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More on the annihilator graph of a commutative ring

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Abstract. Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. The annihilator graph of R is defined as the undirected graph $AG(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. In this paper, we study the affinity between annihilator graph and zero-divisor graph associated with a commutative ring. For instance, for a non-reduced ring R, it is proved that the annihilator graph and the zero-divisor graph of R are identical to the join of a complete graph and a null graph if and only if $ann_R(Z(R))$ is a prime ideal if and only if R has at most two associated primes. Among other results, under some assumptions, we give necessary and sufficient conditions under which $AG(R)$ is a star graph.

Key words: Annihilator graph, Zero-divisor graph, Associated prime ideal.

1. Introduction

Usually, after translating of algebraic properties of rings into graph-theoretic language, some problems in ring theory might be more easily solved. When one assigns a graph to a ring, numerous interesting algebraic problems arise from the translation of some graph-theoretic parameters such as clique number, chromatic number, diameter, radius and so on. There are many extensive studies of this topic, see for example [1], [2], [3], [5] and [7].

Throughout this paper, all rings are assumed to be non-domain commutative rings with identity. We denote by $Min(R)$, $Nil(R)$ and $Z(R)$, the set of all minimal prime ideals, the set of all nilpotent elements and the set of zero-divisors elements of R, respectively. Let $A \subseteq R$. The set of annihilators of A is denoted by $ann_R(A)$ and by A^*, we mean $A \setminus \{0\}$. The ring R is said to be *reduced*, if $Nil(R) = 0$. A prime ideal P of R is called an *associated prime ideal*, if $ann_R(x) = P$, for some non-zero element $x \in R$. The set of all associated prime ideals of R is denoted by $Ass(R)$. For any undefined notation or terminology in ring theory, we refer the reader to [4], [8].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of

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G. The girth of a graph G is denoted by $gr(G)$. We write $u - v$, to denote an edge with ends u, v. A graph $H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \,|\, u, v \in V_0\}$. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \,|\, u \in V(G_1), v \in V(G_2)\}$. Also G is called a *null graph* if it has no edge. For a vertex x in G, we denote the set of all vertices adjacent to x by $N_G(x)$. A complete bipartite graph of part sizes m, n is denoted by $K^{m,n}$. If $m = 1$, then the complete bipartite graph is called *star graph.* Also, a complete graph of n vertices is denoted by K^n. For any undefined notation or terminology in graph theory, we refer the reader to [9].

The *annihilator graph* of a ring R is defined as the graph $AG(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. This graph was first introduced and investigated in [5] and many of interesting properties of annihilator graph were studied. For example, it was proved the annihilator graph is a connected graph of diameter at most 2. Also, the author in [5], studied some relations between two graphs $AG(R)$ and $\Gamma(R)$, where $\Gamma(R)$ is the zero-divisor graph of a ring R. The *zero- divisor graph* of a ring R, denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this article, we continue the study of annihilator graphs associated with commutative rings. Especially, we focus on the conditions under which the annihilator graph is identical to the zero-divisor graph. For instance, for a non-reduced ring R, it is proved that the annihilator graph and the zero-divisor graph of R are identical to the join of a complete graph and a null graph if and only if $ann_R(Z(R))$ is a prime ideal if and only if R has at most two associated primes.

2. Main Results

We begin with the following lemma.

Lemma 2.1 *Let R be a ring.*

(1) *Let x, y be distinct elements of $Z(R)^*$, and suppose that $Z(R) =$*

$ann_R(x) \cup ann_R(y)$. Then $x - y$ is an edge of $\Gamma(R)$ if and only if $x - y$ is an edge of $AG(R)$.

(2) *Let x, y, z be elements of $Z(R)^*$, and suppose that $ann_R(x) = ann_R(y)$. Then $x - z$ is an edge of $AG(R)$ if and only if $y - z$ is an edge of $AG(R)$.*

(3) *Let $\Gamma(R) = K^{1,n}$ for some $n \geq 1$ such that x is adjacent to every other vertex. If $ann_R(x) = ann_R(y)$ for some $y \in Z(R)^*$, then either $x = y$, or $\Gamma(R) = AG(R) = K^{1,1}$.*

Proof. (1) If $x - y$ is an edge of $\Gamma(R)$, then by Part (2) of [5, Lemmad 2.1], $x - y$ is an edge of $AG(R)$. To prove the converse, assume that $x - y$ is an edge of $AG(R)$. It is enough to show that $xy = 0$. Assume to the contrary, $xy \neq 0$. Since $ann_R(x) \cup ann_R(y) \subseteq ann(xy)$, the equality $Z(R) = ann_R(x) \cup ann_R(y)$ implies that $ann_R(xy) = ann_R(x) \cup ann_R(y)$. This means that $x - y$ is not an edge of $AG(R)$, a contradiction.

