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Lowerable vector fields for a finitely $\mathcal{L}$-determined multigerm

Yusuke MIZOTA and Takashi NISHIMURA

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Abstract. We show that the module of lowerable vector fields for a finitely $\mathcal{L}$-determined multigerm is finitely generated in a constructive way.

Key words: Lowerable vector field, finitely $\mathcal{L}$-determined multigerm, preparation theorem, finitely generated module.

1. Introduction

Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. Throughout this paper, all mappings are of class C^∞ for $\mathbb{K} = \mathbb{R}$, and are holomorphic for $\mathbb{K} = \mathbb{C}$, unless otherwise stated.

Let S be a finite set consisting of r distinct points in $\mathbb{K}^n$. A map-germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is called a *multigerm*. When $r = 1$, f is called a *monogerm*. A multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ can be identified with $\{f_k : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0) \mid 1 \leq k \leq r\}$. Each f_k is called a *branch* of f.

Let $C_{n,S}$ (resp., $C_{p,0}$) be the $\mathbb{K}$-algebra of all function-germs on $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$) and $m_{n,S}$ (resp., $m_{p,0}$) be the ideal of $C_{n,S}$ (resp., $C_{p,0}$) consisting of function-germs $(\mathbb{K}^n, S) \to (\mathbb{K}, 0)$ (resp., $(\mathbb{K}^p, 0) \to (\mathbb{K}, 0)$). For a non-negative integer i, let $m^i_{n,S}$ (resp., $m^i_{p,0}$) denote the ideal of $C_{n,S}$ (resp., $C_{p,0}$) consisting of those function-germs on $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$) whose Taylor series vanish up to degree $i - 1$.

For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, let $f^* : C_{p,0} \to C_{n,S}$ be the $\mathbb{K}$-algebra homomorphism defined by $f^*(\psi) = \psi \circ f$. Set $Q(f) = C_{n,S}/f^* m_{p,0} C_{n,S}$ and $\delta(f) = \dim_{\mathbb{K}} Q(f)$.

For a map-germ $f : (\mathbb{K}^n, S) \to \mathbb{K}^p$, let $\theta(f)$ be the set of germs of vector fields along f. The set $\theta(f)$ has the natural $C_{n,S}$-module structure and is identified with the direct sum of p copies of $C_{n,S}$. Set $\theta_S(n) = \theta(\mathrm{id}_{(\mathbb{K}^n,S)})$ and $\theta_0(p) = \theta(\mathrm{id}_{(\mathbb{K}^p,0)})$, where $\mathrm{id}_{(\mathbb{K}^n,S)}$ (resp., $\mathrm{id}_{(\mathbb{K}^p,0)}$) is the germ of the identity mapping of $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$). For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, following Mather [4, p. 141], define $tf : \theta_S(n) \to \theta(f)$ (resp., $\omega f : \theta_0(p) \to \theta(f)$) as $tf(\xi) = df \circ \xi$ (resp., $\omega f(\eta) = \eta \circ f$), where df

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is the differential of f. Following Wall [5, p. 485], set $T\mathcal{R}_e(f) = tf(\theta_S(n))$ and $T\mathcal{L}_e(f) = \omega f(\theta_0(p))$. For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, a vector field $\xi \in \theta_S(n)$ is said to be *lowerable* if $df \circ \xi$ belongs to $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. Let $\mathrm{Lower}(f)$ be the set of all lowerable vector fields for the multigerm f. Then, $\mathrm{Lower}(f)$ has a $C_{p,0}$-module structure via f. The notion of lowerable vector field, which was introduced by Arnol'd [1] for studying bifurcations of wave front singularities, is significant in Singularity Theory (for instance, see [3]).

In the paper, we investigate the following problem.

Problem 1 Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a multigerm satisfying $\delta(f) < \infty$. Then, is the module $\mathrm{Lower}(f)$ finitely generated? In the case that $\mathrm{Lower}(f)$ is finitely generated, prove it in a constructive way.

Our first result is the following Proposition 2, which reduces Problem 1 to that of the finite generation on $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$.

