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Differential systems associated with partial differential equations of several unknown functions

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Abstract. From Realization Lemma established by N. Tanaka, differential systems may be regarded as systems of first order differential equations. We characterize the geometric structure of systems of second order partial differential equations of several unknown functions in terms of differential systems and seek a system of equations the Lie algebra of all infinitesimal automorphisms of which is simple.

Key words: differential system, partial differential equation, several unknown functions, characterization.

1. Introduction

We will treat with differential systems associated with systems of second order partial differential equations of several unknown functions. A differential system D on a manifold M is a subbundle of the tangent bundle TM of M. Here, we assume that equations never contain an equation of first order (Condition (**R.0**) in Section 2.4). According to Realization Lemma, which is established by N. Tanaka ([19]), any differential system corresponds to a system of differential equations of first order (Section 2.2). Regarded as a system of equations of first order, a system of partial differential equations of second order is expected to be characterized as a structure of differential systems with some conditions. K. Yamaguchi provided, in terms of differential systems, a geometric formulation of systems of second and higher order partial differential equations of one unknown function satisfying "submersion condition" such as Condition (**R.0**), where this geometric structure is called a PD-manifold $(R; D^1, D^2)$ ([19]). In contrast, we formulate the structure of systems of second order equations of several unknown functions satisfying Condition (**R.0**) and characterize as a triplet $(R; D^1, D^2)$ of differential systems D^1 and D^2 on a manifold R, called a PD-manifold, satisfying the conditions from (**R.1**) to (**R.6**) in Section 2.4. There exist subbundles F of D^1 satisfying $\partial F \subset D^1$ in each

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case. As compared with one unknown function, such system F is uniquely determined for each system of equations of several unknown functions, and moreover F is completely integrable if the number of unknown functions are more than three. We give an example of PD-manifolds of finite type and show that it is associated with a pseudo-product GLA of irreducible type $(\mathfrak{l}, S)$, which was introduced by Y. Se-ashi ([13]). Finally, from the viewpoint of parabolic geometry ([17], [26]), we seek a PD-manifold the Lie algebra of all infinitesimal automorphisms of which is a simple graded Lie algebra of type (X_l, Δ_1) over $\mathbb{C}$. Such simple graded Lie algebras $\mathfrak{g} = \mathfrak{m} \oplus \bigoplus_{p=0}^{\mu} \mathfrak{g}_p$ of type (X_l, Δ_1) must satisfy that $\mu = 3$ and $\dim \mathfrak{g}_{-3} \geq 2$, where $\mathfrak{m} = \bigoplus_{p=-\mu}^{-1} \mathfrak{g}_p$ is the negative part of $\mathfrak{g}$. From Dynkin diagram, we see that the following types satisfy this necessary condition: $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$ $(1 \leq i < j < k \leq l,\ (i, k) \neq (1, l))$, $(B_l, \{\alpha_1, \alpha_i\})$ $(3 \leq i \leq l)$, $(C_l, \{\alpha_i, \alpha_l\})$ $(2 \leq i \leq l-1)$, $(D_l, \{\alpha_1, \alpha_i\})$ $(3 \leq i \leq l-2)$, $(D_l, \{\alpha_i, \alpha_l\})$ $(3 \leq i \leq l-2)$, $(D_l, \{\alpha_1, \alpha_{l-1}, \alpha_l\})$, $(E_6, \{\alpha_4\})$, $(E_6, \{\alpha_1, \alpha_3\})$, $(E_6, \{\alpha_1, \alpha_5\})$, $(E_7, \{\alpha_3\})$, $(E_7, \{\alpha_5\})$, $(E_7, \{\alpha_2, \alpha_7\})$, $(E_7, \{\alpha_6, \alpha_7\})$, $(E_8, \{\alpha_2\})$, $(E_8, \{\alpha_7\})$, $(F_4, \{\alpha_2\})$ and $(G_2, \{\alpha_1\})$ up to Dynkin diagram automorphism. For $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$ type, the differential system of type $\mathfrak{m}$ corresponds to a system of second order equations of $i(l-k+1)$ (≥ 2) unknown functions that does not satisfy Condition (**R.0**), namely contains a system of equations of only first order. We construct a model equation of this type in Section 2.6.1. The simple Lie algebras of type $(C_l, \{\alpha_i, \alpha_l\})$, $(D_l, \{\alpha_i, \alpha_l\})$, $(E_6, \{\alpha_1, \alpha_3\})$ and $(E_7, \{\alpha_6, \alpha_7\})$ have pseudo-product structures ([26]). For the simple Lie algebras of type $(B_l, \{\alpha_1, \alpha_i\})$, $(D_l, \{\alpha_1, \alpha_i\})$, $(D_l, \{\alpha_1, \alpha_{l-1}, \alpha_l\})$, $(E_6, \{\alpha_4\})$, $(E_6, \{\alpha_1, \alpha_5\})$, $(E_7, \{\alpha_3\})$, $(E_7, \{\alpha_5\})$, $(E_7, \{\alpha_2, \alpha_7\})$, $(E_8, \{\alpha_2\})$, $(E_8, \{\alpha_7\})$, $(F_4, \{\alpha_2\})$ and $(G_2, \{\alpha_1\})$, some of which belong to G_2-geometry ([22]), differential systems of type $\mathfrak{m}$ correspond to systems of second order equations of several unknown functions that have no solutions (Proposition 2.10). While there are many examples of PD-manifolds of simple graded Lie algebra of type (X_l, Δ_1) among systems of equations of one unknown function, there are not among systems of equations of several unknown function (Theorem 2.7). As can be seen in the uniqueness of the differential system F, that appears to be the cause of the number of unknown functions (Remark 2.5 in Section 2.4). Regardless of Condition (**R.0**), a connection with G_2-geometry of PD-manifolds of one unknown function and the geometry of $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$ type deserve further investigation.

Now let us describe the contents of each sections. In Section 2 we recall the definitions of differential systems and their covariant systems, and the jet space $(J(M,n),C)$ of first order. Realization Lemma plays an important role in the characterization and identification of systems of partial differential equations, which is stated in Section 2.2. In addition we recall the jet space $J^2(M,n)$ of second order, symbol algebras of differential systems, the prolongation of symbol algebras and simple graded Lie algebras in Section 2.3. In Section 2.4 we clarify the conditions that differential systems associated with systems of second order partial differential equations of several unknown functions with Condition (**R.0**) should satisfy, namely the condition from (**R.1**) to (**R.6**). Conversely, these conditions characterize such equations in terms of differential systems (Theorem 2.4). Section 2.5 shows an example of a PD-manifold the prolongation of the symbol algebra of which is a pseudo-product GLA of irreducible type. In Section 2.6, by the description of the gradation of each simple Lie algebra of type (X_l, Δ_1) satisfying the necessary condition $\mu = 3$ and $\dim \mathfrak{g}_{-3} \geq 2$, which are listed above, we proof Theorem 2.7.

Throughout this paper, we work in C^∞-category or complex analytic category.

2. Preliminaries

In this section we recall the definitions and notations of various differential systems and Realization Lemma, which are used in the whole of this paper.

2.1. Differential systems and various systems

A *differential system* D (or (M,D)) is a subbundle of the tangent bundle TM of a (real or complex) manifold M. A differential system D is locally defined by linearly independent 1-forms $\varpi^1, \ldots, \varpi^r$ as follows:

$$D = \left\{\varpi^1 = \cdots = \varpi^r = 0\right\},$$

where r is the codimension of D. An *integral element* v of the differential system D at a point $x \in M$ is a subspace of T_xM such that $\varpi^a|_v = 0$ and $d\varpi^a|_v = 0$ for all $1 \leq a \leq r$. An *integral manifold* of the differential system D is a submanifold $\iota : N \longrightarrow M$ such that $\iota^*\varpi^a = 0$ for all $1 \leq a \leq r$. A function f on M is a *first integral* of D if $df \equiv 0 \pmod{D^\perp}$, where $D^\perp$ is the annihilater subbundle of T^*M defined by

$$D^{\perp}(x) = \{\omega \in T_x^*M \mid \omega(X) = 0 \text{ for } X \in T_xM\} \quad \text{for } x \in M.$$

The *k-th derived system* $\partial^k D$ is defined inductively as follows: If $\partial^{k-1}D$ is a differential system, then

$$\partial^k \mathcal{D} = \partial^{k-1}\mathcal{D} + [\partial^{k-1}\mathcal{D}, \partial^{k-1}\mathcal{D}]$$

where $\partial^k \mathcal{D}$ is the space of sections of $\partial^k D$ and $[\,,\,]$ is Lie bracket for vector fields, and we put $\partial^0 D = D$ for convention. Precisely, $\partial^k D$ is defined in terms of sheaves (see [19]). When ∂D coincides with D, D is said to be *completely integrable*.

On the other hand, the *k-th weak derived system* $\partial^{(k)}D$ of D is defined inductively by

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

where $\partial^{(0)}D = D$ and $\partial^{(k)}\mathcal{D}$ is the space of sections of $\partial^{(k)}D$. Let $D^{-(k+1)} = \partial^{(k)}D$ for $k \geq 0$. Note that $D^{-2} = \partial^{(1)}D = \partial D$. A differential system (M, D) is *regular* if D^{-k} is a differential system on M for all $k \geq 2$. For a regular differential system (M, D), it is known ([16, Proposition 1.1], [22, Section 2.4]) that

1. There exists a unique integer $\mu > 0$ such that

$$D = D^{-1} \subsetneq D^{-2} \subsetneq \cdots \subsetneq D^{-\mu+1} \subsetneq D^{-\mu} = \cdots = D^k$$

   for all $k \geq \mu$,
2. $[\mathcal{D}^{-p}, \mathcal{D}^{-q}] \subset \mathcal{D}^{-(p+q)}$ for all $p, q > 0$,

where $\mathcal{D}^{-p}$ is the space of sections of D^{-p}. Note that $D^{-\mu}$ is the smallest completely integrable differential system that contains D.