(2) Suppose that $x - z$ is an edge of $AG(R)$. Then there exists an element $r \in R$ such that $rxz = 0$, $rx \neq 0$ and $rz \neq 0$. The equality $rxz = 0$ together with the assumption $ann_R(x) = ann_R(y)$ imply that $ryz = 0$. Also, it is clear that $ry \neq 0$. Thus $r \in ann_R(yz) \setminus ann_R(y) \cup ann_R(z)$. Hence $y - z$ is an edge of $AG(R)$. The converse is proved, similarly.

(3) is clear. □

By using Lemma 2.1, we provide a simple proof of [5, Theorem 3.17].

Theorem 2.2 ([5, Theorem 3.17]) *Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$. Then the following statements are equivalent*:

(1) $\Gamma(R)$ *is a star graph*;

(2) $\Gamma(R) = K^{1,2}$;

(3) $AG(R) = K^3$.

Proof. Since $AG(R) \neq \Gamma(R)$, (3) $\Rightarrow$ (1) and (3) $\Leftrightarrow$ (2) are obvious. We have only to prove (1) $\Rightarrow$ (3). Let a be the center of the star graph $\Gamma(R)$. Since $\Gamma(R)$ is a star graph and $AG(R) \neq \Gamma(R)$, we deduce that $|Z(R)^*| \geq 3$ and $ann_R(x) = ann_R(y) = \{0, a\}$, for every $x, y \in Z(R) \setminus \{0, a\}$. Furthermore, by [3, Theorem 2.5] and [5, Theorem 3.6], $Z(R) = ann_R(a)$ for a non-zero element $a \in R$. To complete the proof, we show that $|Z(R)^*| = 3$. Suppose to the contrary, a, b, c, x are distinct elements of $Z(R)^*$. With no loss of generality, one may assume that $b - x$ is an edge of $AG(R)$ ($AG(R) \neq \Gamma(R)$). Since $ann_R(b) = ann_R(c)$, Part (2) of Lemma 2.1 implies that $c - x$ is also an edge of $AG(R)$. Similarly, the equality $ann_R(c) = ann_R(x)$ shows that $c - b$ is an edge of $AG(R)$. Since $bx \neq 0$ and $ann_R(bx) \neq ann_R(b) \cup ann_R(x)$, we have $ann_R(bx) = ann_R(a)$. By Part (3) of Lemma 2.1, $bx = a$. Similarly, $cx = a$ and $cb = a$. Hance $x(b - c) = b(c - x) = c(b - x) = 0$ and so $b - x = c - x = b - c = a$, a contradiction. □

To prove Theorem 2.5, the following lemma is needed.

Lemma 2.3 *Let R be a ring and $x \in Z(R)^*$. Then*

(1) *If $ann_R(x)$ is a prime ideal of R, then $N_{\Gamma(R)}(x) = N_{AG(R)}(x)$.*
(2) *If $x \in Nil(R)^*$ and $N_{\Gamma(R)}(x) = N_{AG(R)}(x)$, then $ann_R(x)$ is a prime ideal of R.*

Proof. (1) By Part (2) of [5, Lemma 2.1], it is enough to show that $N_{AG(R)}(x) \subseteq N_{\Gamma(R)}(x)$. Assume to the contrary, $x - y$ is an edge of $AG(R)$ such that $xy \neq 0$. Therefore, there exists an element $r \in R$ such that $rxy = 0$, $rx \neq 0$ and $ry \neq 0$. Thus $ry \in ann_R(x)$. Since $ann_R(x)$ is a prime ideal of R and $y \notin ann_R(x)$, we have $r \in ann_R(x)$, a contradiction. So $N_{\Gamma(R)}(x) = N_{AG(R)}(x)$.