Proposition 2 *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a multigerm satisfying $\delta(f) < \infty$. Then, tf is injective.*

We see that, in the complex analytic case, $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is finitely generated, since $C_{p,0}$ is Noetherian and $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is a $C_{p,0}$-submodule of the finitely generated module $\theta(f)$. However, the algebraic argument gives no constructive proof. Moreover, the finite generation of $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ has been an open problem in the real C^∞ case, as far as the authors know.

The main purpose of the paper is to give a constructive proof of the following theorem, which works well in both the real C^∞ case and the complex analytic case.

Theorem 3 *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a finitely $\mathcal{L}$-determined multigerm. Then, $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ is finitely generated as a $C_{p,0}$-module via f.*

Here, a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is said to be *finitely $\mathcal{L}$-determined* if there exists a positive integer ℓ such that $m_{n,S}^{\ell}\theta(f) \subset T\mathcal{L}_e(f)$ holds. We easily see that $\delta(f)$ is finite if f is finitely $\mathcal{L}$-determined. Thus, by combining Proposition 2 and Theorem 3, we have the following partial affirmative answer to Problem 1.

Corollary 4 *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a finitely $\mathcal{L}$-determined multigerm. Then,* $\mathrm{Lower}(f)$ *is finitely generated as a $C_{p,0}$-module via f.*

Remark 5 According to Theorem 2.5 in [5, p. 494], a monogerm f is finitely $\mathcal{L}$-determined if and only if f is finitely $\mathcal{A}$-determined and $2n \le p$, or f is an immersion-germ. Therefore, it seems impossible to apply our results to the mappings appearing in [1], [3] unfortunately. However, by using the argument of Theorem 3, it is possible even to construct explicit generators of Lower(f) for a multigerm f. For example, we consider the multigerm $f = (f_1, f_2) : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ defined by

$$f_1(x) = (x^2,\, x^3), \quad f_2(x) = (x^3,\, x^2).$$

It is not hard to show that $m^6_{1,S}\theta(f) \subset T\mathcal{L}_e(f)$ holds. Thus, f is finitely $\mathcal{L}$-determined. By using the argument of Theorem 3, the $C_{2,0}$-module Lower(f) is generated by the following two vector fields:

$$((x^3), (3x^4)), \quad ((3x^4), (x^3)).$$

There seems to be no results so far on lowerable vector fields for a multigerm as far as the authors know.

In Section 2 (resp., Section 3), Proposition 2 (resp., Theorem 3) is proved.

2. Proof of Proposition 2

It suffices to show that if a monogerm $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ satisfies $\delta(f) < \infty$, then tf is injective.

Suppose that for $\xi \in \theta_0(n)$, we have $tf(\xi) = 0$ on an open set U_1 containing 0. Then, f is constant along any integral curve of ξ on U_1. Since $\delta(f) < \infty$ holds, each integral curve of ξ on an open set U_2 containing 0 must consist of a single point by Propositions 2.2 and 2.3 in [2, pp. 167–168]. Therefore, we have $\xi = 0$ on $U_1 \cap U_2$. Thus, tf is injective. □

3. Proof of Theorem 3

Since f is finitely $\mathcal{L}$-determined, there exists a positive integer ℓ such that

$$m^{\ell}_{n,S}\theta(f) \subset T\mathcal{L}_e(f) \tag{1}$$

holds and we have $\delta(f) < \infty$. Thus, $Q(f_k)$ is a finite dimensional $\mathbb{K}$-vector space of dimension $\delta(f_k)$ for every k with $1 \leq k \leq r$, where f_k are the branches of f. Then, there exist $\varphi_{k,j} \in C_{n,0}$, $1 \leq j \leq \delta(f_k)$, such that we have

$$Q(f_k) = \left\langle [\varphi_{k,1}], [\varphi_{k,2}], \ldots, [\varphi_{k,\delta(f_k)}] \right\rangle_{\mathbb{K}}.$$