The *Cauchy characteristic system* $\mathrm{Ch}(D)$ of D is defined by

$$\mathrm{Ch}(D)(x) = \left\{X \in D(x) \mid X \lrcorner\, d\varpi^a \equiv 0 \pmod{\varpi_x^1, \ldots, \varpi_x^r} \text{ for } 1 \leq a \leq r\right\}$$

at each point $x \in R$. If $\mathrm{Ch}(D)$ is a differential system, it is a completely integrable system contained by D. Let $p : R \longrightarrow M$ be a map between manifolds R and M and assume p is of constant rank. Let C be a differential system on M. Differential systems $p_*^{-1}(C)$ and $\operatorname{Ker} p_*$ are defined by

$$p_*^{-1}(C)(x) = \{X \in T_x R \mid p_*(X) \in C(p(x))\},$$
$$\operatorname{Ker} p_*(x) = \{X \in T_x R \mid p_*(X) = 0\},$$

for $x \in R$. Note that $\operatorname{Ker} p_*$ is completely integrable.

2.2. Jet space $(J(M,n), C)$ of first order and Realization Lemma

Let M be a (real or complex) manifold of dimension $m+n$. Let $J(M,n)$ be the Grassmann bundle over M. Namely each fiber $J(M,n)_x$ over $x \in M$ is the Grassmannian $Gr(T_x M, n)$ consisting of all n-dimensional subspaces of $T_x M$:

$$J(M,n) = \bigcup_{x \in M} J(M,n)_x \xrightarrow{\Pi} M$$

where Π is the canonical projection of $J(M,n)$ onto M. *The canonical system* C on $J(M,n)$, which is a differential system of codimension m, is defined by

$$C(u) = \Pi_*^{-1}(u) \quad \text{for } u \in J(M,n),$$

where the right hand side means the inverse image of the n-dimensional subspace u of $T_{\Pi(u)}M$ under the differential of Π at u.

Next we will give a canonical coordinate system $(x^i, z^a, p_i^a\ (1 \leq a \leq m, 1 \leq i \leq n))$ of $J(M,n)$. Let us fix a point u_o of $J(M,n)$. Let $(x^i, z^a\ (1 \leq a \leq m,\ 1 \leq i \leq n))$ be a coordinate system on a neighborhood U of $\Pi(u_o)$ such that $dx^1, \ldots, dx^n$ are linearly independent on u_o. Let $\hat{U}$ be the set of all elements $u \in \Pi^{-1}(U)$ such that $dx^1 \wedge \cdots \wedge dx^n\,|_u \neq 0$, which is a neighborhood of u_o. By taking functions p_i^a on $\hat{U}$ for $1 \leq a \leq m$ and $1 \leq i \leq n$ so that $dz^b|_u - \sum_i p_i^b(u)\, dx^i|_u = 0$ for $u \in \hat{U}$ and $1 \leq b \leq m$, we have achieved an inhomogeneous coordinate system $(x^i, z^a, p_i^a\ (1 \leq a \leq m,\ 1 \leq i \leq n))$ on $\hat{U}$. It follows that the canonical system C restricted on $\hat{U}$ is described by the 1-forms

$$\varpi^a = dz^a - \sum_{i=1}^{n} p_i^a dx^i \quad \text{for } 1 \le a \le m \tag{2.2.1}$$

and the Cauchy characteristic system of C is trivial, i.e. $\mathrm{Ch}(C) = \{0\}$.

The theory of submanifolds of $(J(M,n), C)$ is regarded as the geometry of systems of first order differential equations. Indeed, for a system of (regular) first order differential equations $F^\alpha(x^i, z^a, \partial z^a/\partial x^i) = 0$ $(1 \le \alpha \le N)$, where x^i and z^a are independent variables and unknown functions (dependent variables) respectively, we have the submanifold R of $J(\mathbb{K}^{m+n}, n)$ defined by $F^\alpha(x^i, z^a, p_i^a) = 0$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Conversely, a submanifold R of $J(M,n)$ is locally defined by some functions $F^\alpha(x^i, z^a, p_i^a)$ with a canonical coordinate system (x^i, z^a, p_i^a).

Realization Lemma ([19]) *Let R and M be manifolds and a map $p : R \longrightarrow M$. Let D be a differential system on R. Assume that p is of constant rank and $F = \mathrm{Ker}\, p_*$ is a subbundle of D of codimension n. Then there exists a unique map $\psi : R \longrightarrow J(M,n)$ satisfying $p = \Pi \circ \psi$ and $D = \psi_*^{-1}(C)$. Indeed ψ is defined by*

$$\psi(x) = p_*(D(x)) \quad \textit{for } x \in R \tag{2.2.2}$$

and satisfies

$$\mathrm{Ker}\, \psi_*(x) = F(x) \cap \mathrm{Ch}(D)(x).$$

Here, the right hand side of (2.2.2) *means the image of the differential p_* of the subspace $D(x)$ of T_xR, which is considered as a point of $J(M,n)$.*

Note that Realization Lemma holds in complex analytic category with suitable modification.

This lemma also says "any differential system is considered as a system of differential equations of first order." In fact, for a given differential system (M, D), let us choose p as the identity map $\mathrm{id} : M \longrightarrow M$. Then $\psi : M \longrightarrow J(M,n)$ is defined as (2.2.2), where $n = \mathrm{rank}\, D$, and we have $\mathrm{Ker}\, \psi_* = \{0\}$. Therefore M is immersed into $J(M,n)$ and $\psi_*^{-1}(C) = D$.

2.3. Systems of second order partial differential equations of several unknown functions

2.3.1 Jet space $(J^2(M,n), C^2)$ of second order

First we will recall the jet space $(J^2(M,n), C^2)$ of second order in order to treat with systems of second order partial differential equations of several unknown functions ([20]). For convention, we put $J^0(M,n) = M$ and $(J^1(M,n), C^1) = (J(M,n), C)$, and write the projection $\Pi : J^1(M,n) \longrightarrow M$ as Π^1_0. Let $Q^1 = \mathrm{Ker}(\Pi^1_0)_*$, which is the differential system of codimension mn. $J^2(M,n)$ is a fiber bundle over $J^1(M,n)$ whose fiber $J^2(M,n)_x$ over $x \in J^1(M,n)$ consists of all n-dimensional integral elements u of C^1 at x which is transverse to $Q^1(x) \subset T_x J^1(M,n)$, namely $u \cap Q^1(x) = \{0\}$. The dimension of $J^2(M,n)$ is $n + m + mn + m \cdot {}_n\mathrm{H}_2$, where ${}_m\mathrm{H}_n = \binom{m+n-1}{n}$. The *canonical system* C^2 is defined by

$$C^2(u) = (\Pi^2_1)^{-1}_*(u) \quad \text{for } u \in J^2(M,n),$$

where $\Pi^2_1 : J^2(M,n) \longrightarrow J^1(M,n)$ is the canonical projection. For a point $u_o \in J^2(M,n)$, we have a canonical coordinate system $(x^i, z^a, p^a_i \ (1 \le a \le m,\ 1 \le i \le n))$ on a neighborhood U of $\Pi^2_1(u_o)$ in $J^1(M,n)$. Let $\hat{U}$ be a neighborhood of u_o that consists of all points $u \in \Pi^{-1}(U)$ such that $dx^1 \wedge \cdots \wedge dx^n|_u \neq 0$. Let p^a_{ij} for $1 \le a \le m$ and $1 \le i, j \le n$ be functions on $\hat{U}$ such that $dp^b_j|_u - \sum_i p^b_{ji}(u)\, dx^i|_u = 0$ for $u \in \hat{U}$ and $1 \le b \le m$, $1 \le j \le n$. Since $d\varpi^b|_u = 0$ for $u \in \hat{U}$ and $1 \le b \le m$, we have $p^b_{ij} = p^b_{ji}$ for $1 \le b \le m$, $1 \le i \le j \le n$. Thus $(x^i, z^a, p^a_i, p^a_{ij} \ (1 \le a \le m,\ 1 \le i \le j \le n))$ forms a coordinate system on $\hat{U}$, which is called the *canonical coordinate system* of $J^2(M,n)$. The canonical system C^2 on $\hat{U}$ is given by

$$C^2 = \{\varpi^a = \varpi^a_i = 0 \ (1 \le a \le m,\ 1 \le i \le n)\}, \tag{2.3.3}$$

where $\varpi^a = dz^a - \sum_{i=1}^n p^a_i dx^i$, $\varpi^a_i = dp^a_i - \sum_{k=1}^n p^a_{ik} dx^k$.

2.3.2 Symbol algebras of regular differential systems

We will recall the symbol algebra $\mathfrak{m}(x)$ of a regular differential system (M, D) at $x \in M$, introduced by N. Tanaka ([16]). Let D be a regular differential system on a (real or complex) manifold M such that $TM = D^{-\mu}$. Set

$$\mathfrak{m}(x) = \bigoplus_{p=-\mu}^{-1} \mathfrak{g}_p(x),$$

$$\mathfrak{g}_{-1}(x) = D^{-1}(x), \quad \mathfrak{g}_{-p}(x) = D^{-p}(x)/D^{-p+1}(x)\ (p > 1).$$

Let π_{-p} denote the projection of $D^{-p}(x)$ onto $\mathfrak{g}_{-p}(x)$. For $X \in \mathfrak{g}_{-p}(x)$, $Y \in \mathfrak{g}_{-q}(x)$, the bracket product $[X, Y] \in \mathfrak{g}_{-(p+q)}(x)$ is well-defined by

$$[X, Y] = \pi_{-(p+q)}([\hat{X}, \hat{Y}]_x),$$

where $\hat{X}$ and $\hat{Y}$ are vector fields taking values in $\mathcal{D}^{-p}$ and $\mathcal{D}^{-q}$ respectively such that $\pi_{-p}(\hat{X}_x) = X$ and $\pi_{-q}(\hat{Y}_x) = Y$. Then $\mathfrak{m}(x)$ is a nilpotent graded Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with this bracket operation, such that $\dim \mathfrak{m}(x) = \dim M$ and satisfies the condition, called the generating condition,

$$\mathfrak{g}_{-p}(x) = [\mathfrak{g}_{-p+1}(x), \mathfrak{g}_{-1}(x)] \quad \text{for } p > 1.$$

The graded Lie algebra $\mathfrak{m}(x)$ is called the *symbol algebra of* (M, D) *at* x. Generally, $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called a *fundamental graded Lie algebra of μ-th kind* if $\mathfrak{m}$ is a nilpotent graded Lie algebra satisfying that $\mathfrak{g}_{-\mu} \neq 0$ and $\mathfrak{g}_{-k} = 0$ for all $k > \mu$, and the generating condition

$$\mathfrak{g}_{-p} = [\mathfrak{g}_{-p+1}, \mathfrak{g}_{-1}] \quad \text{for } p > 1.$$

For a fundamental graded Lie algebra $\mathfrak{m}$, (M, D) is *of type* $\mathfrak{m}$ if the symbol algebra $\mathfrak{m}(x)$ of (M, D) is isomorphic to $\mathfrak{m}$ at each $x \in M$.

