$.

(1.3) For each $\rho \in (\Sigma(p) \cap V) \backslash \Sigma_1(p)$, J_ρ has a non-zero real eigenvalue.

Of course, (1.2) is actually independent of the choice of Riemann metric on M.

In general if $W \subset T^*X \backslash 0$ is open, P is said to be effectively hyperbolic in W if it is effectively hyperbolic at each point $\bar\rho \in W \cap (\Sigma(p) \backslash \Sigma_1(p))$ and $\Sigma_1(p)$ consists of non-radial points for p, i.e. dp and α are independent there. In this case, a time function for p at $\bar\rho$ is a function t such that (1.2) holds in some neighbourhood of $\bar\rho$. A ray of P in W is a (continuous) curve

$$\beta:\ (a, b) \longrightarrow W \cap \Sigma(p)$$

such that the following three conditions hold:

(1.4) $\beta^{-1}(\Sigma(p) \backslash \Sigma_1(p)) = \{s_i\}$ is discrete in (a, b).

(1.5) If t is a time function at $\rho_i = \beta(s_i)$, $t \circ \beta$ is monotonic near s_i.

(1.6) β is C^∞ on $[s_i, s_{i+1}]$ (and on $(a, s_{-N}]$ or $[s_N, b)$ if $\{s_i\}$ is finite above or below) and is a reparametrized bicharacteristic of p on (s_i, s_{i+1}) (similarly on (a, s_{-N}) or (s_N, b)).

(1.7) Theorem. *If $P \in \Psi^m_{cl}(X)$ is effectively hyperbolic in $W \subset T^*X \backslash 0$ then for any $u \in C^{-\infty}(X)$, $WF(u) \cap (W \backslash WF(Pu))$ is a union of maximally extended rays of P in $W \backslash WF(Pu)$.*

It is straightforward to give a version of Theorem 1.7 for wavefront sets relative to appropriate Sobolev spaces. Then, following the model of

Duistermaat and Hörmander [1], if one requires that W be suitably convex with repect to rays, one obtains microlocal existence results and in the case $W=T^*\Omega$ semiglobal existence results, modulo finite kernels and cokernels. This is not carried out below. Rather, the main application of Theorem 1.7 is to differential operators. Consider $P\in \mathrm{Diff}^m(X)$ and $t\in C^\infty(X)$. P is (weakly) t-hyperbolic if for each $\bar{x}\in X$ the principal symbol $p(\bar{x}, \cdot)\in C^\infty(T^*_{\bar{x}}X)$ is a hyperbolic polynomial with respect to dt.

(1.8) PROPOSITION. *Suppose* $P\in \mathrm{Diff}^m(X)$ *and* $t\in C^\infty(X)$ *satisfy*:

(1.9) *P is weakly t-hyperbolic.*

(1.10) *P has no radial points, i.e. dp and α are independent on $\Sigma_1(p)$.*

(1.11) *At each $\rho\in\Sigma(p)\backslash\Sigma_1(p)$ the Hamilton map has a non-zero real eigenvalue.*

Then P is effectively hyperbolic in $T^*X\backslash 0$.

The conjecture of Ivrii & Petkov is partially answered in the affirmative by the following result.

(1.12) THEOREM. *Suppose that* $P\in \mathrm{Diff}^m(X)$ *is effectively hyperbolic in the sense that* (1.9), (1.10) *and* (1.11) *hold for some* $t\in C^\infty(X)$. *Then each* $\bar{x}\in X$ *has a neighbourhood* $A\subset X$ *such that if* $A_\pm=\{x\in A;\ \pm t(x)\geq 0\}$ *then P is an isomorphism on* $\{f\in C^\infty(A);\ \mathrm{supp}(f)\subset A_\pm\}$ *and on* $\{f\in C^{-\infty}(A);\ \mathrm{supp}(f)\subset A_\pm\}$, *modulo finite kernels and cokernels.*

This result is proved in Section 5 below, after the microlocal structure of effectively hyperbolic pseudodifferential operators has been considered in Section 2, the microlocal energy estimates following from this have been deduced in Section 3 and the proof of Theorem 1.7 has been given in Section 4. The proof of Proposition 1.8 is straightforward and is left to the reader.

2. Preparation

In this section it is shown that a classical pseudodifferential operator P with real principal symbol, p, satisfying (1.1), (1.2) and (1.3) can be reduced to a special form for which these conditions are transparently valid. This reduced form is extended to the operator itself and is used in the next section for the proof of microlocal energy estimates.

(2.1) LEMMA. *Suppose* $p\in C^\infty(M)$, $M=T^*X\backslash 0$, *is real and satisfies* (1.1), (1.2) *and* (1.3) *near* $\bar{\rho}\in\Sigma(p)\backslash\Sigma_1(p)$. *Then the Hessian polynomial of p at*

$\bar{\rho}$, $p_{\bar{\rho}}\in C^\infty(T_{\bar{\rho}}M)$ *satisfies* (1. 1), (1. 2) *and* (1. 3) *with t replaced by its linearization at* $\bar{\rho}$, $t_{\bar{\rho}}=dt$.

PROOF. The Hamilton map of p at $\bar{\rho}$ depends only on the Hessian at $\bar{\rho}$ so (1. 3) certainly holds for $p_{\bar{\rho}}$ at $0\in\Sigma(p_{\bar{\rho}})\backslash\Sigma_1(p_{\bar{\rho}})$. By translation on $T_{\bar{\rho}}M$ the Hessian of $p_{\bar{\rho}}$ at any point of $\Sigma(p_{\bar{\rho}})\backslash\Sigma_1(p_{\bar{\rho}})$, its null space, is reduced to $p_{\bar{\rho}}$ itself and since each translation is a symplectic map, $p_{\bar{\rho}}$ satisfies (1. 3). For any semidefinite quadratic polynomial the Hamilton map has purely imaginary eigenvalues so it follows from (1. 3) that $p_{\bar{\rho}}$ is not semi-definite. It therefore satisfies condition (1. 1). In any local coordinates x, in M, for which $\bar{\rho}$ is the origin,

$$p_{\bar{\rho}}(x)=q(0,x)\qquad q(s,x)=s^{-2}p(sx)$$

in terms of the induced coordinates on $T_{\bar{\rho}}M$. Here, q is C^∞ in all variables so if x' is a simple zero of $p_{\bar{\rho}}$ $q(s,x)=0$ has a unique C^∞ solution $x=x(s)$, $x(0)=x'$. Condition (1. 2) at $sx(s)$ gives

$$\left|\{p,t\}\left(sx(s)\right|>\delta\left|H_p\left(sx(s)\right)\right|.$$

As $s\to 0$, $s^{-1}dp(sx(s))\to dp_{\bar{\rho}}(x')$, $s^{-1}H_p(sx(s))\to H_{p_{\bar{\rho}}}(x')$ and $dt(sx(s))\to dt_{\bar{\rho}}(x')$, so (1. 2) follows for $p_{\bar{\rho}}$ with the constant $\frac{1}{2}\delta>0$. This completes the proof of the Lemma.

Now, extend t, as in (1. 2), to a Darboux coordinate system t, y, τ, η near $\bar{\rho}\in\Sigma(p)\backslash\Sigma_1(p)$. It can be assumed that $\bar{\rho}=(0,0,0,\bar{\eta})$, $\bar{\eta}=(0,\cdots,0,1)$ and

(2. 2) $$p=a\tau^2+b(t,y,\eta)\tau+c(t,y,\eta)\,,\qquad a\neq 0\,.$$

Indeed suppose that $a=0$. Then $b\equiv 0$ would contradict (1. 1) if c were semidefinite, or (1. 2), of c had a simple zero. Thus $db\neq 0$. Again because of (1. 2) $c=0$ must imply $b=0$, so $c=bm$, with m another linear function. Replacing the original Darboux coordinates by t, y', τ', η' where $d\tau'=d\tau+dm$ at $\bar{\rho}$, and then dropping the primes, reduces $p_{\bar{\rho}}$ to τb. Such a product has nilpotent Hamilton map unless the Poisson bracket $\{\tau,b\}=\partial_t b\neq 0$. Using the freedom to perturb t, say to $t'=t+\varepsilon\tau|\eta|^{-1}$, and extending to new Darboux coordinates t', y', τ, η' enusures (2. 2).