We would like to find a finite set of generators for $T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. Let us take any element $\overline{\eta} = (\overline{\eta}_1, \overline{\eta}_2, \ldots, \overline{\eta}_r) \in T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. Let $(x_1, x_2, \ldots, x_n)$ (resp., $(X_1, X_2, \ldots, X_p)$) be the standard local coordinates of $\mathbb{K}^n$ (resp., $\mathbb{K}^p$) around the origin. For every $k = 1, 2, \ldots, r$, the vector field $\overline{\eta}_k$ can be expressed as

$$\overline{\eta}_k = \begin{pmatrix} \dfrac{\partial(X_1 \circ f_k)}{\partial x_1} & \dfrac{\partial(X_1 \circ f_k)}{\partial x_2} & \cdots & \dfrac{\partial(X_1 \circ f_k)}{\partial x_n} \\ \dfrac{\partial(X_2 \circ f_k)}{\partial x_1} & \dfrac{\partial(X_2 \circ f_k)}{\partial x_2} & \cdots & \dfrac{\partial(X_2 \circ f_k)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial(X_p \circ f_k)}{\partial x_1} & \dfrac{\partial(X_p \circ f_k)}{\partial x_2} & \cdots & \dfrac{\partial(X_p \circ f_k)}{\partial x_n} \end{pmatrix} \begin{pmatrix} \widetilde{\varphi}_{1,k} \\ \widetilde{\varphi}_{2,k} \\ \vdots \\ \widetilde{\varphi}_{n,k} \end{pmatrix}$$

for some $\widetilde{\varphi}_{1,k}, \widetilde{\varphi}_{2,k}, \ldots, \widetilde{\varphi}_{n,k} \in C_{n,0}$.

Then, by the preparation theorem, there exist $\psi_{k,i,j} \in C_{p,0}$ such that we have

$$\widetilde{\varphi}_{i,k} = \sum_{1 \leq j \leq \delta(f_k)} (\psi_{k,i,j} \circ f_k)\varphi_{k,j}.$$

Thus, $\overline{\eta}_k$ can be simplified as follows:

$$\overline{\eta}_k = \sum_{i,\,j} (\psi_{k,i,j} \circ f_k)\xi_{k,i,j},$$

where the symbol $\sum_{i,\,j}$ means the summation taken over all i and j with $1 \leq i \leq n$ and $1 \leq j \leq \delta(f_k)$, respectively, and $\xi_{k,i,j}$ is the transpose of

$$\left(\frac{\partial(X_1 \circ f_k)}{\partial x_i}\varphi_{k,j}, \frac{\partial(X_2 \circ f_k)}{\partial x_i}\varphi_{k,j}, \ldots, \frac{\partial(X_p \circ f_k)}{\partial x_i}\varphi_{k,j} \right).$$

Note that $\xi_{k,i,j} \in T\mathcal{R}_e(f_k)$ holds.

For a p-tuple of non-negative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$, set

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_p,\ X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_p^{\alpha_p},$$

and

$$f_k^\alpha = (X_1 \circ f_k)^{\alpha_1} (X_2 \circ f_k)^{\alpha_2} \cdots (X_p \circ f_k)^{\alpha_p}.$$

Then, the function-germs $\psi_{k,i,j} \in C_{p,0}$ can be written in the form

$$\psi_{k,i,j}(X_1, X_2, \dots, X_p) = \sum_{0 \le |\alpha| \le \ell - 1} c_{k,i,j,\alpha} X^\alpha + \sum_{|\alpha| = \ell} \widetilde{\psi}_{k,i,j,\alpha} X^\alpha$$

for some $c_{k,i,j,\alpha} \in \mathbb{K}$ and $\widetilde{\psi}_{k,i,j,\alpha} \in C_{p,0}$. Recall that ℓ is the positive integer given in (1). We have

$$\overline{\eta}_k = \sum_{i,j} \sum_{0 \le |\alpha| \le \ell - 1} c_{k,i,j,\alpha} (f_k^\alpha \xi_{k,i,j}) + \sum_{i,j} \sum_{|\alpha| = \ell} \big(\widetilde{\psi}_{k,i,j,\alpha} \circ f_k\big)(f_k^\alpha \xi_{k,i,j}).$$

Set

$$\overline{\xi}_{k,i,j,\alpha} = (\underbrace{0,\, 0, \dots,\, 0,\, f_k^\alpha \xi_{k,i,j}}_{k \text{ entries}},\, 0, \dots,\, 0).$$