Conversely, given a fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p=-\mu}^{-1} \mathfrak{g}_p$ of μ-th kind, we can construct a regular differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$, which is called the *standard differential system of type* $\mathfrak{m}$: Let $M(\mathfrak{m})$ be the simply connected Lie group with Lie algebra $\mathfrak{m}$. Then we define a left invariant subbundle $D_{\mathfrak{m}}$ of $TM(\mathfrak{m})$ by $\mathfrak{g}_{-1}$. Then $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is a regular differential system of type $\mathfrak{m}$.

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded Lie algebra of μ-th kind over $\mathbb{K}$. The *prolongation* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p(\mathfrak{m})$ of $\mathfrak{m}$ is defined inductively as follows ([21]):

$$
\begin{aligned}
\mathfrak{g}_{-p}(\mathfrak{m}) &= \mathfrak{g}_{-p} \quad \text{for } p > 0, \\
\mathfrak{g}_0(\mathfrak{m}) &= \left\{ u \in \bigoplus_{p<0} \mathfrak{g}_p \otimes \mathfrak{g}_p^* \;\middle|\; u([X,\, Y]) = [u(X),\, Y] + [X,\, u(Y)] \right\}, \\
\mathfrak{g}_k(\mathfrak{m}) &= \left\{ u \in \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^* \;\middle|\; u([X,\, Y]) = [u(X),\, Y] + [X,\, u(Y)] \right\} \\
&\hspace{18em} \text{for } k > 0.
\end{aligned}
$$

Now we will see that $\mathfrak{g}(\mathfrak{m})$ is a graded Lie algebra. The bracket operation of $\mathfrak{g}(\mathfrak{m})$ is given as follows: First, for $u_0,\, u_0' \in \mathfrak{g}_0$, we define $[u_0,\, u_0'] \in \mathfrak{g}_0$ by

$$[u_0,\, u_0'](X) = u_0(u_0'(X)) - u_0'(u_0(X)) \quad \text{for } X \in \mathfrak{m}.$$

Thus $\mathfrak{g}_0(\mathfrak{m})$ becomes a Lie algebra with this bracket operation. Moreover, putting

$$[u_0,\, X] = -[X,\, u_0] = u_0(X) \quad \text{for } u_0 \in \mathfrak{g}_0(\mathfrak{m}) \text{ and } X \in \mathfrak{m},$$

we see that $\bigoplus_{p\leq 0} \mathfrak{g}_p(\mathfrak{m})$ is a graded Lie algebra.

Similarly, for $u_k \in \mathfrak{g}_k(\mathfrak{m})$ $(k > 0)$ and $X \in \mathfrak{m}$, we put $[u_k,\, X] = -[X,\, u_k] = u_k(X)$. For $u_k \in \mathfrak{g}_k(\mathfrak{m})$ and $u_l \in \mathfrak{g}_l(\mathfrak{m})$ $(k,\, l \geq 0)$, by induction on the integer $k + l \geq 0$, we define $[u_k,\, u_l] \in \mathfrak{g}_{k+l}(\mathfrak{m})$ by

$$[u_k,\, u_l](X) = [u_k,\, [u_l,\, X]] - [u_l,\, [u_k,\, X]] \quad \text{for } X \in \mathfrak{m}.$$

Then it follows easily that $\mathfrak{g}(\mathfrak{m})$ is a graded Lie algebra with this bracket operation.

It is known that the structure of the Lie algebra $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ of all infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ can be described by $\mathfrak{g}(\mathfrak{m})$. Especially, $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ is isomorphic to $\mathfrak{g}(\mathfrak{m})$ when $\mathfrak{g}(\mathfrak{m})$ is finite dimensional. For detail, see [16] and [21].

2.3.3 Symbol algebra $\mathfrak{C}^2(n,m)$ of $(J^2(M,n), C^2)$

We will recall the symbol algebra $\mathfrak{C}^2(n,m)$ of the canonical system $(J^2(M,n), C^2)$ ([19]). Let M be a manifold of dimension $m + n$ and $(J^2(M,n), C^2)$ the jet space of second order. Let us take the canonical coordinate system $(x^i, z^a, p_i^a, p_{ij}^a$ $(1 \leq a \leq m,\, 1 \leq i \leq j \leq n))$ on a neighborhood U as in Section 2.3.1. Then we have a local coframe

$$\left\{\varpi^a,\, \varpi_i^a,\, dx^i,\, dp_{ij}^a\ (1 \le a \le m,\, 1 \le i \le j \le n)\right\},$$

where $\varpi^a = dz^a - \sum_{i=1}^n p_i^a dx^i$, $\varpi_i^a = dp_i^a - \sum_{k=1}^n p_{ik}^a dx^k$. Let us take the dual frame of this coframe

$$\left\{\frac{\partial}{\partial z^a},\, \frac{\partial}{\partial p_i^a},\, \frac{d}{dx^i},\, \frac{\partial}{\partial p_{ij}^a}\ (1 \le a \le m,\, 1 \le i \le j \le n)\right\},$$

where

$$\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_{a=1}^m p_i^a \frac{\partial}{\partial z^a} + \sum_{a=1}^m \sum_{k=1}^n p_{ik}^a \frac{\partial}{\partial p_k^a}.$$

Then we have

$$\begin{aligned}
\left[\frac{\partial}{\partial p_{ij}^a},\, \frac{d}{dx^k}\right] &= \left(\delta_k^j - \frac{1}{2}\delta_i^j\right)\frac{\partial}{\partial p_i^a} + \left(\delta_k^i - \frac{1}{2}\delta_i^j\right)\frac{\partial}{\partial p_j^a},\\
\left[\frac{\partial}{\partial p_i^a},\, \frac{d}{dx^k}\right] &= \delta_k^i \frac{\partial}{\partial z^a}
\end{aligned}$$

and

$$\begin{aligned}
C^2 &= \left\langle \frac{\partial}{\partial p_{ij}^a},\, \frac{d}{dx^i}\ (1 \le a \le m,\, 1 \le i \le j \le n)\right\rangle,\\
\partial^{(1)} C^2 &= \left\langle \frac{\partial}{\partial p_i^a},\, \frac{\partial}{\partial p_{ij}^a},\, \frac{d}{dx^i}\ (1 \le a \le m,\, 1 \le i \le j \le n)\right\rangle,\\
\partial^{(2)} C^2 &= TJ^2(M, n).
\end{aligned}$$

Thus we see that the symbol algebra of $(J^2(M,n), C^2)$ is isomorphic to $\mathfrak{C}^2(n,m)$, which is defined as follows ([19]): Let V and W be vector spaces of dimension n and m respectively. Set

$$\mathfrak{C}^2(V,W) = \mathfrak{C}^2_{-3} \oplus \mathfrak{C}^2_{-2} \oplus \mathfrak{C}^2_{-1},$$
$$\mathfrak{C}^2_{-3} = W, \quad \mathfrak{C}^2_{-2} = W \otimes V^*, \quad \mathfrak{C}^2_{-1} = V \oplus W \otimes S^2(V^*).$$

The bracket operations of $\mathfrak{C}^2(n,m)$ is defined through the pairing between V and V^* such that V and $W \otimes S^2(V^*)$ are abelian subspaces of $\mathfrak{C}^2_{-1}$.

2.3.4 Graded simple Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let Φ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ and choose a simple root system Δ of Φ. Then we have the root decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \left(\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{\alpha}\right),$$

where Φ^+ denotes the set of positive roots and $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [h, X] = \alpha(h)X \text{ for } h \in \mathfrak{h}\}$ for $\alpha \in \Phi$. Let us take a non-empty subset Δ_1 of Δ. Then Δ_1 induces the following gradation:

$$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p, \qquad \mathfrak{g}_{-p} = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_{-\alpha},$$

$$\mathfrak{g}_0 = \left(\bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_{-\alpha}\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_{\alpha}\right), \qquad \mathfrak{g}_p = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_{\alpha},$$

where

$$\Phi^+ = \bigcup_{p\geq 0} \Phi_p^+, \qquad \Phi_p^+ = \left\{\alpha = \sum_{k=1}^{l} n_k\alpha_k \in \Phi^+ \,\middle|\, \sum_{\alpha_k\in\Delta_1} n_k = p\right\}.$$

Moreover, the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$ is a fundamental graded Lie algebra. Let θ denote the highest root of Φ^+. Writing $\theta = \sum_{i=1}^{l} n_i(\theta)\alpha_i$ for some $n_i(\theta) \in \mathbb{Z}_{\geq 0}$, we have

$$\mu = \sum_{\alpha_k\in\Delta_1} n_k(\theta),$$

where μ denotes the integer such that $\mathfrak{g}_{-\mu} \neq 0$ and $\mathfrak{g}_{-(\mu+1)} = 0$, namely $\mathfrak{m}$ is of μ-th kind.

When $\mathfrak{g}$ is a simple Lie algebra of type X_l, let (X_l, Δ_1) denote the simple Lie algebra $\mathfrak{g}$ with the gradation defined by Δ_1.

Conversely, it is known that the gradation of any simple graded Lie algebra over $\mathbb{C}$ satisfying the generating condition is obtained from some $\Delta_1 \subset \Delta$:

Theorem 2.1 ([21]) *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ satisfying the generating condition $\mathfrak{g}_{-(p+1)} = [\mathfrak{g}_{-p}, \mathfrak{g}_{-1}]$ for $p > 0$. Let X_l be the Dynkin diagram of $\mathfrak{g}$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic to a graded Lie algebra (X_l, Δ_1) for some $\Delta_1 \subset \Delta$. Moreover (X_l, Δ_1) and (X_l, Δ_1') are isomorphic if and only if there exists a diagram automorphism φ of X_l such that $\varphi(\Delta_1) = \Delta_1'$.*

In Section 2.6 we will seek a simple graded Lie algebra that is isomorphic to the prolongation of the symbol algebra of second order partial differential equations (or PD-manifolds $(R; D^1, D^2)$) of several unknown functions.