In view of (2. 2) the Malgrange preparation theorem can be applied to p near $\bar{\rho}$, yielding

$$p=q\cdot\left(\tau^2+b'(t,y,\eta)\,\tau+c'(t,y,\eta)\right)$$

where $q\neq 0$. Dropping the elliptic factor q and passing to new Darboux coordinates, always fixing $\bar{\rho}$, t, y', $\tau+\frac{1}{2}b'$, η', $y'=Y(t,y,\eta)$, $\eta'=H(t,y,\eta)$ ensures that

(2.3) $$p=\tau^2-c(t,y,\eta)\,.$$

For the Hessian polynomial of p at $\bar{\rho}$,

(2.4) $$p_{\bar{\rho}}=\tau^2-\bar{c}(t,y,\eta)\,,\qquad \bar{c}=c_{\bar{\rho}}$$

necessarily $\bar{c}>0$. Indeed, $\bar{c}\leq 0$ violates (1.1) for $p_{\bar{\rho}}$ and a simple zero for $\bar{c}$ would violate (1.2). Thus, $p_{\bar{\rho}}$ is a hyperbolic polynomial and the discussion of Ivrii & Petkov [5], or more particularly of Hörmander [2], applies and shows that under condition (1.3), (see [2] Theorem 1.4.6)

(2.5) $$p_{\bar{\rho}}=c^2-d^2-e,\ e\geq 0,\ \{c,d\}\neq 0,\qquad \{c,e\}=\{d,e\}=0\,,$$

for some linear functions c, d. Consider the vector

$$\nu=(\partial_\tau c,\partial_\tau d)=-(H_t c,H_t d)\,.$$

Certainly $\nu\neq 0$ since otherwise, (2.4) could not hold. In (2.5) there is freedom to make a hyperbolic linear transformation in c and d and a conformal change in p. Thus, the orbits of $\nu=(f,g)$ in $\boldsymbol{R}^2$ consist of O, $|f|=|g|$, $|f|>|g|$, $|f|<|g|$, with only the last two stable under perturbation of t. It can therefore be arranged that $\nu=(1,0)$ or $(0,1)$. Making a scale change and a canonical transformation to (t',y',c,η') gives

$$p_{\bar{\rho}}=\tau^2-(t+\alpha(\tau,y,\eta))^2-\gamma(\tau,y,\eta),\qquad \gamma\geq 0,\ \{t+\alpha,\gamma\}=0\,.$$

and a further canonical transformation to (t',y',τ,η'), $t'=t+\alpha(\tau/\eta_n,y,\eta/\eta_n)$ ensures that

(2.6) $$p_{\bar{\rho}}=\tau^2-t^2-\gamma(y,\eta)\,,\qquad \gamma\geq 0\,.$$

Applying the preparation theorem and change of variables as before gives (2.3) in the more restricted from:

(2.7) $$p=\tau^2-h(t,y,\eta)\left(t+a(y,\eta)\right)^2-c(y,\eta),\ da=\alpha=0\ \ at\ \bar{\rho},\ h(\bar{\rho})=1,$$
$$c\geq 0,\ c_{\bar{\rho}}=\gamma\,.$$

If a further change of Darboux coordinates is made to $t'=t+a(y,\eta)$, $\tau'=\tau$, $y=Y(\tau,y',\eta')$, $\eta=H(\tau,y',\eta')$ then

$$\partial_\tau^2 c(Y,H)=\partial_\tau c(Y,H)=0\ \text{ at }\ \bar{\rho}$$

because $\partial_\tau^2 c=2c_{\bar{\rho}}(\partial_\tau(Y,H),\partial_\tau(Y,H))=H_\alpha^2\gamma$ at $\bar{\rho}$. Thus, in the new coordinates

$$p|_{t=0}=\tau^2-c(Y,H)$$

has two real (or one double) solution for each (y', η') near the base point. Dropping the primes and applying the preparation theorem again

$$p=\left(\tau-b(t, y, \eta)\right)^{2}-h(t, y, \eta)t^{2}-c(y, \eta)$$

where, as before, $h(\bar{\rho})=1$, $c\geq 0$ and now, $db(\bar{\rho})=0$. Finally changing to new coordinates t, y', $\tau'=\tau-b$, η' where $y'=Y(t, y, \eta)$, $\eta'=H(t, y, \eta)$ $dy=dy$, $d\eta'=d\eta$ at $\bar{\rho}$, leads to the desired special form for p:

(2.8) $\quad p=\tau^{2}-h(t, y, \eta)\,t^{2}-g(t, y, \eta),\ h(\bar{\rho})=1,\ g\geq 0,\ \partial_{t}g=\partial_{t}^{2}g=0$ at $\bar{\rho}$.

Proceeding to the quantization of this normal form for the principal symbol gives:

(2.9) PROPOSITION. *Suppose $P\in\Psi^{m}(X)$ is a classical pseudodifferential operator with real principal symbol satisfying* (1.1), (1.2) *and* (1.3). *Then, there are Fourier integral operators*

$$G,\ F:\ C_{c}^{\infty}(\boldsymbol{R}^{d})\longrightarrow C_{c}^{\infty}(X)$$

associated to a local canonical diffeomorphism $\chi:\ T^{*}\boldsymbol{R}^{d}\rightarrow T^{*}X$ defined near $(0, \bar{\xi})$, $\bar{\xi}=(0, \cdots, 0, 1)$ with $\chi(0, \bar{\xi})=\bar{\rho}$, G, F being elliptic at $(0, \bar{\xi})$ and such that

(2.10) $\quad F\cdot P\equiv S\cdot Q\cdot G \quad$ *near* $\left((0, \bar{\xi}), \bar{\rho}\right)$

where $S\in\Psi^{0}(\boldsymbol{R}^{d})$ is elliptic and properly supported and

(2.11) $\quad Q=D_{1}^{2}-H(x_{1}, x', D_{x'})\,x_{1}^{2}-C(x_{1}, x', D_{x'})$

with H, C classical pseudodifferential operators in $\boldsymbol{R}^{d-1}$ depending on x_{1} as a C^{∞} parameter, of order 2, *and having symbols*

(2.12) $\quad h\geq\dfrac{1}{2}|\xi'|^{2},\ g(x, \xi')\geq 0,\ D_{x_{1}}g(0, 0, \bar{\xi}')=D_{x_{1}}^{2}g(0, 0, \bar{\xi}')=g(0, 0, \bar{\xi})=0$

$\bar{\xi}'=(0, \cdots, 0, 1)\in\boldsymbol{R}^{d-1}$.