(2) Assume that $x \in Nil(R)^*$ and $N_{\Gamma(R)}(x) = N_{AG(R)}(x)$. Then by [5, Theorem 3.10], $Nil(R)^* \subseteq N_{AG(R)}(x)$. If $x^2 \neq 0$, then $x^3 = 0$ and $x(x + x^2) = 0$. Thus $x^2 + x^3 = x^2 = 0$, which is impossible. So $x^2 = 0$. Now, we show that $ann_R(x)$ is a prime ideal of R. To prove this, let $ab \in ann_R(x)$, $a \notin ann_R(x)$ and $b \notin ann_R(x)$. Thus $x \neq a$ and $x \neq b$. Since $xab = 0$ and $ax \neq 0$ and $bx \neq 0$, we have $a, b \in Z(R)^*$. If $ab \neq 0$, then x is adjacent to a in $AG(R)$ which contradicts the assumption $N_{\Gamma(R)}(x) = N_{AG(R)}(x)$. Hence $ab = 0$ and so $b \in ann_R(a) \setminus ann_R(x)$. Since $x \in ann_R(x) \setminus ann_R(a)$, by Part (4) of [5, Lemma 2.1], $x - a$ is an edge of $AG(R)$, a contradiction. □

In light of Lemma 2.3, we have the following corollary.

Corollary 2.4 *Let R be a ring. If $\Gamma(R) = AG(R)$, then for every $x \in Nil(R)^*$, $ann_R(x) \in Ass(R)$.*

Let R be a ring and $\Sigma = \{ann_R(x) \mid 0 \neq x \in R\}$. Recall that the set of all maximal elements of Σ (under $\subseteq$) is a subset of $Ass(R)$. We set $\Sigma^* = \Sigma \setminus \{(0)\}$. Now, we are ready to present the following result.

Theorem 2.5 *Let R be a ring such that for every edge of $AG(R)$, say $x - y$, either $ann_R(x) \in Ass(R)$ or $ann_R(y) \in Ass(R)$. Then $\Gamma(R) = AG(R)$.*

Proof. It follows from Part (1) of Lemma 2.3. □

Corollary 2.6 *Let R be a ring. If $\Sigma^* = Ass(R)$, then $\Gamma(R) = AG(R)$.*

Proposition 2.7 *Let R be a non-reduced ring such that $Z(R)$ is not an ideal of R. Then $\Sigma^* \neq Ass(R)$.*

Proof. The result follows from Corollary 2.6 and [5, Theorem 3.15]. □

If R is a reduced ring, then the converse of Theorem 2.6 is also true (see [5, Theorem 3.6]). The annihilator graph of a reduced ring has been studied extensively in [5] and it has been characterized all reduced rings R that $\Gamma(R) = AG(R)$. So in the rest of this paper, almost everywhere, we assume that R is a non-reduced ring. We are interested in characterize non-reduced rings whose annihilator and zero-divisor graphs are identical. Therefore, the following question is posed:

Question 2.8 *Let R be a non-reduced ring and $x-y$ be an edge of $AG(R)$. If $\Gamma(R) = AG(R)$, then is it true either $ann_R(x) \in Ass(R)$ or $ann_R(y) \in Ass(R)$?*

In what follow, we provide some examples for which the Question 2.8 has an affirmative answer.

Example 2.9 (1) [5, Example 2.7] Let $R = \mathbb{Z}_8$. Then $2-6$ is an edge of $AG(R)$ and $ann_R(2) = ann_R(6) \notin Ass(R)$. On the other hand, $\Gamma(R) \neq AG(R)$.

(2) [5, Example 2.8] Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and let $a = (0,1)$ and $b = (1,2)$. Then $a-b$ is an edge of $AG(R)$, $ann_R(a) \notin Ass(R)$ and $ann_R(b) \notin Ass(R)$. Also, it is known that $\Gamma(R) \neq AG(R)$ and so the Question 2.8 has an affirmative answer.

(3) [5, Example 3.22] Let $D = \mathbb{Z}_2[X,Y,W]$, $I = (X^2,Y^2,XY,XW)D$ be an ideal of D, and let $R = D/I$. It is not hard to check that if $a-b$ is an edge of $AG(R)$, then either $ann_R(a) \in Ass(R)$ or $ann_R(b) \in Ass(R)$. Since $\Gamma(R) = AG(R)$, the Question 2.8 has an affirmative answer.