2.4. Characterization of partial differential equations

Let M be a (real or complex) manifold of dimension $m+n$ (m, $n \geq 2$). Let R be a submanifold of $J^2(M,n)$ satisfying the condition

$$\rho : R \longrightarrow J^1(M,n) \textit{ is submersion} \qquad (\mathbf{R.0})$$

where ρ is the restriction of the projection $\Pi : J^2(M,n) \longrightarrow J^1(M,n)$ to R. This condition implies that the system of second order partial differential equations never contains an equation of only first order. Let $\iota : R \longrightarrow J^2(M,n)$ be the inclusion. Let D^1 and D^2 be differential systems on R defined by the pullback by ι of ∂C^2 and C^2 respectively. Let $\varpi^1, \dots, \varpi^m$ and $\varpi^1_1, \dots, \varpi^m_n$ be 1-forms on $J^2(M,n)$ such that $\partial C^2 = \{\varpi^a = 0\ (1 \leq a \leq m)\}$ and $C^2 = \{\varpi^a = \varpi^a_i = 0\ (1 \leq a \leq m,\ 1 \leq i \leq n)\}$. Then it follows from Condition (**R.0**) that these forms ϖ^a, ϖ^a_i are linearly independent at each point of R and that

$$\begin{aligned} D^1 &= \{\varpi^a = 0\ (1 \leq a \leq m)\}, \\ D^2 &= \{\varpi^a = \varpi^a_i = 0\ (1 \leq a \leq m,\ 1 \leq i \leq n)\}. \end{aligned} \qquad (2.4.4)$$

Here, by our abuse of notation, we write $\iota^*\varpi$ as ϖ. Thus we see that

$$\begin{aligned} &D^1 \textit{ and } D^2 \textit{ are differential systems of codimension} \\ &m \textit{ and } m+mn \textit{ respectively.} \end{aligned} \qquad (\mathbf{R.1})$$

From (2.3.3), there exist 1-forms $\omega^1, \dots, \omega^n$ on R such that the forms ϖ^a, ϖ^a_i and ω^i are linearly independent at each point and $d\varpi^a \equiv \sum_i \omega^i \wedge \varpi^a_i$ (mod ϖ^b $(1 \leq b \leq m)$). Therefore we have

$$\partial D^2 \subset D^1. \tag{\textbf{R.2}}$$

Since $\mathrm{Ch}(D^1) = \{\varpi^a = \varpi^a_i = \omega^i = 0\,(1 \le a \le m,\, 1 \le i \le n)\}$,

$$\mathrm{Ch}(D^1) \textit{ is a subbundle of } D^2 \textit{ of codimension } n. \tag{\textbf{R.3}}$$

Since $d\varpi^a \wedge \omega^1 \wedge \cdots \wedge \omega^n \equiv 0 \pmod{\varpi^b\ (1 \le b \le m)}$ for $1 \le a \le m$, we have

$$D^1 \textit{ is of Cartan rank } n. \tag{\textbf{R.4}}$$

By applying Realization Lemma to $\rho : R \longrightarrow J^1(M,n)$ and D^2, we obtain

$$\mathrm{Ch}(D^1) \cap \mathrm{Ch}(D^2) = \{0\}. \tag{\textbf{R.5}}$$

Because, we have the unique map $\psi : R \longrightarrow J^1(J^1(M,n),n)$ such that $\rho = \Pi' \circ \psi$ and $D^2 = \psi_*^{-1}(C)$, where $\Pi' : J^1(J^1(M,n),n) \longrightarrow J^1(M,n)$ is the projection and C is the canonical system of $J^1(J^1(M,n),n)$. In fact, the structure equation of D^1 and Condition (**R.0**) yield that $\mathrm{Ker}\,\rho_* = \mathrm{Ch}(D^1) \subset D^2$. By definition, for $v \in R$, we see that $\psi(v)$ is a n-dimensional integral element of $(J^1(M,n), C^1)$ and transverse to $Q^1 = \mathrm{Ker}(\Pi^1_0)_*(\rho(v))$. Namely $\psi(v) \in J^2(M,n)$. From the uniqueness of ψ, it follows that $\psi = \iota$. Therefore we obtain (**R.5**) from Realization Lemma.

Furthermore we will see that there exists an additional differential system F in the following lemma:

Lemma 2.2 *Let R be a (real or complex) manifold and D^1, D^2 differential systems satisfying four conditions from* (**R.1**) *to* (**R.4**)*. Then there exists a unique subbundle F of D^1 of codimension n such that $\partial F \subset D^1$. Moreover, F satisfies $F \cap D^2 = \mathrm{Ch}(D^1)$ and is, if $m \ge 3$, completely integrable.*

Proof. D^1 and D^2 are locally expressed as follows:

$$D^1 = \{\varpi^a = 0\,(1 \le a \le m)\},$$
$$D^2 = \{\varpi^a = \varpi^a_i = 0\,(1 \le a \le m,\, 1 \le i \le n)\},$$

g where ϖ^a, ϖ^a_i are linearly independent 1-forms. Condition (**R.2**) implies $d\varpi^a \equiv 0 \pmod{\varpi^b,\ \varpi^b_j\ (1 \le b \le m,\ 1 \le j \le n)}$ for $1 \le a \le m$, and thus $d\varpi^a$ expressed as

$$d\varpi^a \equiv \sum_{b,\,j} \pi_b^{aj} \wedge \varpi_j^b \quad (\mathrm{mod}\ \varpi^c\,(1 \le c \le m)) \tag{2.4.5}$$

with some 1-forms π_b^{aj}. It follows from Condition (**R.3**) that $\mathrm{Ch}(D^1) = \left\{\varpi^a = \varpi_i^a = \pi_b^{ai} = 0\,(1 \le a,\, b \le m,\, 1 \le i \le n)\right\}$. Now we will choose 1-forms ω^i from $\{\pi_b^{ai}\}$ in order that ϖ^a, ϖ_i^a, ω^i are linearly independent. From Condition (**R.4**), we may take 1-forms $\omega^1, \dots, \omega^n$ such that $\varpi^1 \wedge \cdots \wedge \varpi^m \wedge \omega^1 \wedge \cdots \wedge \omega^n \neq 0$ and $d\varpi^a \equiv 0$ $(\mathrm{mod}\ \varpi^b,\, \omega^j\ (1 \le b \le m,\, 1 \le j \le n))$. Substituting Equation (2.4.5) into the second equation, we obtain $\pi_b^{aj} \equiv 0$ $(\mathrm{mod}\ \varpi^c,\, \varpi_k^c,\, \omega^k\ (1 \le c \le m,\, 1 \le k \le n))$. Thus, from Condition (**R.3**), we achieve

$$\mathrm{Ch}(D^1) = \left\{\varpi^a = \varpi_i^a = \omega^i = 0\,(1 \le a \le m,\, 1 \le i \le n)\right\} \tag{2.4.6}$$

and see that ϖ^a, ϖ_i^a, ω^i are linearly independent. On the other hand, Condition (**R.4**) allows us to write

$$d\varpi^a \equiv \sum_i \omega^i \wedge \pi_i^a \quad (\mathrm{mod}\ \varpi^b\,(1 \le b \le m)) \tag{2.4.7}$$

with some 1-forms π_i^a. From Condition (**R.2**), we have $\pi_i^a \equiv 0$ $(\mathrm{mod}\ \varpi^b,\, \varpi_j^b,\, \omega^j\ (1 \le b \le m,\, 1 \le j \le n))$. It follows from (2.4.6) and (2.4.7) that $\mathrm{Ch}(D^1) = \left\{\varpi^a = \pi_i^a = \omega^i = 0\,(1 \le a \le m,\, 1 \le i \le n)\right\}$, which implies that ϖ^a, π_i^a, ω^i are linearly independent. Compared with (2.4.6), we may write $\pi_i^a \equiv \sum_{b,\,j} A_{ib}^{aj}\, \varpi_j^b + \sum_j B_{ij}^a\, \omega^j$ $(\mathrm{mod}\ \varpi^c\ (1 \le c \le m))$ with some functions A_{ib}^{aj}, B_{ij}^a. Substituting this into $\sum_i \omega^i \wedge \pi_i^a \equiv 0$ $(\mathrm{mod}\ \varpi^b,\, \varpi_j^b\ (1 \le b \le m,\, 1 \le j \le n))$, we obtain $B_{ji}^a = B_{ij}^a$. Replacing $A_{ib}^{aj}\, \varpi_j^b$ by ϖ_i^a, we achieve

$$d\varpi^a \equiv \sum_i \omega^i \wedge \varpi_i^a \quad (\mathrm{mod}\ \varpi^b\,(1 \le b \le m)). \tag{2.4.8}$$

Let F be a subbundle of D^1 of codimension n defined by

$$F = \left\{\varpi^a = \omega^i = 0\,(1 \le a \le m,\, 1 \le i \le n)\right\},$$

which satisfies $\partial F \subset D^1$. Then it follows that $F \cap D^2 = \mathrm{Ch}(D^1)$. Now we will see the uniqueness and complete integrability of F. Let $\hat{F}$ be an another subbundle of D^1 of codimension n satisfying $\partial \hat{F} \subset D^1$. Write $\hat{F} =$

$\{\varpi^a = \hat{\omega}^i = 0\,(1 \leq a \leq m,\, 1 \leq i \leq n)\}$ with some 1-forms $\hat{\omega}^i$. Since $\partial\hat{F} \subset D^1$, we have $d\varpi^a \equiv 0 \pmod{\varpi^b,\, \hat{\omega}^j\ (1 \leq b \leq m,\, 1 \leq j \leq n)}$. By Equation (2.4.8), we obtain $\omega^i \equiv 0 \pmod{\varpi^b,\, \varpi^b_j,\, \hat{\omega}^j\ (1 \leq b \leq m,\, 1 \leq j \leq n)}$. Thus we may write $\omega^i \equiv \sum_{b,\,j} B^{ij}_b\, \varpi^b_j \pmod{\varpi^c,\, \hat{\omega}^j\ (1 \leq c \leq m,\, 1 \leq j \leq n)}$ with some functions B^{ij}_b. Substituting them into Equation (2.4.8), we have, for $1 \leq a \leq m$, $B^{ji}_a = B^{ij}_a$ and $B^{ij}_b = 0$ if $b \neq a$. It follows from $m \geq 2$ that $B^{ij}_a = 0$ for $1 \leq a \leq m$ and $1 \leq i,\, j \leq n$. Thus we achieve $F = \hat{F}$.