PROOF. In the discussion above, leading to (2.8), it has been shown how to construct the canonical diffeomorphism χ. If F, G are chosen to have essential supports very near $((0, \bar{\xi}), \bar{\rho})$ and to be elliptic, with symbols arranged to absorb the elliptic factors dropped in deriving (2.8), then (2.10) holds for some Q' having principal symbol as in (2.11), (2.12) and $S=Id$. To remove the lower order terms, $Q_{1}\in\Psi^{1}(\boldsymbol{R}^{d})$, near $(0, \bar{\xi})$, it is enough to write, using the quantized from of the preparation theorem of Boutet de Monvel

$$Q_{1}=\left(q_{0}(x_{1}, x', D_{x})D_{x_{1}}+q_{1}(x_{1}, x', D_{x'})+Q\right)R \qquad \text{near } (0, \bar{\xi})$$

with $R\in\Psi^{-1}(\boldsymbol{R}^d)$. Absorbing the factor $Id+R$ into G in (2.10) reduces the first order terms of Q_1 to a first order differential operator in x_1, with pseudodifferential operator coefficients. Even the term $q_0(x, D')D_{x_1}$ can be removed by commuting $(Id-q(x, D')x_1)$ through Q, for an appropriate choice of q. This gives the desired form (2.11), with $S=(Id-q(x, D')x_1)$ and G absorbing the further elliptic factors.

3. Energy estimates

For a model operator (2.11) satisfying (2.12) the derivation of a priori estimates is relatively straightforward following the classical methods and incorporating ideas of Oleinik and Ivrii. Since the first variable, x_1, is distinguished below it is denoted by t, the remaining variables are written $y_1, \cdots, y_n$, $n=d-1$, and the corresponding canonical dual variables are denoted τ, $\eta_1, \cdots, \eta_n$.

Thus, consider the t-differential operator with classical pseudodifferential coefficients of order two:

$$P=D_t^2-t^2H^2(t, y, D_y)-G(t, y, D_y)-F(t, y, D_y) \tag{3.1}$$

where F, G, H all have Schwartz' kernels supported in $|y-y'|\leq\frac{1}{4}$, H and G are selfadjoint, F is of order one and

$$\sigma_1(H)\geq|\eta|+O(t),\ \sigma_2(G)\geq 0,\ \sigma_2(G)(0, 0, \bar{\eta})=D_t^2\sigma_2(G)(0, 0, \bar{\eta})=0 \tag{3.2}$$

where $\bar{\eta}=(0, 0, \cdots, 0, 1)$. The estimtaes in $t\leq 0$ and $t\geq 0$ will be handled separately, first in $t\leq 0$.

As a suitable microlocalizing symbol near $(t, y, \eta)=(0, 0, \bar{\eta})$ consider

$$\mu_\varepsilon(t, y, \eta)=\varphi\left(\frac{t}{\varepsilon}+\frac{1}{\varepsilon}\left(|y|^2+\left|\frac{\eta}{|\eta|}-\bar{\eta}\right|^2\right)\right)\rho(|\eta|). \tag{3.3}$$

Here $\rho\in C^\infty(\boldsymbol{R};\ [0, 1])$, with $\rho(r)=1$ in $r\geq 1$, $\rho(r)=0$ in $r<\frac{1}{2}$, serves to make μ_ε smooth near $\eta=0$ whereas $\varphi\in C^\infty(\boldsymbol{R};\ [0, 1])$ has the properties

$$\operatorname{supp}(\varphi)\subset\left(-\infty, \frac{3}{4}\right),\ \varphi'\leq 0,\ \varphi(r)=1 \text{ in } r\leq\frac{1}{2}, \tag{3.4}$$
$$\varphi^{\frac{1}{2}},\ (-\varphi')^{\frac{1}{2}}\in C^\infty(\boldsymbol{R}).$$

Thus, μ_ε is constant on the parabolic surfaces $t=a\varepsilon-\varepsilon\left(|y|^2+\left|\frac{\eta}{|\bar{\eta}|}-\eta\right|^2\right)$, $|\eta|\geq 1$, and is decreasing in t.

To allow for regularization and iteration in the estimates below the test operator will depend on five parameters. Of these $m\in\boldsymbol{R}$ is an overall order parameter, $N\in\boldsymbol{N}$ is a regularizing parameter, $\varepsilon>0$ controls the parabolic form of the (micro-) support, $\nu>0$ further cuts off the support near $t=-\nu$ and $k\in\boldsymbol{Z}$ is a singular order parameter. For notational simplicity the dependence on the parameters will be suppressed, in large part, below. Set

$$\chi_{\nu,N,m}(t,\eta)=\left(1+\left(1-sgn(m)\cdot\frac{|\eta|}{\nu}\right)^{m}\cdot\exp\left(-\frac{1}{N}\left(1+\frac{t}{\nu}|\eta|\right)\right)\right) \tag{3.5}$$

and let

$$Tv(t,y)=(2\pi)^{-n}\int e^{i(y-y')\cdot\eta}am(T)(t,y,y',\eta)\,v(t,y')\,dy'd\eta \tag{3.6}$$

be specified by the amplitude

$$am(T)=\mu_{\varepsilon}(t,y,\eta)\cdot\chi_{\nu,N,m}(t,\eta)\cdot\varphi\left(-\frac{t}{\nu}\right)\cdot\varphi(|y'|)\,. \tag{3.7}$$

For m fixed ε, ν in a compact subset of $(0,\infty)\,x\,(0,\infty)$ and $N\geq 1$ T lies in a bounded subset of $C_c^{\infty}(\boldsymbol{R}_t;\,\Psi_c^m(\boldsymbol{R}^n))$, having in particular uniformly compact support in all variables. Moreover, as $N\to\infty$ T converges in $C_c^{\infty}(\boldsymbol{R};\,\Psi_c^{m'}(\boldsymbol{R}^n))$ for any $m'>m$. Such a family of operators will simply be called a bounded family of order m. The basic test operators are given by

$$S=S_k=(-t)^{2k+1}A^2,\ A=\frac{1}{2}(T+T^*),\qquad k\in\boldsymbol{N}, \tag{3.8}$$

a bounded selfadjoint family of order $2m$.

Now, proceed to compute the form of

$$Q=D_t\cdot S\cdot P-P^*\cdot S\cdot D_t\,. \tag{3.9}$$

Inserting the formula (3.1) for P shows that

$$Q=Q_1+Q_2+Q_3+Q_4+Q_5 \tag{3.10}$$

where

$$Q_1=-D_t\cdot[D_t,S_k]\cdot D_t \tag{3.11}$$

$$Q_2=-H\cdot[D_t,S_{k+1}]\cdot H \tag{3.12}$$

$$Q_3=-D_t\cdot S_k\cdot G\cdot S_k\cdot D_t \tag{3.13}$$

$$Q_4=-D_t\cdot[S_{k+1},H]\cdot H+H\cdot[H,S_{k+1}]\cdot D_t-[D_t,H]\cdot S_{k+1}\cdot H-H\,S_{k+1}\cdot[D_t,H] \tag{3.14}$$

$$Q_5=-D_t\cdot S_kF+F^*S_k\cdot D_t\,. \tag{3.15}$$

Let B' be the bounded family of order m defined by the amplitude

$$am(B') = \mu_\varepsilon(t, y, \eta) \cdot \chi_{\nu,N,m}(t, \eta) \cdot \varphi\left(\frac{t}{\nu}\right) \cdot \varphi\left(|y'|\right),$$

where $\psi \in C_c^\infty(\boldsymbol{R}; [0, 1])$ has support in $\left[\frac{1}{4}, 1\right]$ and is identically equal to one on supp (φ'). Set $B = \frac{1}{2}(B' + (B')^*)$. Thus, B localizes near $t = -\nu$. Note also that using the strict Gårding inequality, and modifying F in (3. 1), it can always be assumed that

$$\langle Gu, u \rangle \geq 0 .$$

(3. 16) Proposition. *There exists* $\bar{\varepsilon} > 0$ *and* $\bar{k} \geq 2$ *such that if* $u \in C^\infty([-\nu, 0]; C^{-\infty}(\boldsymbol{R}^n))$ $k \geq \bar{k}$, $\bar{\varepsilon} \geq \varepsilon' > \varepsilon > 0$, $\bar{\varepsilon} \geq \nu > 0$, $m' \in \boldsymbol{R}$ *then*