Note that $\overline{\xi}_{k,i,j,\alpha} \in T\mathcal{R}_e(f)$ holds. Then, we have

$$\overline{\eta} = \sum_{1 \le k \le r} \sum_{i,j} \sum_{0 \le |\alpha| \le \ell - 1} c_{k,i,j,\alpha} \overline{\xi}_{k,i,j,\alpha} + \sum_{1 \le k \le r} \sum_{i,j} \sum_{|\alpha| = \ell} \big(\widetilde{\psi}_{k,i,j,\alpha} \circ f\big) \overline{\xi}_{k,i,j,\alpha}.$$

We define the finite sets L and H of $T\mathcal{R}_e(f)$ as follows:

$$L = \big\{\overline{\xi}_{k,i,j,\alpha} \mid 0 \le |\alpha| \le \ell - 1,\, 1 \le k \le r,\, 1 \le i \le n,\, 1 \le j \le \delta(f_k)\big\},$$

$$H = \big\{\overline{\xi}_{k,i,j,\alpha} \mid |\alpha| = \ell,\, 1 \le k \le r,\, 1 \le i \le n,\, 1 \le j \le \delta(f_k)\big\}.$$

Then, $H \subset T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$ by (1). Therefore,

$$\sum_{1\le k\le r}\sum_{i,j}\sum_{0\le|\alpha|\le\ell-1} c_{k,i,j,\alpha}\overline{\xi}_{k,i,j,\alpha}$$

belongs to $V = T\mathcal{R}_e(f)\cap T\mathcal{L}_e(f)\cap L_{\mathbb{K}}$.

The set V is a finite dimensional $\mathbb{K}$-vector space. Set $\dim_{\mathbb{K}} V = m$. Then, there exist $\underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m \in T\mathcal{R}_e(f)\cap T\mathcal{L}_e(f)$ such that we have

$$V = \left\langle \underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m \right\rangle_{\mathbb{K}}.$$

Clearly, we have $V \subset \left\langle \underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m \right\rangle_{f^*C_{p,0}}$. Therefore, we see that

$$\overline{\eta} \in \left\langle \underline{\xi}_1, \underline{\xi}_2 \ldots, \underline{\xi}_m \right\rangle_{f^*C_{p,0}} + H_{f^*C_{p,0}}.$$

Thus, we have

$$T\mathcal{R}_e(f)\cap T\mathcal{L}_e(f) \subset \left\langle \underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m \right\rangle_{f^*C_{p,0}} + H_{f^*C_{p,0}}.$$

The converse inclusion also holds, since $\left\{\underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_m\right\}\cup H$ is contained in $T\mathcal{R}_e(f)\cap T\mathcal{L}_e(f)$. Thus, $T\mathcal{R}_e(f)\cap T\mathcal{L}_e(f)$ is finitely generated as a $C_{p,0}$-module via f. $\square$

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References

[1] Arnol'd V. I., *Wave front evolution and equivariant Morse lemma.* Comm. Pure Appl. Math. **29** (1976), 557–582.

[2] Golubitsky M. and Guillemin V., *Stable mappings and their singularities.* Graduate Texts in Mathematics **14**, Springer-Verlag, New York-Heidelberg, 1973.

[3] Ishikawa G., *Openings of differentiable map-germs and unfoldings.* Topics on Real and Complex Singularities, Proceedings of the 4th Japanese-Australian Workshop (JARCS4), Kobe 2011, World Scientific, 2014, pp. 87–113.

[4] Mather J., *Stability of C^∞ mappings, III. Finitely determined map-germs.* Publ. Math. Inst. Hautes tudes Sci. **35** (1968), 127–156.

[5] Wall C. T. C., *Finite determinacy of smooth map-germs.* Bull. London Math. Soc. **13** (1981), 481–539.

Yusuke Mizota
Academic Support Center
Kyushu Sangyo University
Fukuoka 813-8503, Japan
E-mail: mizota@ip.kyusan-u.ac.jp

Takashi Nishimura
Research Group of Mathematical Sciences
Research Institute of Environment and Information Sciences
Yokohama National University
Yokohama 240-8501, Japan
E-mail: nishimura-takashi-yx@ynu.jp