Assume $m \geq 3$. Since

$$\begin{aligned} 0 &= d^2\varpi^a \\ &\equiv \sum_i d\omega^i \wedge \varpi^a_i \pmod{\varpi^b,\, \omega^j\ (1 \leq b \leq m,\, 1 \leq j \leq n)}, \end{aligned}$$

we have $d\omega^i \equiv 0 \pmod{\varpi^b,\, \varpi^a_j,\, \omega^j\,(1 \leq b \leq m,\, 1 \leq j \leq n)}$ for each $1 \leq a \leq m$ and $1 \leq i \leq n$. For $m \geq 3$, $d\omega^i \equiv 0 \pmod{\varpi^b,\, \omega^j\,(1 \leq b \leq m,\, 1 \leq j \leq n)}$, which implies that F is completely integrable. $\square$

The above discussion yields a formulation of systems of second order partial differential equations of several unknown functions:

Definition 2.3 Let D^1, D^2 be differential systems on a (real or complex) manifold R satisfying the conditions from (**R.1**) to (**R.5**) and the condition

$$F \textit{ is completely integrable}, \tag{\textbf{R.6}}$$

where the differential system F was defined in Lemma 2.2. Then we call the triplet $(R; D^1, D^2)$ a *PD-manifold*.

Note that Condition (**R.6**) is satisfied automatically from (**R.1**) to (**R.4**) unless $m = 2$. The following theorem implies that a PD-manifold $(R; D^1, D^2)$ is regarded as a submanifold of $(J^2(M, n), C^2)$:

Theorem 2.4 *Let $(R; D^1, D^2)$ be a PD-manifold. Let F be the differential system on R in Lemma 2.2. Assume that the space $M = R/F$ of leaves of the foliation is a manifold of dimension $m+n$. Then there exists an immersion $\varphi : R \longrightarrow J^2(M, n)$ satisfying $D^2 = \varphi_*^{-1}(C^2)$.*

$$
\begin{tikzcd}
 & J^1(M,n) \arrow[d, "\Pi^1_0"] \\
R \arrow[ur, "\psi"] \arrow[r, "p"'] & M = R/F
\end{tikzcd}
\qquad
\begin{tikzcd}
 & J^1(J^1(M,n),n) \arrow[d] \\
R \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & J^1(M,n)
\end{tikzcd}
$$

Proof. Let p denote the canonical projection of R onto M. Then $\operatorname{Ker} p_* = F$ is a subbundle of D^1 of codimension n. By applying Realization Lemma to p and D^1, we see that there exists a unique map $\psi : R \longrightarrow J^1(M,n)$ satisfying $p = \Pi^1_0 \circ \psi$ and $D^1 = \psi_*^{-1}(C^1)$, where $\Pi^1_0 : J^1(M,n) \longrightarrow M$ is the projection, and see that $\operatorname{Ker}\psi_* = \operatorname{Ch}(D^1)$. Moreover, by applying Realization Lemma to the map ψ and D^2, we see that there exists a map $\varphi : R \longrightarrow J^1(J^1(M,n),n)$ as in the lemma. It follows from Condition (**R.5**) that φ is an immersion. Finally we will show that $\varphi(v)$ is a n-dimensional integral element of C^1 and $\varphi(v) \cap \operatorname{Ker}(\Pi^1_0)_*(\psi(v)) = \{0\}$ for each $v \in R$. Since $\varphi(v) = \psi_*(D^2(v))$ and $\partial D^2 \subset D^1$, we see that $\varphi(v)$ is a n-dimensional integral element of C^1. It follows from $F \cap D^2 = \operatorname{Ch}(D^1)$ that $\varphi(v) \cap \operatorname{Ker}(\Pi^1_0)_*(\psi(v)) = \{0\}$. $\square$

Remark 2.5 In the case of one unknown function, i.e. $m = 1$, we can also consider a submanifold R of $J^2(M,n)$ satisfying Condition (**R.0**), where $\dim M = n+1$. Let D^1 and D^2 be the restriction to R of ∂C^2 and C^2 respectively. Then D^1 and D^2 also satisfy the conditions from (**R.1**) to (**R.5**) ([19]). However there are many subbundles F of D^1 of codimension n such that $\partial F \subset D^1$. In fact, there are independent 1-forms $\varpi, \varpi_1, \ldots, \varpi_n, \omega^1, \ldots, \omega^n$ such that $D^1 = \{\varpi = 0\}$, $D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}$ and $d\varpi \equiv \sum_i \omega^i \wedge \varpi_i \pmod{\varpi}$. Set $\hat{F} = \left\{\varpi = \hat{\omega}^1 = \cdots = \hat{\omega}^n = 0\right\}$, where $\hat{\omega}^i = \omega^i - \sum_j B^{ij}\varpi_j$ for any functions B^{ij} satisfying $B^{ji} = B^{ij}$. Then $\hat{F}$ satisfies $\partial\hat{F} \subset D^1$ (of course, D^2 also satisfies $\partial D^2 \subset D^1$). That is occurred from the symmetric property of contact manifold.

Conversely, let R be a manifold and let D^1 and D^2 be differential systems on R satisfying the following conditions:

(R.1) D^1 and D^2 are differential systems of codimension 1 and $n+1$ respectively.
(R.2) $\partial D^2 \subset D^1$.
(R.3) $\operatorname{Ch}(D^1)$ is a subbundle of D^2 of codimension n.

(R.4) $\mathrm{Ch}(D^1)(v) \cap \mathrm{Ch}(D^2)(v) = \{0\}$ at each $v \in R$.

Then $(R; D^1, D^2)$ is called a PD-manifold of second order, which is a characterization of the structure of systems of second order partial differential equations of one unknown function ([19], [22]). Compared with equations of several unknown functions, those of one unknown function satisfy the condition that D^1 is of Cartan rank n.

2.5. PD manifolds of finite type

We will seek an example of PD-manifolds of finite type by utilizing fundamental graded Lie algebras and representation theory.

2.5.1 Symbol algebra of PD-manifold $(R; D^1, D^2)$

We will define the *symbol algebra* $\mathfrak{s}(x) = \mathfrak{s}_{-3}(x) \oplus \mathfrak{s}_{-2}(x) \oplus \mathfrak{s}_{-1}(x)$ of a PD-manifold $(R; D^1, D^2)$ at a point $x \in R$, following [19]. Let us fix a point $x \in R$ and put $D^{-1} = D^2$, $D^{-2} = D^1$ and $D^{-3} = TR$. We set

$$\mathfrak{s}_{-3}(x) = D^{-3}(x)/D^{-2}(x), \quad \mathfrak{s}_{-2}(x) = D^{-2}(x)/D^{-1}(x), \quad \mathfrak{s}_{-1}(x) = D^{-1}(x).$$

The bracket operation of $\mathfrak{s}(x)$ is defined as follows: Let π_{-p} denote the projection of $D^{-p}(x)$ onto $\mathfrak{s}_{-p}(x)$ for $1 \le p \le 3$. For $X \in \mathfrak{s}_{-p}(x)$, $Y \in \mathfrak{s}_{-q}(x)$, the bracket product $[X, Y] \in \mathfrak{s}_{-(p+q)}(x)$ is well-defined by

$$[X, Y] = \pi_{-(p+q)}([\hat{X}, \hat{Y}]_x),$$

where $\hat{X}$ and $\hat{Y}$ denote vector fields taking values in $D^{-p}(y)$ and $D^{-q}(y)$ at each point $y \in R$ respectively such that $\pi_{-p}(\hat{X}_x) = X$ and $\pi_{-q}(\hat{Y}_x) = Y$. Let $\mathfrak{f}(x) = \mathrm{Ch}(D^1)(x)$. It follows from (**R.3**) that $\mathfrak{f}(x)$ is a subspace of $\mathfrak{s}_{-1}(x)$ of codimension n. For $X \in \mathfrak{s}_{-1}(x)$, since $d\varpi^a(X, Y) = 0$ for all $Y \in D^1(x)$ if and only if $[X, \mathfrak{s}_{-2}(x)] = 0$, we obtain

$$\mathfrak{f}(x) = \{X \in \mathfrak{s}_{-1}(x) \mid [X, \mathfrak{s}_{-2}(x)] = 0\}. \tag{2.5.9}$$

Let ϖ^a, ϖ^a_i $(1 \le a \le m,\ 1 \le i \le n)$ denote 1-forms defining D^1 and D^2 as in (2.4.4). Since they are the restriction of the defining 1-forms of C^2, we see that $\mathfrak{s}(x)$ is isomorphic to a graded Lie subalgebra of $\mathfrak{C}^2(n, m)$ satisfying $\mathfrak{s}_{-3}(x) \simeq \mathfrak{C}^2_{-3}$, $\mathfrak{s}_{-2}(x) \simeq \mathfrak{C}^2_{-2}$ and $\mathfrak{f}(x) = \mathrm{Ch}(\partial C^2)(x) \cap T_x R$.

We assume $\mathrm{Ch}(D^1) \neq \{0\}$, namely $\mathfrak{f}(x) \neq \{0\}$ at each point $x \in R$ in what follows.

If $\mathrm{Ch}(D^1) = \{0\}$, applying Realization Lemma to the projection $\pi : R \longrightarrow M = R/F$ and D^1, we have a map $\psi : R \longrightarrow J^1(M,n)$ such that $\pi = \Pi^1_0 \circ \psi$ and $D^1 = \psi_*^{-1}(C^1)$. Since $\mathrm{Ker}\,\psi_*(v) = F(v) \cap \mathrm{Ch}(D^1)(v) = \{0\}$ for $v \in R$ and $\dim R = \mathrm{codim}\,\mathrm{Ch}(D^1) = \dim J^1(M,n)$, R is locally diffeomorphic to $J^1(M,n)$. Therefore D^2 is completely integrable if R is integrable.

Now we assume that there exists a n-dimensional integral element V of (R, D^2) at each point $x \in R$ such that

$$\mathfrak{s}_{-1}(x) = V \oplus \mathfrak{f}(x),$$

where V is an abelian subalgebra in $\mathfrak{s}(x)$. By fixing a basis of $\mathfrak{s}_{-3}(x)$, we identify $\mathfrak{s}_{-3}(x)$ with a m-dimensional vector space W. It follows from $V \cap \mathfrak{f}(x) = \{0\}$ and (2.5.9) that $\mathfrak{s}_{-2}(x)$ is identified with $W \otimes V^*$ through the bracket product $[\ ,\ ] : \mathfrak{s}_{-2}(x) \times \mathfrak{s}_{-1}(x) \longrightarrow \mathfrak{s}_{-3}(x)$. Let $\mu : \mathfrak{f}(x) \longrightarrow W \otimes S^2(V^*)$ be a linear map defined by

$$\mu(f)(v_1, v_2) = [[f,\, v_1],\, v_2] \in \mathfrak{s}_{-3}(x) \simeq W \quad \text{for } f \in \mathfrak{f}(x) \text{ and } v_1,\, v_2 \in V,$$

which implies $\mu(f)(v_1, v_2) = \mu(f)(v_2, v_1)$. Moreover, we see easily that μ is injective.