$$\begin{aligned} &\left\|(-t)^k A_{m,\varepsilon,N} D_t u\right\|^2 + \left\|(-t)^{k+1} A_{m+1,\varepsilon,N} u\right\|^2 + \left\langle (-t)^{2k} G A_{m,\varepsilon,N} u, A_{m,\varepsilon,N} u \right\rangle \\ &+ \operatorname{Im} \langle u, Qu \rangle \leq C \Big\{ \left\|B_{m,\varepsilon',2N} D_t u\right\|^2 + \left\|B_{m+1,\varepsilon',2N} u\right\|^2 + \|u\|_{m'}^2 + \|D_t u\|_{m'}^2 \\ &+ \left\|(-t)^k A_{m-\frac{1}{2},\varepsilon',2N} D_t u\right\|^2 + \left\|(-t)^{k+1} A_{m+\frac{1}{2},\varepsilon',2N} u\right\|^2 \\ &+ \left\langle (-t)^{2k} G A_{m-\frac{1}{2},\varepsilon',2N} u, A_{m-\frac{1}{2},\varepsilon',2N} u \right\rangle \Big\} . \end{aligned} \tag{3.17}$$

Here, norms and inner products are in $L^2((-\nu, 0) \times \{|y| < 1\})$ *except that the subscript* m' *indicates the norm in* $L^2((-\nu, 0); H^{m'}(|y| < 1))$. *The constant* C *in in* (3. 17) *can be taken independent of* N *and also of* ε, ε', ν, m' *over a compact subset of the parameter ranges.*

In the proof, the imaginary part of the inner product $\langle u, -Qu \rangle$ will be estimated from below. In this estimation systematic use is made of the uniformity, on bounded sets, of the symbol calculus, of L^2 boundedness of operators of order zero and of the strict Gårding inequality. Thus, suppose that R is a bounded family of order $2m$ with amplitude

$$am(R) = \sum_{finite} a_p \cdot \left(\frac{|\eta|}{N}\right)^p \cdot \chi_{\nu,N,2m} \tag{3.18}$$

where a_p is independent of N, of order zero, with support in

$$\operatorname{supp}\left(am(A_{m,\varepsilon})\right) \cap \operatorname{supp}\left(\varphi'\left(-\frac{t}{\nu}\right)\right).$$

Then, using the symbol calculus, for any $\varepsilon' > \varepsilon$,

$$R = B_{m,\varepsilon' 2N} \cdot M \cdot B_{m,\varepsilon',2N} + R_{m'} \tag{3.19}$$

where M is a bounded family of order zero and $R_{m'}$ is a bounded family of the preassigned order m'. Similarly, if R' is a bounded family of order $2m-1$, with amplitude of the form (3.18), with $2m$ replaced by $2m-1$, and the a_p's supported in $\operatorname{supp}(am(A_{m,\varepsilon}))$ then, again for $\varepsilon'>\varepsilon$,

$$R'=B_{m-\frac{1}{2},\varepsilon',2N}\cdot M'\cdot B_{m-\frac{1}{2},\varepsilon',2N}+A_{m-\frac{1}{2},\varepsilon',2N}\cdot M'\cdot A_{m-\frac{1}{2}\varepsilon',2N}+R'_{m'} \tag{3.20}$$

with M', M'' bounded families of order zero. From this it follows that terms such as $\langle(-t)^{2k}LD_tu, D_tu\rangle$, $L=R$ or R' are bounded by the right side of (3.17). Such division will be used freely below. In general, J will be used to denote terms which can be bounded above by the right of (3.17).

The other way in which terms are shown to be in J is that although of order $2m$ they have the support properties of R' above, but have negative symbols. Then in the division to get (3.20), the symbols of M' and M'' can be taken negative, modulo terms of order zero. The bound from above is now a consequence of the strict Gårding inequality. Note also that integration by parts, even in the t-variable, is possible in the inner products obtained from (3.11)-(3.15) since the presence of the factor $(-t)^{2k+1}$, $k\geq 2$, annihilates all boundary terms at $t=0$. As a simple example of the type of estimates derived below consider the interpolation result:

(3.21) LEMMA. *If* $u\in C^\infty([-\nu,0]; C^{-\infty}(\boldsymbol{R}^n))$ *then*

$$(2k+1)\left\|(-t)^kA_{m+\frac{1}{2},\varepsilon,N}u\right\|^2\leq\left\|(-t)^kA_{m,\varepsilon,N}D_tu\right\|^2+\left\|(-t)^{k-1}A_{m+1,\varepsilon,}Nu\right\|^2+J.$$

PROOF. Consider

$$\begin{aligned}U&=-i\left[D_t,(-t)^{2k+1}A^2_{m+\frac{1}{2},\varepsilon,N}\right]\\&=(2k+1)(-t)^{2k}A^2_{m+\frac{1}{2}}-i(-t)^{2k+1}[D_t,A_{m+\frac{1}{2}}]\,A_{m+\frac{1}{2}}-i(-t)^{2k+1}\\&\qquad A_{m+\frac{1}{2}}\cdot[D_t,A_{m+\frac{1}{2}}].\end{aligned}\tag{3.22}$$

Since $A^2_{m+\frac{1}{2}}=A_m\cdot A_{m+\frac{1}{2}}+R_{2m}$, with R_{2m} of the form (3.18), it follows from Schwartz' inequality that

$$\left|\langle Uu,u\rangle\right|\leq\left\|(-t)^kA_{m,\varepsilon,N}D_tu\right\|^2+\left\|(-t)^{k+1}A_{m+1,\varepsilon,N}u\right\|^2+J.$$

Now, the symbol of $-i[D_t,A_{m+\frac{1}{2}}]$ is positive, except for the term arising from differentiation of $\varphi\left(-\dfrac{t}{\nu}\right)$, so applying the estimates discussed above,

$$-Im\left\langle(-t)^{2k+1}[D_t,A_{m+\frac{1}{2}}]\,A_{m+\frac{1}{2}}u,u\right\rangle=J,$$

which proves the Lemma.

PROOF OF PROPOSITION 3.16. Fixing $m'<<m$, $\langle u, -Qu\rangle$ can be decomposed using (3.10) and (3.11)-(3.15). The positive contributions come from Q_1, Q_2 and Q_3. As in (3.22),

$$-i\left[D_t, (-t)^{2k+1}A^2\right]=(2k+1)(-t)^{2k}A_m^2+(-t)^{2k+1}F+(-t)^{2k+1}R \tag{3.23}$$

where $F=F_{(1)}{}^2+F_{(2)}{}^2+F_{(3)}{}^2$, with $F_{(j)}=\frac{1}{2}(E_{(j)}+E_{(j)}{}^*)$ and $E_{(j)}$ defined by

$$\begin{aligned} am(E_{(1)})^2 &= \frac{2}{\varepsilon}(-\varphi)\left(\frac{t}{\varepsilon^2}+\frac{1}{\varepsilon}\left(|y|^2+\left|\frac{\eta}{|\eta|}-\bar{\eta}\right|^2\right)\right)\cdot\rho|\eta|\cdot\mu_\varepsilon\cdot\chi_{\nu,N,2m}\cdot \\ &\quad \left(-\frac{t}{\nu}\right)\varphi(|y'|) \\ am(E_{(2)})^2 &= \mu_\varepsilon^2\cdot\left|\frac{2m|\eta|}{\nu}\right|\cdot\chi_{\nu,N,2m-1}\cdot\varphi\left(-\frac{t}{\nu}\right)\cdot\varphi(|y'|) \\ am(E_{(3)})^2 &= \mu_\varepsilon^2\cdot\left(\frac{|\eta|}{N\nu}\right)\cdot\chi_{\nu,N,2m}\cdot\varphi\left(-\frac{t}{\nu}\right)\varphi\cdot(|y'|) \end{aligned} \tag{3.24}$$

and R contains the terms arising from differentiation of $\varphi\left(-\frac{t}{\nu}\right)$ as well as a bounded family of order $2m-1$ arising from the extraction of the square roots, using the calculus. Using (3.19) and (3.20) the terms from R can be absorbed into J. Thus,

$$\begin{aligned} Im\langle u, -Q_1u\rangle &= (2k+1)\left\|(-t)^kA_{m,\varepsilon,N}D_tu\right\|^2 \\ &\quad +\sum_{j=1}^{3}\left\|(-t)^{k+\frac{1}{2}}F_{(j)}D_tu\right\|^2-J\,. \end{aligned} \tag{3.25}$$