(4) [5, Example 3.23] Let $D = \mathbb{Z}_2[X,Y,W]$, $I = (X^2,Y^2,XY,XW, YW^3)D$ be an ideal of D, and let $R = D/I$. Then let $x = X+I$, $y = Y+I$ and $w = W+I$ be elements of R. We have $w-w^2$ is an edge of $AG(R)$, $ann_R(w) \notin Ass(R)$ and $ann_R(w^2) \notin Ass(R)$. Moreover, it is known that $\Gamma(R) \neq AG(R)$.

In the following theorem, for a non-reduced ring R, we provide conditions under which $\Gamma(R) = AG(R)$.

Theorem 2.10 *Let R be a non-reduced ring. Then the following statements are equivalent*:

(1) $\Gamma(R) = AG(R) = K^n \vee \overline{K}^m$, *where* $n = |Nil(R)^*|$ *and* $m = |Z(R) \setminus Nil(R)|$;
(2) $ann_R(Z(R))$ *is a prime ideal of* R;
(3) $\Sigma^* = Ass(R)$ *and* $|\Sigma^*| \leq 2$.

Proof. (1) $\Rightarrow$ (2) With no loss of generality, one may assume that $m \neq 0$. Since $\Gamma(R) = K^n \vee \overline{K}^m$, every vertex of K^n is adjacent to all other vertices of $\Gamma(R)$ and there is no edge between vertices of $\overline{K}^m$. Thus $ann_R(Z(R)) = V(K^n) \cup \{0\}$, $xy \neq 0$ and $ann_R(x) = ann_R(y) = ann_R(Z(R))$, for every $x, y \in V(\overline{K}^m)$. Now, we show that $ann_R(Z(R))$ is a prime ideal of R. To see this, let $xy \in ann_R(Z(R))$, $x \notin ann_R(Z(R))$ and $y \notin ann_R(Z(R))$. Thus $x \neq y$, and hence $Z(R) = ann_R(xy) \neq ann_R(x) \cup ann_R(y) = ann_R(Z(R))$. Therefore, $x - y$ is an edge of $AG(R)$, a contradiction. So, $ann_R(Z(R))$ is a prime ideal of R.

(2) $\Rightarrow$ (1) Assume that $ann_R(Z(R))$ is a prime ideal of R. Thus $xy = 0$, for all $x, y \in ann_R(Z(R))$, and $xy \neq 0$, for all $x, y \in Z(R) \setminus ann_R(Z(R))$. Now, it is not hard to see that $\Gamma(R)[ann_R(Z(R))^*]$ and $\Gamma(R)[Z(R) \setminus ann_R(Z(R))]$ are two subgraph of $\Gamma(R)$ such that $\Gamma(R)[ann_R(Z(R))^*]$ is complete, $\Gamma(R)[Z(R) \setminus ann_R(Z(R))]$ is null and $\Gamma(R) = \Gamma(R)[ann_R(Z(R))^*] \vee \Gamma(R)[Z(R) \setminus ann_R(Z(R))]$. To complete the proof, we have only to show that $\Gamma(R) = AG(R)$. Let x, y be non-adjacent vertices of $\Gamma(R)$. Then $x, y, xy \in Z(R) \setminus ann_R(Z(R))$. Since $ann_R(Z(R))$ is a prime ideal of R, we conclude that $\mathrm{ann}(x) = \mathrm{ann}(y) = ann_R(xy) = ann_R(Z(R))$, i.e., x, y are not adjacent in $AG(R)$, as desired.

(2) $\Rightarrow$ (3) Since $ann_R(Z(R))$ is a prime ideal of R, for every $x \in Z(R)^*$, either $ann_R(x) = ann_R(Z(R))$ or $ann_R(x) = Z(R)$. Hence $\Sigma^* = \{ann_R(Z(R)), Z(R)\}$ and so $\Sigma^* = Ass(R)$ and $|\Sigma^*| \leq 2$.