Thus, we obtain

$$\mathfrak{s}_{-3}(x) \simeq W, \quad \mathfrak{s}_{-2}(x) \simeq W \otimes V^*, \quad \mathfrak{s}_{-1}(x) = V \oplus \mathfrak{f}(x),$$
$$\mathfrak{f}(x) \subset W \otimes S^2(V^*).$$

In consequent two sections we will seek a PD-manifold $(R; D^1, D^2)$ of type $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$, that is, PD-manifolds whose symbol algebra is isomorphic to

$$\begin{aligned} &\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}; \\ &\qquad \mathfrak{s}_{-3} = W, \quad \mathfrak{s}_{-2} = W \otimes V^*, \quad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}, \end{aligned} \tag{2.5.10}$$

where W and V are vector spaces of dimension m and n, and $\mathfrak{f}$ is a non-zero subspace of $W \otimes S^2(V^*)$. Here, especially we have

$$\dim \mathfrak{s}_{-2} = \dim \mathfrak{s}_{-3} \cdot (\dim \mathfrak{s}_{-1} - \dim \mathfrak{f}),$$

which will be utilized in Section 2.6.

2.5.2 Example of partial differential equations of finite type

We will consider an example of PD-manifolds $(R; D^1, D^2)$ of finite type and see that it has a pseudo-product structure of irreducible type $(\mathfrak{l}, S)$ ([13], [25]):

Example 2.6 Let us consider the following system of second order partial differential equations of two unknown functions z^1, z^2 with three independent variables x_1, x_2, x_3 :

$$\frac{\partial^2 z^a}{\partial x_1 \partial x_1} = \frac{\partial^2 z^a}{\partial x_2 \partial x_2} = \frac{\partial^2 z^a}{\partial x_2 \partial x_3} = \frac{\partial^2 z^a}{\partial x_3 \partial x_3} = 0 \quad \text{for } a = 1,\, 2. \tag{2.5.11}$$

Note that the system of equations of one unknown function

$$\frac{\partial^2 z}{\partial x_1 \partial x_1} = \frac{\partial^2 z}{\partial x_2 \partial x_2} = \frac{\partial^2 z}{\partial x_2 \partial x_3} = \frac{\partial^2 z}{\partial x_3 \partial x_3} = 0$$

is known as a model equation of type $(A_4, \{\alpha_1, \alpha_2, \alpha_4\})$ ([22, Section 5.3]). The system of equations (2.5.11) defines the submanifold R of $J^2(\mathbb{R}^5, 2)$ and differential system D on R as follows:

$$\begin{aligned} R &= \{p^a_{11} = p^a_{22} = p^a_{23} = p^a_{33} = 0 \, (a = 1,\, 2)\} \\ D &= \left\{\varpi^1 = \varpi^2 = \varpi^1_1 = \varpi^1_2 = \varpi^1_3 = \varpi^2_1 = \varpi^2_2 = \varpi^2_3 = 0\right\} \end{aligned}$$

where $(x^i, z^a, p^a_i, p^a_{ij} \; (1 \le a \le 2,\, 1 \le i \le j \le 3))$ is the canonical coordinate system of $J^2(\mathbb{R}^5, 2)$ and

$$\begin{aligned} \varpi^1 &= dz^1 - p^1_1\, dx^1 - p^1_2\, dx^2 - p^1_3\, dx^3, \\ \varpi^2 &= dz^2 - p^2_1\, dx^1 - p^2_2\, dx^2 - p^2_3\, dx^3, \\ \varpi^1_1 &= dp^1_1 - p^1_{12}\, dx^2 - p^1_{13}\, dx^3, \\ \varpi^1_2 &= dp^1_2 - p^1_{12}\, dx^1, \\ \varpi^1_3 &= dp^1_3 - p^1_{13}\, dx^1, \\ \varpi^2_1 &= dp^2_1 - p^2_{12}\, dx^2 - p^2_{13}\, dx^3, \\ \varpi^2_2 &= dp^2_2 - p^2_{12}\, dx^1, \\ \varpi^2_3 &= dp^2_3 - p^2_{13}\, dx^1. \end{aligned}$$

We obtain the symbol algebra $\mathfrak{m}(x) = \mathfrak{m}$ of (R, D) at each point x as follows:

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \qquad \mathfrak{g}_{-3} = W, \quad \mathfrak{g}_{-2} = W \otimes V^*, \quad \mathfrak{g}_{-1} = V \oplus \mathfrak{f},$$
$$\mathfrak{f} = W \otimes \langle e^1 \circledcirc e^2, \, e^1 \circledcirc e^3 \rangle \subset W \otimes S^2(V^*),$$

where W and V is vector spaces of dimension 2 and 3 respectively, and $\{e^1, e^2, e^3\}$ is a basis of V^*. Now we will see that the prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$ is isomorphic to a pseudo-product GLA of irreducible type $(\mathfrak{l}, S)$ for some simple graded Lie algebra $\mathfrak{l}$ of depth 1 and irreducible $\mathfrak{l}$-module S.

Let $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{b} = \mathfrak{sl}(3, \mathbb{R})$. Let us fix a Cartan subalgebra $\mathfrak{h}^{\mathfrak{a}}$ of $\mathfrak{a}$ (resp. $\mathfrak{h}^{\mathfrak{b}}$ of $\mathfrak{b}$) and let $\Phi^{\mathfrak{a}}$ (resp. $\Phi^{\mathfrak{b}}$) be a root system of $\mathfrak{a}$ (resp. $\mathfrak{b}$) relative to $\mathfrak{h}^{\mathfrak{a}}$ (resp. $\mathfrak{h}^{\mathfrak{b}}$). Let us fix a simple root system $\Delta^{\mathfrak{a}} = \{\alpha_1\}$ of $\Phi^{\mathfrak{a}}$ (resp. $\Delta^{\mathfrak{b}} = \{\beta_1, \beta_2\}$ of $\Phi^{\mathfrak{b}}$). Then we have $\Phi^{\mathfrak{a}} = \{\pm\alpha_1\}$, $\Phi^{\mathfrak{b}} = \{\pm\beta_1, \pm\beta_2, \pm(\beta_1 + \beta_2)\}$ and the root decomposition of $\mathfrak{a}$ (resp. $\mathfrak{b}$) relative to $\Delta^{\mathfrak{a}}$ (resp. $\Delta^{\mathfrak{b}}$):

$$\mathfrak{a} = \mathfrak{h}^{\mathfrak{a}} \oplus \bigoplus_{\alpha \in \Phi^{\mathfrak{a}}} \mathfrak{g}^{\mathfrak{a}}_{\alpha}, \qquad \mathfrak{b} = \mathfrak{h}^{\mathfrak{b}} \oplus \bigoplus_{\beta \in \Phi^{\mathfrak{b}}} \mathfrak{g}^{\mathfrak{b}}_{\beta},$$

where $\mathfrak{g}^{\mathfrak{a}}_{\alpha} = \{X \in \mathfrak{a} \mid [H, X] = \alpha(H)X \text{ for } H \in \mathfrak{h}^{\mathfrak{a}}\}$ (resp. $\mathfrak{g}^{\mathfrak{b}}_{\beta}$) is the root space for $\alpha \in \Phi^{\mathfrak{a}}$ (resp. $\beta \in \Phi^{\mathfrak{b}}$). Let $\Delta^{\mathfrak{a}}_1 = \{\alpha_1\} = \Delta^{\mathfrak{a}}$ and $\Delta^{\mathfrak{b}}_1 = \{\beta_1\} \subset \Delta^{\mathfrak{b}}$. They define gradations of $\mathfrak{a}$ and $\mathfrak{b}$ of depth 1 as follows:

$$\mathfrak{a} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1, \qquad \mathfrak{a}_{\pm 1} = \mathfrak{g}^{\mathfrak{a}}_{\pm\alpha_1}, \quad \mathfrak{a}_0 = \mathfrak{h}^{\mathfrak{a}},$$
$$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1, \qquad \mathfrak{b}_{\pm 1} = \mathfrak{g}^{\mathfrak{b}}_{\pm\beta_1} \oplus \mathfrak{g}^{\mathfrak{b}}_{\pm(\beta_1+\beta_2)}, \quad \mathfrak{b}_0 = \mathfrak{h}^{\mathfrak{b}} \oplus \mathfrak{g}^{\mathfrak{b}}_{-\beta_2} \oplus \mathfrak{g}^{\mathfrak{b}}_{\beta_2}.$$

Let U be a vector space over $\mathbb{R}$ of dimension 2. Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be the reductive graded Lie algebra of depth 1 defined by

$$\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{gl}(U), \quad [\mathfrak{a}, \mathfrak{gl}(U)] = [\mathfrak{b}, \mathfrak{gl}(U)] = 0,$$
$$\mathfrak{l}_{\pm 1} = \mathfrak{a}_{\pm 1} \oplus \mathfrak{b}_{\pm 1}, \quad \mathfrak{l}_0 = \mathfrak{a}_0 \oplus \mathfrak{b}_0 \oplus \mathfrak{gl}(U).$$

Note that the semisimple ideal $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ of $\mathfrak{l}$ coincides with $\mathfrak{a} \oplus \mathfrak{b}$, which is not simple (cf. [13], [25]).