The same type of argument applies easily to Q_2, except that k is increased to $k+1$, so

$$\begin{aligned} Im\langle u, -Q_2u\rangle &= (2k+3)\left\|(-t)^{k+1}A_{m,\varepsilon,N}Hu\right\|^2 \\ &\quad +\sum_{j=1}^{3}\left\|(-t)^{k+3/2}F_{(j)}Hu\right\|^2-J\,. \end{aligned} \tag{3.26}$$

Similar arguments can again be applied to Q_3, but some care needs to be exercised with the commutators. To do this, write

$$-Q_3=\left[D_t, (-t)^{2k+1}AGA\right]+D_t\cdot(-t)^{2k+1}R_1-(-t)^{2k+1}R_1^*\cdot D_t \tag{3.27}$$

where $R_1=A\cdot A$, G is a bounded family of order $2m+1$. Setting

$$Q_3'=-i(2k+1)\,AGA,\quad Q_3''=Q_3-Q_3'\,, \tag{3.28}$$

$$Im\langle u, -Q_3'u\rangle=(2k+1)\langle Au, GAu\rangle\,. \tag{3.29}$$

Let π stand for the sum of the positive terms on the right in (3.25), (3.26) and (3.29); then if $\bar{\varepsilon}>0$ is small enough,

$$(3.30)\qquad Im\langle u, -Q_3''u\rangle \geq -\frac{1}{4}\pi - J .$$

Consider first the terms coming from R_1 in (3.27). The part of R_1 of order $2m$ clearly gives a contribution in J, so it suffices to replace R_1 by another operator with the same principal symbol, uniformly in N; this can be taken of the form:

$$(3.31)\qquad R_1' = \sum_{j=1}^{3} F_{(j)}G_{(j)}\cdot A ,$$

where $G_{(j)}$ is bounded of order one, with $(\sigma_1(G_{(j)}))^2 \leq \delta\ \sigma_2(G)$, where $\delta\downarrow 0$ as $\bar{\varepsilon}\downarrow 0$ and the inequality for a smooth positive function, f,

$$|df|^2 \leq Cf$$

has been used. Thus, using the strict Gårding inequality,

$$\left|\left\langle D_t u, (-t)^{2k+1}R_1 u\right\rangle\right| \leq r^2 \sum_{j=1}^{3}\left\|(-t)^{k+\frac{1}{2}}F_{(j)}u\right\|^2 + \frac{3\delta}{r^2}\left\langle(-t)^{2k+1}Au, GAu\right\rangle + \frac{3\delta}{r^2}C\left\|(-t)^k A_{m+\frac{1}{2}}u\right\|^2 + J ,$$

which, in view of Lemma 3.21, gives a bound of the tye type (3.30) for this part of Q_3'' for any positive constant replacing $\frac{1}{4}$, if $\bar{\varepsilon}>0$ is small enough.

The part remaining of Q_3'' to be estimated is

$$(3.32)\qquad \begin{aligned} Q_3'' &= -(-t)^{2k+1}[D_t, A]\,GA - (-t)^{2k+1}A\,[D_t, G]\,A \\ &\quad -(-t)^{2k+1}AG\,[D_t, A] . \end{aligned}$$

The corresponding contribution to (3.30) of the first and third terms has positive principal symbol, except for a term which can be absorbed in J using (3.19). For the second term in (3.32) Schwartz' inequality gives

$$\left|\left\langle(-t)^{2k+1}Au, G_t'Au\right\rangle\right| \leq r^2\left\|(-t)^{k+1}AHu\right\|^2 + \frac{1}{r^2}\left\langle(-t)^k(G_t')^2Au, Au\right\rangle + J .$$

Using the strict Gårding inequality and the bound on the symbol of $(G_t')^2$ in terms of the symbol of G, and also Lemma 3.21, this shows

$$\left|\left\langle(-t)^{2k+1}Au, C_t'Au\right\rangle\right| \leq \delta(\bar{\varepsilon})\,\pi + J$$

where $\delta(\bar{\varepsilon})\downarrow 0$ as $\bar{\varepsilon}\downarrow 0$.

Consider then the first and third terms in (3. 32). The symbol calculus shows that these can be rewritten in the form

$$(3.33) \qquad (-t)^{2k+1}\Big(\sum_{j=1}^{3} F_{(j)}GF_{(j)}+\sum_{j,p=1}^{3} F_{(p)}L_{pj}F_{(j)}+LA_{m+\frac{1}{2}}+R'\Big)$$

where L_{pj}, L and R' are bounded families of orders 1, $m+\frac{1}{2}$ and $2m$ respectively. The first sum is positive and the third term is bounded by

$$\delta\Big\|(-t)^k A_{m+\frac{1}{2}}u\Big\|^2+\frac{C}{\delta}\Big\|(-t)^{k+1}Lu\Big\|^2$$

and since Lemma 3. 21 applies to the first term, this in turn is bounded by $\delta\pi+J$. Observe from the proof of Lemma 3. 21 that the additional positivity in U gives a bound

$$\sum_{j=1}^{3}\Big\|(-t)^{k+\frac{1}{2}}F_{(j)}H^{\frac{1}{2}}u\Big\|^2\leq\delta\pi+J\,.$$

Using this the second sum in (3. 33) can be bounded above. The final term in (3. 33) is in J, so (3, 30) is established.

The final two terms in Q, Q_4, Q_5 are easily bounded in the form

$$(3.34) \qquad \big|\langle u, -Q_j u\rangle\big|\leq\frac{C}{k}\pi+J \qquad k\geq 2,\ j=4,\ 5,$$

for some C independent of k, $\bar{\varepsilon}$. This completes the proof of (3. 17), if $\bar{k}>4C$, and $\bar{\varepsilon}>0$ is small enough.

(3. 35) REMARK. Examination of the estimates above shows that the choice of $\bar{k}$ is governed by the ratio of the subprincipal symbol of P to the effectively hyperbolic eigenvalue of the Hamilton map.