(3) $\Rightarrow$ (2) Let $ann_R(x)$ and $ann_R(y)$ be elements of Σ^*. Since $\Sigma^* = Ass(R)$, by Corollary 2.6, $\Gamma(R) = AG(R)$ and hence it follows from [5, Theorem 3.15] that $Z(R)$ is an ideal of R. This, together with the fact $Z(R) = ann_R(x) \cup ann_R(y)$ imply that either $ann_R(x) \subseteq ann_R(y)$ or $ann_R(y) \subseteq ann_R(x)$. With no loss of generality, suppose that $ann_R(x) \subseteq ann_R(y)$.

Thus $Z(R) = ann_R(y)$. Now, we have only to show that $ann_R(x) = ann_R(Z(R))$. We consider the following two cases:

Case 1. Let $a, b \in ann_R(x)$. Then either $ann_R(a) = ann_R(x)$ or $ann_R(a) = Z(R)$. Thus $ab = 0$.

Case 2. Let $a \in ann_R(x)$ and $b \notin ann_R(x)$. Then it is easily seen that $ann_R(b) = ann_R(x)$ and so $ab = 0$.

The proof is complete. □

Theorem 2.11 *Let R be a non-reduced ring and $|\Sigma^*| \leq 2$. If $\Gamma(R) = AG(R)$, then $\Sigma^* = Ass(R)$.*

Proof. Assume that $x, y \in Z(R)^*$ and $\Sigma^* = \{ann_R(x), ann_R(y)\}$. So $Z(R) = ann_R(x) \cup ann_R(y)$. Since $\Gamma(R) = AG(R)$, by [5, Theorem 3.15], $Z(R)$ is an ideal of R and so either $ann_R(x) \subseteq ann_R(y)$ or $ann_R(y) \subseteq ann_R(x)$. With no loss of generality, suppose that $ann_R(x) \subseteq ann_R(y)$. Since $Z(R) = ann_R(y)$, we have only to show that $ann_R(x)$ is a prime ideal of R. Let $ab \in ann_R(x)$, $a \notin ann_R(x)$ and $b \notin ann_R(x)$. If $ab \neq 0$, then $x - a$ is an edge of $AG(R)$, by definition, and thus $xa = 0$ (since $\Gamma(R) = AG(R)$), which is impossible. So $a \in ann_R(b)$. On the other hand, we know that $ann_R(b) = ann_R(x)$ or $ann_R(b) = ann_R(y)$. If $ann_R(b) = ann_R(x)$, then $ax = 0$, a contradiction. If $ann_R(b) = ann_R(y)$, then it is easily seen that $bx = 0$, again we get a contradiction. Hence $\Sigma^* = Ass(R)$. □

To characterize non-reduced rings whose annihilator graphs are star, the following lemma is needed.

Lemma 2.12 *Let R be a non-reduced ring and $x \in Z(R) \setminus Nil(R)$. If $x^n = x^{n+1}$, where n is a positive integer, then $gr(AG(R)) \leq 4$.*

Proof. Since $x^n = x^{n+1}$ for some $x \in Z(R) \setminus Nil(R)$, there exists an element $e \in Z(R)^*$ such that $e = e^2$. So by Brauer's Lemma (see [6, 10.22]), $R \cong Re \times R(1-e)$. Hence we may assume that $R \cong R_1 \times R_2$. With no loss of generality, one may assume that there exists $a \in Nil(R_2)^*$ and $a^2 = 0$. Therefore, $(1, 0)(0, a) = (1, 0)(0, 1) = (0, 0)$ and $(1, a)(0, 1) = (0, a)$. Thus $ann_R((0, a)) \neq ann_R((1, a)) \cup ann_R((0, a))$. So $(1, 0) - (0, 1) - (1, a) - (0, a) - (1, 0)$ is a cycle of length four. □

Theorem 2.13 *Let R be a non-reduced ring such that R is not ring-isomorphic to $\mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $B = \mathbb{Z}_2[X]/(X^2)$. Then the following statements are equivalent:*

(1) *$gr(AG(R)) = \infty$;*
(2) *$AG(R)$ is a star graph;*
(3) *$AG(R)$ is a bipartite graph;*
(4) *$AG(R)$ is a complete bipartite graph;*
(5) *$\Sigma^* = Ass(R) = \{ann_R(x), ann_R(y)\}$, for some $x, y \in Z(R)$. Furthermore, if $ann_R(x) = ann_R(y)$, then $|ann_R(x)| = |Z(R)| = 3$. And if $ann_R(x) \neq ann_R(y)$, then $\Sigma^* = \{Z(R), ann_R(Z(R))\}$ and $|ann_R(Z(R))^*| = 1$.*

Proof. (2) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (4) follows from [5, Theorem 2.2].