Let $\{\varpi^{\mathfrak{a}}_1\}$ and $\{\varpi^{\mathfrak{b}}_1, \varpi^{\mathfrak{b}}_2\}$ be fundamental weights relative to $\Delta^{\mathfrak{a}}$ and $\Delta^{\mathfrak{b}}$. Let $T^{\mathfrak{a}}$ (resp. $T^{\mathfrak{b}}$) be the irreducible $\mathfrak{a}$-module (resp. $\mathfrak{b}$-module) with highest weight $\varpi_1^{\mathfrak{a}}$ (resp. $\varpi_2^{\mathfrak{b}}$). Then $S = T^{\mathfrak{a}} \otimes T^{\mathfrak{b}} \otimes U$ is a faithful irreducible $\mathfrak{l}$-module and decomposed as follows:

$$S = \bigoplus_{p=-3}^{-1} S_p, \qquad S_{-3} = V_1^{\mathfrak{a}} \otimes V_1^{\mathfrak{b}} \otimes U, \quad S_{-2} = \left(V_1^{\mathfrak{a}} \otimes V_0^{\mathfrak{b}} \oplus V_0^{\mathfrak{a}} \otimes V_1^{\mathfrak{b}}\right) \otimes U,$$
$$S_{-1} = V_0^{\mathfrak{a}} \otimes V_0^{\mathfrak{b}} \otimes U,$$

where $V_0^{\mathfrak{a}} = V(\varpi_1^{\mathfrak{a}})$, $V_1^{\mathfrak{a}} = V(\varpi_1^{\mathfrak{a}} - \alpha_1)$, $V_0^{\mathfrak{b}} = V(\varpi_2^{\mathfrak{b}}) \oplus V(\varpi_2^{\mathfrak{b}} - \beta_2)$, $V_1^{\mathfrak{b}} = V(\varpi_2^{\mathfrak{b}} - (\beta_1 + \beta_2))$ and $V(\lambda)$ is the weight space with weight λ. Since $\dim S_{-3} = 2$, $\dim S_{-2} = 6$ and $\dim \mathfrak{l}_{-1} = 3$, it follows from the property of S (see [13, Proposition 4.3.1] or [25, Lemma 2.1 (4)]) that S_{-2} is isomorphic to $W \otimes V^*$, where $W = S_{-3}$ and $V = \mathfrak{l}_{-1}$. Namely $\mathfrak{l}_{-1} \oplus S$ is isomorphic to $\mathfrak{m}$. Moreover, by direct calculation, we can see that the prolongation of $\mathfrak{m}$ is isomorphic to $\mathfrak{l} \oplus S$.

2.6. Partial differential equations of simple type

In this section everything will be considered in complex analytic category. We will seek a simple graded Lie algebra of type (X_l, Δ_1) that the negative part $\mathfrak{m}$ is isomorphic to the symbol algebra of PD-manifolds of m (≥ 2) unknown functions. A necessary condition for this is that $\mathfrak{m}$ is of third kind and $\dim \mathfrak{g}_{-3} \geq 2$. From extended Dynkin diagrams (see Figure 2.1 at page 63), the following are the simple graded Lie algebras of type (X_l, Δ_1) satisfying this necessary condition (cf. [4]): $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$ ($1 \leq i < j < k \leq l$, $(i, k) \neq (1, l)$), $(B_l, \{\alpha_1, \alpha_i\})$ ($3 \leq i \leq l$), $(C_l, \{\alpha_i, \alpha_l\})$ ($2 \leq i \leq l-1$), $(D_l, \{\alpha_1, \alpha_i\})$ ($3 \leq i \leq l-2$), $(D_l, \{\alpha_i, \alpha_l\})$ ($3 \leq i \leq l-2$), $(D_l, \{\alpha_1, \alpha_{l-1}, \alpha_l\})$, $(E_6, \{\alpha_4\})$, $(E_6, \{\alpha_1, \alpha_3\})$, $(E_6, \{\alpha_1, \alpha_5\})$, $(E_7, \{\alpha_3\})$, $(E_7, \{\alpha_5\})$, $(E_7, \{\alpha_2, \alpha_7\})$, $(E_7, \{\alpha_6, \alpha_7\})$, $(E_8, \{\alpha_2\})$, $(E_8, \{\alpha_7\})$, $(F_4, \{\alpha_2\})$, $(G_2, \{\alpha_1\})$ up to Dynkin diagram automorphism.

However, we will see that there exist no such simple graded Lie algebras. Precisely, we state as follows:

Theorem 2.7 *Let* $\mathfrak{s} = \bigoplus_{p=-3}^{-1} \mathfrak{s}_p$ *be a fundamental graded Lie algebra satisfying* (2.5.10). *Then, for any simple graded Lie algebra* $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ *of type* (X_l, Δ_1), $\mathfrak{s}$ *is never isomorphic to the negative part* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *of* $\mathfrak{g}$. *In other words, there are no PD-manifolds of type* $\mathfrak{s}$ *such that the prolongation of* $\mathfrak{s}$ *is isomorphic to some simple graded Lie algebra.*

Note that, among (X_l, Δ_1) listed above, $(C_l, \{\alpha_i, \alpha_l\})$ ($2 \leq i \leq l-1$),

$(D_l, \{\alpha_i, \alpha_l\})$ $(3 \leq i \leq l-2)$, $(E_6, \{\alpha_1, \alpha_3\})$ and $(E_7, \{\alpha_6, \alpha_7\})$ appeared in Theorem 2.3 (a) of [26]. That is, they are the prolongation of $\mathfrak{m} = \mathfrak{l}_{-1} \oplus S$ for some pseudo-product graded Lie algebra of type $(\mathfrak{l}, S)$. In the case of $(C_l, \{\alpha_i, \alpha_l\})$ $(2 \leq i \leq l-1)$ and $(D_l, \{\alpha_i, \alpha_l\})$ $(3 \leq i \leq l-2)$, according to Case (3) and (9) in Section 3 of [26], since $\dim \mathfrak{g}_{-2} = \dim \mathfrak{l}_{-1} (= \dim V)$ and $\mathfrak{f} = S_{-1}$, $\mathfrak{m}$ cannot be isomorphic to $\mathfrak{s}$ satisfying (2.5.10). In the case of $(E_6, \{\alpha_1, \alpha_3\})$, according to Case (2) in Section 4 of [26], since $\dim \mathfrak{g}_{-3} \cdot (\dim \mathfrak{g}_{-1} - \dim \mathfrak{f}) - \dim \mathfrak{g}_{-2} = |\Phi_3^+| \cdot (|\Phi_1^+| - |\Psi^1|) - |\Phi_2^+| > 0$, $\mathfrak{m}$ cannot be isomorphic to $\mathfrak{s}$ satisfying (2.5.10). In the case of $(E_7, \{\alpha_6, \alpha_7\})$, according to Case (4) in Section 4 of [26], since $\dim \mathfrak{g}_{-3} \cdot (\dim \mathfrak{g}_{-1} - \dim \mathfrak{f}) - \dim \mathfrak{g}_{-2} = |\Phi_3^+| \cdot (|\Phi_1^+| - |\Psi^7|) - |\Phi_2^+| > 0$, $\mathfrak{m}$ cannot be isomorphic to $\mathfrak{s}$ satisfying (2.5.10).

Thus it is enough to investigate the other types: $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$ $(1 \leq i < j < k \leq l, (i, k) \neq (1, l))$, $(B_l, \{\alpha_1, \alpha_i\})$ $(3 \leq i \leq l)$, $(D_l, \{\alpha_1, \alpha_i\})$ $(3 \leq i \leq l-2)$, $(D_l, \{\alpha_1, \alpha_{l-1}, \alpha_l\})$, $(E_6, \{\alpha_4\})$, $(E_6, \{\alpha_1, \alpha_5\})$, $(E_7, \{\alpha_3\})$, $(E_7, \{\alpha_5\})$, $(E_7, \{\alpha_2, \alpha_7\})$, $(E_8, \{\alpha_2\})$, $(E_8, \{\alpha_7\})$, $(F_4, \{\alpha_2\})$, $(G_2, \{\alpha_1\})$.

Remark 2.8 Especially $(B_l, \{\alpha_1, \alpha_3\})$, $(D_l, \{\alpha_1, \alpha_3\})$, $(D_4, \{\alpha_1, \alpha_3, \alpha_4\})$, $(E_6, \{\alpha_4\})$, $(E_7, \{\alpha_3\})$, $(E_7, \{\alpha_6, \alpha_7\})$, $(E_8, \{\alpha_7\})$, $(F_4, \{\alpha_2\})$, $(G_2, \{\alpha_1\})$ are appeared in G_2-geometry ([22, Section 6.2]). They arise as reductions of some PD-manifolds associated with systems of second order partial differential equations of one unknown function. We will see that they has no solutions as systems of second order partial differential equations of two unknown functions in Proposition 2.10 at page 74.

Now we begin to prove Theorem 2.7. Let (X_l, Δ_1) be one of the other types.

Let $\Phi_{\mathfrak{f}} = \left\{\alpha \in \Phi_1^+ \mid \mathfrak{g}_{-\alpha} \subset \mathfrak{f}\right\}$. Note that $\mathfrak{f} = \bigoplus_{\alpha \in \Phi_{\mathfrak{f}}} \mathfrak{g}_{-\alpha}$. Indeed, let $\alpha' \in \Phi_1^+ \setminus \Phi_{\mathfrak{f}}$ and let $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be a non-zero vector for $\alpha \in \Phi_1^+$. By definition, there exists $\beta \in \Phi_2^+$ such that $\alpha' + \beta \in \Phi_3^+$. Taking a non-zero vector $Y \in \mathfrak{g}_{-\beta}$, we have $[X_{-\alpha'}, Y] \neq 0$. If $X = \sum_{\alpha \in \Phi_1^+} A^\alpha X_{-\alpha} \in \mathfrak{f}$, then we have $A^{\alpha'} = 0$ since $[X, Y] = 0$. Therefore $X \in \bigoplus_{\alpha \in \Phi_{\mathfrak{f}}} \mathfrak{g}_{-\alpha}$.