(3. 36) PROPOSITION. *There exist* $\bar{\varepsilon}>0$ *and* $k'\geq 2$ *such that if* $u\in C^{\infty}([0,\nu]\,;\,C^{-\infty}(\boldsymbol{R}^n))$ *vanishes at* $t=0$ *to order* $k'+2$, $k=-k'$ *in* (3. 8), $\bar{\varepsilon}\geq\varepsilon>\varepsilon'>0$, $\bar{\varepsilon}\geq\nu>0$, $m'\in\boldsymbol{R}$ *then*

$$\begin{aligned}&\Big\|t^k A_{m,\varepsilon,N}D_t u\Big\|^2+\Big\|t^{k+1}A_{m+1,\varepsilon,N}u\Big\|^2+\Big\langle t^{2k}GA_{m,\varepsilon,N}u,\ A_{m,\varepsilon,N}u\Big\rangle\\ (3.37)\qquad &+Im\,\langle u, Qu\rangle\leq C\Big\{\Big\|t^k A_{m-\frac{1}{2},\varepsilon',2N}D_t u\Big\|^2+\Big\|t^{k+1}A_{m+\frac{1}{2},\varepsilon',2N}u\Big\|^2\\ &+\Big\langle t^{2k}GA_{m-\frac{1}{2},\varepsilon',2N}u,\ A_{m-\frac{1}{2},\varepsilon',2N}u\Big\rangle+\|u^2_{m'}\|+\|D_t u\|^2_{m'}\Big\}\,.\end{aligned}$$

Here, the norms and inner products are over $(0,\nu)\times\{|y|\leq 1\}$, *and are in* L^2 *except for the last two which are in* $L^2((0,\nu)\,;\,H^{m'})$. *The constant C can be taken independent of N and of the other marameters on compact sets.*

PROOF. The proof of the basic estimate (3. 37) is left to the reader, since it very similar to the proof of (3. 17). The absence of the initial value terms, appearing as B in (3. 17), is reflected in the assumption that u vanishes to high order at $t=0$. It is important to note that the order of vanishing is determined by the choice of k, not by the choice of order of the remainder, m'.

4. Propagation of singularities

In order to prove Theorem 1. 7, using the microlocal energy estimates deduced in Section 3, it is necessary to have a qualitative description of the rays of P. Note that the definition of a ray is invariant under symplectic transformation so it can be assumed that p takes the form (2. 8). Away from the zeros of g, which parametrize the double characteristics of p, the rays are bicharacteristics. Set

$$\Sigma_{\pm}(p)=\left\{(t, y, \tau, \eta)\,;\ p=0,\ \ \pm\tau\geq 0\right\}.$$

Then, $\Sigma_{+}\cap\Sigma_{-}=\Sigma_{2}(p)=\{(0, y, 0, \eta)\,;\ g(0, y, \eta)\}=0\}$.

Consider the local bicharacteristic flow on $\Sigma_{\pm}\backslash\Sigma_{2}$, reparameterized by t. Since $\tau=\pm(ht^{2}+g)^{\frac{1}{2}}$ on $\Sigma_{\pm}$, the Hamilton vector field can be reduced to

$$\text{(4. 1)}\qquad \partial_{t}\pm\frac{1}{2}(ht^{2}+g)^{-\frac{1}{2}}H_{c}',\ H_{c}'=\sum_{j=1}^{p}\partial_{\eta_{j}}c\partial_{y_{j}}-\partial_{y_{j}}c\partial_{\eta_{j}},\qquad c=t^{2}h+g\,.$$

This is C^{∞} in $t\neq 0$ and integration of it defines diffeomorphisms

$$\text{(4. 2)}\qquad F_{\pm}(t, s):\ \boldsymbol{R}^{2n}\rightarrow\boldsymbol{R}^{2n}\qquad t,\ s<0\ \text{ or }\ t,\ s>0\,,$$

by $F_{\pm}(t, s)(y, \eta)=(y', \eta')$ if the integral curve of (4. 1) with initial point (s, y, η) passes through (t, y', η').

(4. 3) LEMMA. *The limits* $\lim_{t\uparrow 0}F_{\pm}(t, s):\ \boldsymbol{R}^{2n}\rightarrow\boldsymbol{R}^{2n},\ s<0$ *and* $\lim_{t\downarrow 0}F_{\pm}(t, s):$ $\boldsymbol{R}^{2n}\rightarrow\boldsymbol{R}^{2b}m\ s>0$, *are locally uniform and define surjective maps.*

PROOF. Let $\gamma(t)=(t, Y(t), H(t))$ be the integral curve of (4. 1) with initial point (s, y, η), $s<0$. Certainly $\gamma(t)$ is C^{∞} in $t<0$ and since

$$\left|\frac{d}{dt}(Y, H)\right|\leq C|dc|/(t^{2}+g)^{\frac{1}{2}}\leq C'$$

is uniformly bounded in $t<0$. It follows that $\lim_{t\downarrow 0}\gamma(t)$ exists, and that $\gamma:$ $[s, 0]\rightarrow\boldsymbol{R}^{2n}$ is continuous. Clearly the curve depends continuously on the initial point (y, η), proving the existence and uniformity of the limits. Surjectivity follows too since each $F_{\pm}(t, s)$, $t<0$, is surjective.

In particular this shows that through each point of Σ_2 pass at least four distinct rays.

With $\nu>0$ fixed, consider the microlocal influence domain:

$$D_{-\nu}(\rho)=\left\{\rho'\in\Sigma(p)\,;\ t(\rho')=-\nu \text{ and there is a ray from } \rho' \text{ to } \rho\right\}.$$

As a consequence of Lemma 4.3,

$$\lim_{d\downarrow 0}\bigcup_{|\rho-\bar\rho|<d} D_{-\nu}(\rho)=D_{-\nu}(\bar\rho)\,; \tag{4.4}$$

so that if N is an open neighbourhood of $D_{-\nu}(\bar\rho)$ there is a neighbourhood N' of $\bar\rho$ such that $D_{-\nu}(\rho)\subset N$ for all $\rho\in N'$.

Proof of Theorem 1.7. It is enough to show that if $\bar\rho\in\Sigma_2(p)$, $\bar\rho\notin WF(Pu)$ and for some $\nu>0$, sufficiently small, $D_{-\nu}(\bar\rho)\cap WF(u)=\phi$ then $\bar\rho\notin WF(u)$. In Proposition 3.16 the operator representing the initial data, B, has essential support in $-\nu\le t\le -\frac{1}{4}\nu$, $|y|^2+\left|\frac{|\eta|}{\eta}-\bar\eta\right|^2<\nu$. The rays through this set meet Σ_2 in a neighbourhood of $(0, 0, 0, \bar\eta)$, the radius of which tends to 0 with ν. Thus, using (4.4) for $\nu>0$ sufficiently small,

$$D_{-\nu}(\bar\rho)\cap WF(u)=\phi\Rightarrow Bu,\ BD_t u\in C^\infty,$$

uniformly as $N\to\infty$.

It can always be assumed that $WF(u)$ lies in a small neighbourhood of $(0, 0, 0, \bar\eta)$, so $u\in C^\infty([-\nu, 0]\,;\ C^{-\infty}(B_1))\cap C^2([-\nu, 0]\,;\ H^{m'}(B_1))$ for some m'. Now, Proposition 3.16 can be applied and iterated, showing that for any m,

$$\left\|(-t)^k A_{m,\varepsilon,N}D_t u\right\|,\ \ \left\|(-t)^{k+1}A_{m+1,\varepsilon,N}u\right\|\le C,$$

uniformly in N. The limit as $N\to\infty$ shows that

$$\left\|(-t)^k A_{m,\varepsilon,\infty}D_t u\right\|,\ \ \left\|(-t)^{k+1}A_{m+1,\varepsilon,\infty}u\right\|<\infty.$$

Now, from the pseudodifferential equation, $Pu=f$, it follows that

$$(-t)^{k+1}A_{m,\varepsilon,\infty}D_t^p u\in L^2\big((-\nu, 0)\times B_1\big) \text{ for all } p \text{ and } m$$

and hence that $A_{m,\varepsilon,\infty}u\in C^\infty([-\nu, 0]\times B_1)$.

In particular this shows that all the Cauchy data of $A_{m,\varepsilon,\infty}u$ at $t=0$ are C^∞, so can be removed to any order without affecting the singularities. Thus, Proposition 3.36 can be applied in the same way to conclude that

$$A_{m,\varepsilon,\infty}u \in C^\infty\big([-\nu, \nu] \times B_1\big) \text{ for } \varepsilon,\ \nu>0 \text{ sufficiently small.}$$

Thus, $\bar{\rho} \notin WF(u)$ proving the local, and hence the global, form of the propagation of singularities stated in Theorem 1. 7.