(5) $\Rightarrow$ (1) If $|Z(R)| = 3$, then obviously $AG(R) = K^2$. Moreover, if $|ann_R(Z(R))^*| = 1$, then the result follows from Theorem 2.10.

(1) $\Rightarrow$ (2). By [5, Theorem 3.10], $|Nil(R)^*| \leq 2$. First assume that $|Nil(R)^*| = 2$ and $Nil(R)^* = \{a, b\}$, for some elements $a, b \in R$. It is easy to see that $a = -b$, and thus $ann_R(a) = ann_R(b)$. Since $gr(AG(R)) = \infty$, by Part (2) of Lemma 2.1, $AG(R) = K^{1,1}$. Now, assume that $Nil(R)^* = \{a\}$, for some $a \in R$. Thus $Nil(R)$ is a minimal ideal of R and so $ann_R(a)$ is a maximal ideal. We show that $Z(R) = ann_R(a)$. Assume to the contrary, there exists $x \in Z(R) \setminus ann_R(a)$. Since $ann_R(a)$ is a prime ideal, $ann_R(x) \subseteq ann_R(a)$. Let $y \in ann_R(x)$ (since $xa \neq 0$ so $y \neq a$). If $y^n = y^{n+1}$, for some positive integer n, then Lemma 2.12 implies that $gr(AG(R)) \leq 4$, which is a contradiction. Also, if $y^n \neq y^{n+1}$, then $x - y^n - a - y^{n+1} - x$ is a cycle of length four, a contradiction. So $Z(R) = ann_R(a)$ and hence a is adjacent to all other vertices. This, together with $gr(AG(R)) = \infty$, implies that $AG(R)$ is a star graph.

(4) $\Rightarrow$ (5). Let $AG(R)$ be complete bipartite. By [5, Corollary 2.10], $\Gamma(R) = AG(R)$. It follows from the proof of (1) $\Rightarrow$ (2) that $|Nil(R)^*| \leq 2$. If $|Nil(R)^*| = 2$, then it is easy to see that $\Sigma^* = Ass(R) = \{ann_R(x)\}$ and $|ann_R(x)| = |Z(R)| = 3$. So, assume that $|Nil(R)^*| = 1$. For the unique element $a \in Nil(R)^*$, by Theorem 2.3 (2), $ann_R(a)$ is a prime ideal of R. Now, let $x \in Z(R) \setminus ann_R(a)$. Since $AG(R)$ is a complete bipartite graph and $\Gamma(R) = AG(R)$, we infer that $ann_R(a) = ann_R(x)$. Since $|Nil(R)^*| = 1$ and $xa \neq 0$, we conclude that $xa = a$ and so $x - 1 \in ann_R(a) = ann_R(x)$. Thus $x = x^2$ and hence by Brauer's Lemma (see [6, 10.22]), R is a decomposable ring. This contradicts [5, Theorem 3.15]. Therefore, $Z(R) = ann_R(a)$. Now, it is not hard to see that $ann_R(x) = ann_R(Z(R))$, for every $a \neq x \in Z(R)^*$, and $|\Sigma^*| = 2$. Thus by

Theorem 2.11, $\Sigma^* = Ass(R) = \{Z(R), ann_R(Z(R))\}$. □

Theorem 2.13, [5, Theorem 3.6] and [5, Theorem 3.16] lead to the following corollaries.

Corollary 2.14 *Let R be a ring. Then $AG(R)$ is a complete bipartite graph if and only if one of the following statements holds*:

(1) $Nil(R) = (0)$ *and* $|Min(R)| = 2$;
(2) $Nil(R) \neq (0)$ *and either* $AG(R) = K^{1,n}$, *or* $AG(R) = K^{2,3}$, *where* $1 \leq n \leq \infty$.

Theorem 2.13 provides an alternate proof for [5, Theorem 3.18].