5. Cauchy problem

Suppose that $P \in \mathrm{Diff}^m(X)$ is effectively hyperbolic at $\bar{x}$ and hence in an open neighbourhood of $\bar{x}$. We shall first compactify the problem by modifying P suitably away from $\bar{x}$. Observe that, because of (1. 9), it can be assumed that

$$P = D_t^m + \sum_{j=1}^{m} P_j(t, x, D_x)\, D_t^{m-j}, \tag{5. 1}$$

where P is weakly t-hyperbolic ; i. e. the symbol of P has real roots

$$p = \sigma_m(P) = \tau^m + \sum_{j=1}^{m} p_j(t, x, \xi)\, \tau^{m-j} = \prod_{k=1}^{m} \big(\tau - \mu_j(t, x, \xi)\big), \tag{5. 2}$$

with the μ_j real throughout the region $|t|<\varepsilon$, $|x_e|<\varepsilon$, $\xi \in \boldsymbol{R}^n$.

Consider the projection of the set of doubly characteristic points :

$$\Sigma_2' = \Big\{(t, x, \xi) \in [-\varepsilon, \varepsilon]^{n+1} \times S^{n-1};\ \mu_j(t, x, \xi) = \mu_k(t, x, \xi) \text{ some } j \neq k\Big\}. \tag{5. 3}$$

By assumption (1. 11), p has at most double roots. Select, using the compactness of Σ_2', a finite covering by open sets

$$\Sigma_2' = \bigcup_{r=1}^{R} U_r \tag{5. 4}$$

where each U_r as here has a base point ρ_r at which $\mu_j(\rho_r) = \mu_k(\rho_r)$ if and only if $k=j+1$ and $j \in J_r$, which defines the index set $J_r \subset \{1, \cdots, -1\}$, and in addition $\mu_j < \mu_{j+1}$ throughout $\bar{U}_r$ if $j \notin J_r$. Thus in $\bar{U}_r$ the roots are separated either singly or in pairs. Srandard arguments with the pseudodifferential operator calculus give a factorization of P microlocally over each U_r, in the sense that

$$P \equiv \big(D_t - M_1(t, x, D_x)\big) \cdots \Big(\big(D_t - A_j(t, x, D_x)\big)^2 - R_j(t, x, D_x)\Big) \cdots \tag{5. 5}$$

where each simple root μ_j correspond to a first order factor with $M_j \in \Psi^1$ depending smoothly on t and each $j \in J$ corresponds to a second order factor :

$$\sigma_1(M_j)=\mu_j(t,x,\xi)\ j,\ j-1\notin J_r\,;\ \sigma_1(A_j)=\frac{1}{2}(\mu_j+\mu_{j-1}),\tag{5.6}$$
$$\sigma_2(R_j)=\frac{1}{4}(\mu_j-\mu_{j-1})^2,\ j\in J_r$$

Equality in (5.5) is modulo operators of the form

$$\sum_{j=0}^{m-1}Q_j(t,x,D_x)D_t^j$$

with coefficients $Q_j\in\Psi^{m-j}$ depending smoothly on t and of order $-\infty$ in a conic neighbourhood of $\bar{U}_r$. The order of the factors in (5.5) is fixed so the full factorization is determined modulo errors of the same type.

By refining the partition of unity (5.4) if necessary we can assume that for some $1\leq R'\leq R$, the U_r with $r\geq R'$ cover the boundary $t=\pm\varepsilon$ or $x_j=\pm\varepsilon$ and that $|t|>\frac{1}{2}\varepsilon$, or $|x_j|>\frac{1}{2}\varepsilon$ for some j on these U_r. Choosing a positive elliptic operator $Q\in\Psi^2(\boldsymbol{R}^n)$ and a microlocal partition of unity $\varphi_r\in\Psi^0(\boldsymbol{R}^n)$, depending smoothly on t and subordinate to the covering U_r the operator P_r given by the product on the right in (5.5) can be modified by replacing R_j by $R_j+\delta Q\varphi_r$, $\delta>0$, $j\in J_r$. Calling this modified operator $P_{r,\delta,j}$ set first

$$P''\equiv\sum_{r>R'}\varphi_rP_r+\sum_{r<R'}\varphi_rP_r+\varphi_{R'}P_{r,\delta,j}\tag{5.6}$$

for $j\in J_{R'}$ the smallest element. Repeat the factorization and next modify the second R_j in $P_{R'}$, keeping δ small. Repeat the process successively for all $j\in J_r$, $\gamma\geq R'$, eventually giving the effectively hyperbolic operator P''. By construction $P=P''$ in a neighbourhood of $\bar{x}$ and P'' has only simple roots near the boundary. We next extend P'' by extending the coefficients be periodic of period say $T=3\varepsilon$ and so that on the torus P'' has simple zeros outside the set $\bar{D}_{\frac{1}{2}\varepsilon}$,

$$D_{\frac{1}{2}\varepsilon}=\left\{|t|,\ |x_j|<\frac{1}{2}\varepsilon\right\}.$$

Finally, cutting off the supports of the coefficients of P'' near the diagonal and adding a suitable term of order $-\infty$ in the coefficients we can insure that the resulting operator P' is of the form (5.1), is strictly hyperbolic outside D_ε and extends P in the sense that

$$P\varphi=P'\varphi\ \text{ in } D_{\frac{1}{2}\varepsilon}\ \text{ if } \operatorname{supp}(\varphi)\subset D_{\frac{1}{2}\varepsilon}\,.\tag{5.7}$$

Repeating this procedure, but in addition adding terms $\varepsilon^2\varphi_rQ$ to the R_j in the factorization of P over U_r, $r<R'$, we obtain an operator $P\varepsilon$ of the form (5.1) which is strictly hyperbolic as a periodic t-differential operator

with x-pseudodifferential coefficients for $\varepsilon\neq 0$ on the torus and which is C^∞ in ε and has $P_0=P'$.

(5.8) PROPOSITION. *If P' is as above, the extension of an effectively hyperbolic differential operator* (5.1), *then there exists L and for each $s, s'\in \boldsymbol{R}$ $C_{s,s'}$ such that if $u\in C^{-\infty}(\boldsymbol{R}\times \boldsymbol{T}^n)$ has support in $[-R, \infty)\times \boldsymbol{T}^n$ and $P'u\in H^s$ $((-\infty, T)\times \boldsymbol{T}^N)$ then $u\in H^{s-L}((-\infty, T)\times \boldsymbol{T}^n)$ and*

$$||u||_{s-L}\leq C_{s,s'}\left(||P'u||_s+||u||_{s'}\right) \tag{5.9}$$

with the Sobolev norms taken over $(-\infty, T)\times \boldsymbol{T}^n$.

PROOF. Since P' is everywhere effectively hyperbolic and strictly hyperbolic outside a compact set it is clear from the discussion in Section 3, 4 above that the regularity of u microlocally near any point (t, x, τ, ξ) with $t<T$ depends only on the regularity of $P'u$ and of u along the backward rays (if any) through this point, at least if $\xi\neq 0$. Since t increases strictly along a ray there can be only a fixed finite number of crossings of the doubly characteristic variety along such a ray between $t=-T$ and $t=T$, and therefore only a fixed finite loss of derivatives, L. This gives the estimate (5.9), microlocally in $t<T$, $\xi\neq 0$. Although P' is not quite a pseudodifferential operator near $\xi=0$, not even microlocal, it is very much elliptic there so no difficulty arises from that region. Finally near $t=T$ the estimates (5.9), which involve Sobolev norms on half-spaces, are the standard estimates for strictly hyperbolic operators (5.1). Thus, the derivation of (5.9) is completely standard.

We further remark that the estimates (5.9) hold with P' replaced by P_ε and the constants $C_{s,s'}$ independent of $|\varepsilon|<1$; it is only necessary to observe that the reductions on Section 2 and subsequent estimates in Section 3 can be made uniform in ε. Of course, for $\varepsilon\neq 0$ estimates (5.9) for P_ε holds with $L=1-m$, but not uniformly as $\varepsilon\downarrow 0$.

An immediate consequence of (5.9) is that

$$\begin{aligned}\ker(P')=\Big\{&u\in C^{-\infty}(-\infty, T)\times \boldsymbol{T}^n;\ \operatorname{supp}(u)\subset[-T, T)\\ &\times \boldsymbol{T}^n,\ Pu=0\Big\}\end{aligned} \tag{5.10}$$

is a finite dimensional subspace of $C^\infty((-\infty, T]\times \boldsymbol{T}^n)$ and that

$$\begin{aligned}\operatorname{ran}(P')=\Big\{&\in C^{-\infty}(-\infty, T)\times \boldsymbol{T}^n;\ Pu=f \text{ where } \operatorname{supp}(u)\\ &\subset[-T, T)\times \boldsymbol{T}^n\Big\}\end{aligned} \tag{5.11}$$

is a closed subspace of finite codimension. Indeed, $\ker(P')$ is closed in $C^{-\infty}$ and by repeated application of (5.9), a subsapce of $C^{\infty}((-\infty, T]\times \boldsymbol{T}^n)$ hence has finite dimension. Similarly, if $f_n\in\operatorname{ran}(P')$ then there is a unique solution of $Pu_n=f_n$, with u_n supported in $t\geq -T$ and in some fixed complement to $\ker(P')$. If $f_n\to f$ in $C^{-\infty}$ then $f_n\to f$ in some space $H^s((-\infty, T)\times \boldsymbol{T}^n)$, it follows that u_n is bounded in H^{s-L-1} since if not $v_n=u_n/\|u_n\|_{s-L-1}$ would have a weakly convergent subsequence in H^{s-L-1} with limit v, $P'v=\lim f_n/\|u_n\|=0$, so $v\in\ker(P')$. However, from (5.9) it follows that the sequence v_n is bounded in H^{s-L}, so strongly convergent in H^{s-L-1} and therfore, $\|v\|_{s-L-1}=1$ which contradicts the assumption that each u_n is in the fixed complement to $\ker(P')$. These results and duality show that P' is a Fredholm operator on the space of distributions on $(-\infty, T)\times \boldsymbol{T}^n$ with support in $t\geq -T$, whereas to complete the proof of Theorem 1.12 we need to show this in $t\geq\bar{t}$.

Apply the remarks above to $(P')^*$ with the t variable reversed, thus

$$\operatorname{ran}\left((P')^*\right)=(P')^*\left\{w\in C^{-\infty}\left((-T, \infty)\times \boldsymbol{T}^n\right);\ w=0 \text{ in } t>T\right\}$$

has a finite dimensional complement $A\subset\{w\in C^{\infty}([-T, \infty)\times \boldsymbol{T}^n);\ w=0$ in $t>T\}$. For each $\varepsilon\neq 0$ set

$$K_\varepsilon=(P_\varepsilon^*)^{-1}A\,, \tag{5.12}$$

a subspace of C^{∞} of fixed dimension $k=\dim A$, for $\varepsilon\neq 0$, since the Cauchy problem for P_ε^* is uniquely solvable. Now, we wish to show that

(5.13) if $f\in\operatorname{ran}(P')$ has $f=0$ in $t<\bar{t}$ then u with $u=0$ in $t<\bar{t}$ and $Pu=f$.

To see this, choose an orthonormal basis of K_ε in $L^2((-T, T)\times \boldsymbol{T}^n)$, e_j^ε, $j=1, \cdots, k$. Passing to a sequence $\varepsilon(m)\to 0$, we can ensure that $e_j^{\varepsilon(m)}\to e_j$ weakly in L^2. Now,

$$P_\varepsilon^* e_j^\varepsilon=\sum_{j'=1}^{k} c_{j',j}^\varepsilon a_{j'}$$

if $\{a_j\}$ is a basis of A. The topologies induced on A from the various Sobolev spaces are all the same, so $a_{j',j}^{\varepsilon(m)}$ converge as $\varepsilon(m)\to 0$, in fact they converge to zero since A does not meet $\operatorname{ran}(P'^*)$; in any case it follows from the estimates (5.9) applied to P_ε^* uniformly in ε that $e_j^{\varepsilon(m)}\to e_j$ in C^{∞}. Of course, the e_j are just a basis of $\ker(P^*)$. Given $\bar{t}\in(-T, T)$ choose $\bar{t}'\in(-T, T)$ with $\bar{t}'<\bar{t}$, consider the restriction of K_ε to $[\bar{t}', T]\times \boldsymbol{T}^n$ and repeat the orthonormalization and convergence arguments, only now (5.9) is not available so we only know that $e_j^{\varepsilon(m)}(\bar{t}')\to e_j(\bar{t}')$ weakly in $L^2((\bar{t}', T)\times \boldsymbol{T}^n)$

but uniformly with all derivatives on any compact subset of $(\bar{t}', T]\times \boldsymbol{T}^n$. The dimension of the corresponding space of constraints is $k(\bar{t}')\leq k$, i.e. $j=1, \cdots, k(\bar{t}')$.

Now, if f is as in (5.13) and in addition $f(e_j(\bar{t}'))=0$, $j=1, \cdots, k(\bar{t}')$, which makes sense since $e_j(\bar{t}')\in C^\infty((\bar{t}', \infty)\times \boldsymbol{T}^n)$ vanishes in $t>T$, then we can find $f_m\to f$ such that $\operatorname{supp}(f_m)\subset\{t\geq\bar{t}'\}$ and $f_m(e_j^{(m)}(\bar{t}'))=0$. Indeed, simply set

$$f_m=f-\sum_j f\left(e_j^{(m)}(\bar{t}')\right)\bar{e}_j^{(m)}(\bar{t}')$$

where $\bar{e}_j^{(m)}(\bar{t}')$ is $e_j^{(m)}(\bar{t}')$ extended to be zero in $t<\bar{t}'$, still in L^2. Set $u_m=P_\epsilon^{-1}f_m$, $u_m=0$ in $t<-T$, and note from the construction above that

$$u_m(A)=0\,. \tag{5.14}$$

To see this just note that the $P_\epsilon^*e_j^\epsilon$ span A, by definition, and u_m has support in $t<\bar{t}'$, from the properties of strictly hyperbolic iperators, and is in C^∞ near $t=\bar{t}'$. Thus,

$$u_m(P^*e_j^{(m)})=u_m\left(P^*\bar{e}_j^{(m)}(\bar{t}')\right)=f_m\left(\bar{e}_j^{(m)}(\bar{t}')\right)=0\,.$$

Now, (5.14) shows that the u_m lie in a fixed complement to $\ker(P)$, namely the annihilator of A, so from (5.9) applied to P_ϵ, uniformly in ε, it follows that as $m\to\infty$, $\varepsilon(m)\to 0$, $u_m\to u$ in $C^{-\infty}$. Since $f_m\to f$ we have shown that if $u\in\operatorname{ran}(P')$ has support in $t\geq\bar{t}$ and $\bar{t}'<\bar{t}$ then there is a solution u of $P'u=f$ with u supported in $t\geq t'$. The finiteness of the kernel of P' shows that this solution stabilizes as $\bar{t}'\uparrow\bar{t}$, so we have proved (5.13).

The remainder of the proof of Theorem 1.12 is now straightforward.

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