Corollary 2.15 ([5, Theorem 3.18]) *Let R be a non-reduced commutative ring with $|Z(R)^*| \geq 2$. Then the following statements are equivalent*:

(1) $AG(R)$ *is a star graph*;
(2) $gr(AG(R)) = \infty$;
(3) $AG(R) = \Gamma(R)$ *and* $gr(\Gamma(R)) = \infty$;
(4) $Nil(R)$ *is a prime ideal of R and either* $Z(R) = Nil(R) = \{0, -w, w\}$ $(w \neq -w)$ *for some nonzero* $w \in R$ *or* $Z(R) \neq Nil(R)$ *and* $Nil(R) = \{0, w\}$ *for some non-zero* $w \in R$ (*and hence* $wZ(R) = \{0\}$);
(5) *Either* $AG(R) = K^{1,1}$ *or* $AG(R) = K^{1,\infty}$;
(6) *Either* $\Gamma(R) = K^{1,1}$ *or* $\Gamma(R) = K^{1,\infty}$.

Proof. (1) ⇒ (2) and (1) ⇒ (3) are clear and (2) ⇒ (1) and (3) ⇒ (1) follow from Theorem 2.13.

(1) ⇔ (4) It is easy to see that $AG(R)$ is a star graph if and only if either $\Gamma(R) = AG(R) = K^2 \vee \overline{K}^0$ or $\Gamma(R) = AG(R) = K^1 \vee \overline{K}^\infty$. Now, by Theorem 2.10, $AG(R)$ is a star graph if and only if $Nil(R)$ is a prime ideal of R and one of the following holds:

(i) $Z(R) = Nil(R) = \{0, -w, w\}$ $(w \neq -w)$, for some non-zero $w \in R$.
(ii) $Z(R) \neq Nil(R)$ and $Nil(R) = \{0, w\}$, for some non-zero $w \in R$ (put $|Nil(R)^*| = n$ in Theorem 2.10).

(5) ⇒ (1) is obvious and (1) ⇒ (5) follows from the proof of Theorem 2.13.

(1) ⇒ (6) is easily obtained by Theorem 2.13 and its proof.

(6) ⇒ (1) is obvious by [5, Theorem 3.17]. □

Let x be a vertex of $AG(R)$ which is adjacent to every other vertex. In the following theorem, we provide conditions under which x is adjacent to every other vertex in $\Gamma(R)$.

Theorem 2.16 *Let R be a ring and $\Sigma = \{ann_R(x) \,|\, 0 \neq x \in R\}$. Then the following statements are equivalent:*

(1) *x is adjacent to every other vertex in $\Gamma(R)$;*
(2) *$ann_R(x)$ is a maximal element of Σ and x is adjacent to every other vertex in $AG(R)$.*

Proof. (1) ⇒ (2) Suppose that x is adjacent to every other vertex in $\Gamma(R)$. Then by Part (2) of [5, Lemma 2.1], x is adjacent to every other vertex in $AG(R)$. Also, by [3, Theorem 2.5], $ann_R(x)$ is a maximal element of Σ.

(2) ⇒ (1) Suppose that $ann_R(x)$ is a maximal element of Σ and x is adjacent to every other vertex in $AG(R)$. To complete the proof, we consider the following two cases:

Case 1. Let $x \in ann_R(x)$. We claim that $Z(R) = ann_R(x)$. Assume to the contrary, $y \in Z(R) \setminus ann_R(x)$. So $xy \neq 0$ and since $ann_R(x)$ is a maximal element of Σ, we conclude that $ann_R(xy) = ann_R(y) \cup ann_R(x)$, a contradiction. Hence $Z(R) = ann_R(x)$ and so the claim is proved. Thus x is adjacent to every other vertex in $\Gamma(R)$.

Case 2. Let $x \notin ann_R(x)$. Since $ann_R(x)$ is a prime ideal of R, $x^n \neq 0$, for every positive integer n. If $x \neq x^2$, then $ann_R(x) \subsetneqq ann_R(x^3)$, a contradiction. Thus $x = x^2$ and so $R \cong Rx \times R(1-x)$. Hence we may assume that $R \cong R_1 \times R_2$ where $(1,0)$ adjacent to every other vertex. Now, we show that $R_1 \cong \mathbb{Z}_2$ and R_2 is an integral domain. To see this, let $a \in R_1 \setminus \{0,1\}$. Obviously, $(1,0)(a,0) = (a,0)$, i.e., $(1,0)$ is not adjacent to $(a,0)$, a contradiction. Also, if $Z(R_2) \neq 0$, then for any $x \in Z(R_2)^*$, $(1,0)(1,x) = (1,0)$. That means $(1,0)$ is not adjacent to $(1,x)$ which is impossible. Thus $R_1 \cong \mathbb{Z}_2$ and R_2 is an integral domain. Now, by [3, Theorem 2.5], x is adjacent to every other vertex in $\Gamma(R)$. □

Proposition 2.17 *Let R be a non-reduced ring and for every $x \in Z(R)^*$, set $\Sigma_x = \{ann_R(x^i)\}$, where $i \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *$AG(R)$ is a complete graph and $|\Sigma_x| < \infty$, for every $x \in Z(R)^*$;*
(2) *$Z(R) = Nil(R)$.*

Proof. (2) $\Rightarrow$ (1) is easily obtained by [5, Theorem 3.10].

(1) $\Rightarrow$ (2) Suppose that $x \in Z(R) \setminus Nil(R)$. If $x^n = x^{n+1}$, where n is a positive integer, then by the proof of Lemma 2.12, $AG(R)$ is not a complete graph, a contradiction. So $x^i \neq x^{i+1}$, for every $i \in \mathbb{N}$. Since $|\Sigma_x| < \infty$, $ann_R(x^i) = ann_R(x^{i+1})$, for some $i \in \mathbb{N}$. This implies that $x - x^i$ is not an edge of $AG(R)$, unless $xx^i = x^{i+1} = 0$, a contradiction. Thus $Z(R) = Nil(R)$. □

In view of above proposition, we have the following corollary.

Corollary 2.18 *Let R be a non-reduced ring such that $Z(R) \neq Nil(R)$, and let $AG(R)$ be a complete graph. Then:*

(1) $\Gamma(R) \neq AG(R)$;
(2) *R is not a Noetherian ring.*

Proof. (1) By Theorem 2.17, there is an element $x \in Z(R)^*$ such that $|\Sigma_x| = \infty$, and so $x \notin Nil(R)$. If $x^n = x^{n+1}$, where n is a positive integer, then by proof of Lemma 2.12, $AG(R)$ is not a complete graph, a contradiction. So $x^i \neq x^{i+1}$, for every $i \in \mathbb{N}$. Now, $x - x^i$ is not an edge of $\Gamma(R)$. Hence $\Gamma(R) \neq AG(R)$

(2) Suppose that $x \in Z(R) \setminus Nil(R)$. Since $AG(R)$ is a complete graph, $|\Sigma_x| = \infty$, and so the chain $ann_R(x) \subseteq ann_R(x^2) \subseteq \cdots \subseteq ann_R(x^i) \subseteq \cdots$ will not stabilize. Thus R is not a Noetherian ring. □

We close this paper with the following example which is devoted to the study of relation between two graphs $\Gamma(\mathbb{Z}_n)$ and $AG(\mathbb{Z}_n)$.

Example 2.19 Let $R = \mathbb{Z}_n$. If $\mathbb{Z}_n$ is not local, then $\Gamma(\mathbb{Z}_n) = AG(\mathbb{Z}_n)$ if and only if $n = pq$ for distinct prime numbers p, q. Moreover in this case $\Gamma(\mathbb{Z}_n) = K^{p-1,q-1}$. If $\mathbb{Z}_n$ is local, then $\Gamma(\mathbb{Z}_n) = AG(\mathbb{Z}_n)$ if and only if $n = p^2$, where p is a prime number. Moreover in this case $\Gamma(\mathbb{Z}_n) = K^{p-1}$. For instance it is easy to see that $\Gamma(\mathbb{Z}_{10}) = K^{1,4} = AG(\mathbb{Z}_{10})$. Also, for local rings $\mathbb{Z}_{25}$ and $\mathbb{Z}_8$, we can easily check that $\Gamma(\mathbb{Z}_{25}) = K^4 = AG(\mathbb{Z}_{25})$, but $\Gamma(\mathbb{Z}_8) = K^{1,2} \neq K^3 = AG(\mathbb{Z}_8)$.

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