We divide each cases (X_l, Δ_1) into sequent subsections:

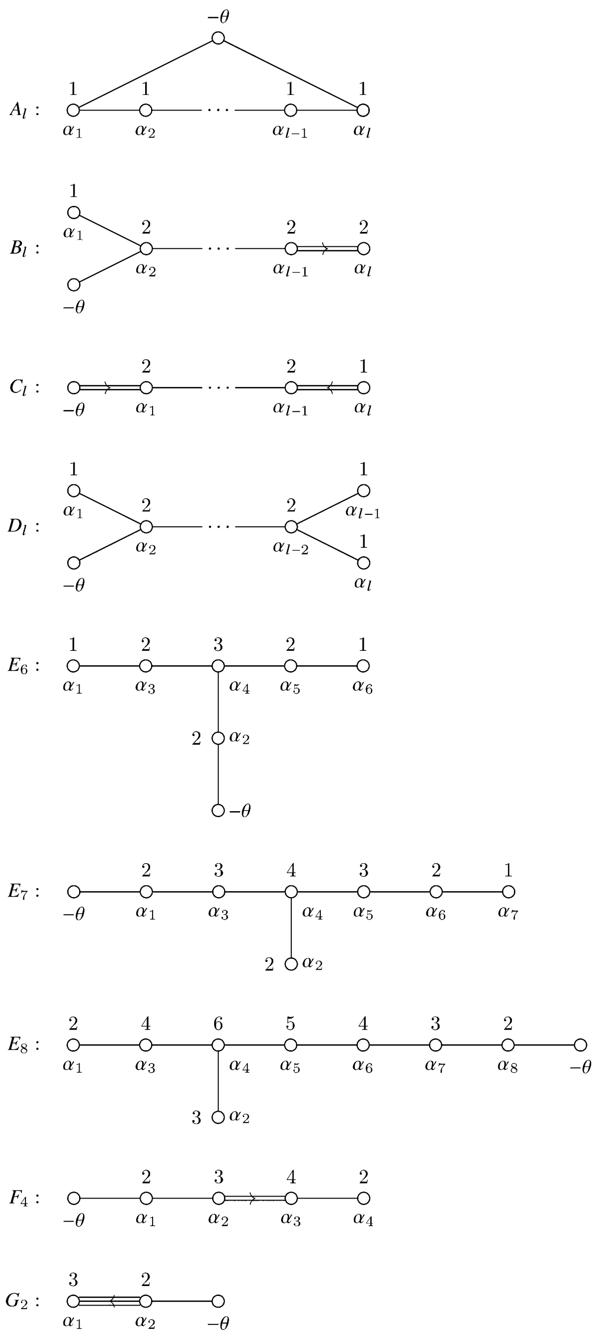

Figure 2.1. The Extended Dynkin Diagrams

2.6.1 $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$-type $(1 \le i < j < k \le l,\ (i,\, k) \ne (1,\, l))$

We easily see that the decomposition of Φ^+ with respective to Δ_1 is $\Phi^+ = \Phi_3^+ \cup \Phi_2^+ \cup \Phi_1^+$, where

$$\begin{aligned}
\Phi_3^+ &= \{\alpha_p + \cdots + \alpha_i + \cdots + \alpha_j + \cdots + \alpha_k + \cdots + \alpha_q \mid 1 \le p \le i,\ k \le q \le l\},\\
\Phi_2^+ &= \{\alpha_p + \cdots + \alpha_i + \cdots + \alpha_j + \cdots + \alpha_q \mid 1 \le p \le i,\ j \le q < k\}\\
&\quad \cup \{\alpha_p + \cdots + \alpha_j + \cdots + \alpha_k + \cdots + \alpha_q \mid i < p \le j,\ k \le q \le l\},\\
\Phi_1^+ &= \{\alpha_p + \cdots + \alpha_i + \cdots + \alpha_q \mid 1 \le p \le i \le q < j\}\\
&\quad \cup \{\alpha_p + \cdots + \alpha_j + \cdots + \alpha_q \mid i < p \le j \le q < k\}\\
&\quad \cup \{\alpha_p + \cdots + \alpha_k + \cdots + \alpha_q \mid j < p \le k \le q \le l\},\\
\Phi_{\mathfrak{f}} &= \{\alpha_p + \cdots + \alpha_j + \cdots + \alpha_q \mid i < p \le j \le q < k\}.
\end{aligned}$$

Then we obtain $|\Phi_3^+| = i(l-k+1)$, $|\Phi_2^+| = i(k-j) + (j-i)(l-k+1)$, $|\Phi_1^+| = i(j-i) + (j-i)(k-j) + (k-j)(l-k+1)$ and $\dim \mathfrak{f} = (j-i)(k-j)$. Therefore we have

$$\begin{aligned}
&|\Phi_3^+| \cdot (|\Phi_1^+| - \dim \mathfrak{f}) - |\Phi_2^+|\\
&\quad = (i-1)(i+1)(j-i)(l-k+1) + i(k-j)(l-k)(l-k+2) > 0,
\end{aligned} \tag{2.6.12}$$

which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Now we will describe a model equation of the PD-manifold of type $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$. We have the following matrix representation of $\mathfrak{sl}(l+1, \mathbb{C}) = \bigoplus_{p=-3}^{3} \mathfrak{g}_p$ of type $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$:

$$\mathfrak{g}_{-3} = \left\{ \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline Z & 0 & 0 & 0 \end{array} \right) \;\middle|\; Z \in M(l-k+1, i) \right\},$$

$$\mathfrak{g}_{-2} = \left\{ \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline P_1 & 0 & 0 & 0 \\ \hline 0 & P_2 & 0 & 0 \end{array} \right) \;\middle|\; \begin{array}{l} P_1 \in M(k-j, i), \\[1ex] P_2 \in M(l-k+1, j-i) \end{array} \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline X_1 & 0 & 0 & 0 \\ \hline 0 & F & 0 & 0 \\ \hline 0 & 0 & X_2 & 0 \end{array} \right) \;\middle|\; \begin{array}{l} X_1 \in M(j-i,i), \\ X_2 \in M(l-k+1,k-j), \\ F \in M(k-j,j-i) \end{array} \right\},$$

$$\mathfrak{g}_{0} = \left\{ \left(\begin{array}{c|c|c|c} L_1 & 0 & 0 & 0 \\ \hline 0 & L_2 & 0 & 0 \\ \hline 0 & 0 & L_3 & 0 \\ \hline 0 & 0 & 0 & L_4 \end{array} \right) \;\middle|\; \begin{array}{l} L_1 \in M(i,i),\, L_2 \in M(j-i,j-i), \\ L_3 \in M(k-j,k-j), \\ L_4 \in M(l-k+1,l-k+1), \\ \sum_{i=1}^{4} \operatorname{tr} L_i = 0 \end{array} \right\},$$

$$\mathfrak{g}_i = \left\{ {}^tX \mid X \in \mathfrak{g}_{-i} \right\} \quad \text{for } 1 \le i \le 3,$$

where $M(a,b)$ denotes the space of all $a \times b$ matrices.

Now we recall the formula for the Maurer-Cartan form on $M(\mathfrak{m})$ by N. Tanaka ([16, Section 2.3]):

Proposition 2.9 *Let $\mathfrak{m} = \bigoplus_{p=-3}^{-1} \mathfrak{g}_p$ be a fundamental graded Lie algebra of third kind over $\mathbb{R}$ and $(M(\mathfrak{m}), D_{\mathfrak{m}})$ the standard differential system of type $\mathfrak{m}$. Let u_{-p} denote the projection of $\mathfrak{m}$ onto $\mathfrak{g}_{-p}$ for $p = 1,\, 2,\, 3$, which may be regarded as a $\mathfrak{g}_{-p}$-valued function on $\mathfrak{m}$. Let η_{-p} be the $\mathfrak{g}_{-p}$-component of the Maurer-Cartan form of $M(\mathfrak{m})$. Then η_{-p} is expressed as follows*:

$$\begin{aligned} \eta_{-3} &= du_{-3} - \frac{1}{3}\,[u_{-2},\, du_{-1}] - \frac{2}{3}\,[u_{-1},\, du_{-2}] + \frac{1}{6}\,[u_{-1},\, [u_{-1},\, du_{-1}]], \\ \eta_{-2} &= du_{-2} - \frac{1}{2}\,[u_{-1},\, du_{-1}], \qquad (2.6.13) \\ \eta_{-1} &= du_{-1}. \end{aligned}$$

Here, $M(\mathfrak{m})$ is identified with $\mathfrak{m}$ by $f = \rho \circ S$, where ρ denotes the projection of the affine transformation group $AF(\mathfrak{m})$ of $\mathfrak{m}$ onto $\mathfrak{m}$ and $S : M(\mathfrak{m}) \longrightarrow AF(\mathfrak{m})$ is the injective homomorphism induced by the injective homomorphism s of $\mathfrak{m}$ into the Lie algebra $\mathfrak{af}(\mathfrak{m})$ of all infinitesimal affine transformations of $\mathfrak{m}$ defined by

$$s(X)(Y) = X + \sum_{p,\,q<0} \frac{q}{p+q}\,[u_p(X),\, u_q(Y)] \quad \textit{for } X,\, Y \in \mathfrak{m}.$$

Note that the same can apply to a fundamental graded Lie algebra of third kind over $\mathbb{C}$ with suitable modifications. Now we consider the holomorphic differential system $D_{\mathfrak{m}}$ on the simply connected complex Lie group $M(\mathfrak{m})$. By definition, we have the standard differential system $D_{\mathfrak{m}}$ of type $\mathfrak{m}$ as follows:

$$D_{\mathfrak{m}} = \{\eta_{-3} = \eta_{-2} = 0\}\,.$$

With respect to the matrix representation of $\mathfrak{sl}(l+1,\mathbb{C})$, we may write the holomorphic functions u_{-p} as

$$u_{-3} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline Z & 0 & 0 & 0 \end{array}\right), \qquad u_{-2} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline P_1 & 0 & 0 & 0 \\ \hline 0 & P_2 & 0 & 0 \end{array}\right),$$

$$u_{-1} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline X_1 & 0 & 0 & 0 \\ \hline 0 & F & 0 & 0 \\ \hline 0 & 0 & X_2 & 0 \end{array}\right).$$

Substituting them for the formula, we have

$$\eta_{-3} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \Theta_0 & 0 & 0 & 0 \end{array}\right), \qquad \eta_{-2} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \Theta_1 & 0 & 0 & 0 \\ \hline 0 & \Theta_2 & 0 & 0 \end{array}\right),$$

and

$$D_{\mathfrak{m}} = \{\Theta_0 = \Theta_1 = \Theta_2 = 0\}\,,$$

where

$$\begin{aligned}\Theta_0 = dZ &- \frac{1}{3}P_2 dX_1 + \frac{1}{3}dX_2 P_1 - \frac{2}{3}X_2 dP_1 + \frac{2}{3}dP_2 X_1 \\ &+ \frac{1}{6}X_2(F dX_1 - dF X_1) - \frac{1}{6}(X_2 dF - dX_2 F)X_1,\end{aligned}$$

$$\Theta_1 = dP_1 - \frac{1}{2}FdX_1 + \frac{1}{2}dFX_1,$$
$$\Theta_2 = dP_2 - \frac{1}{2}X_2dF + \frac{1}{2}dX_2F.$$

The exterior derivative of Θ_0 is

$$d\Theta_0 = -dX_2 \wedge d\left(P_1 + \frac{1}{2}FX_1\right) - d\left(P_2 - \frac{1}{2}X_2F\right) \wedge dX_1.$$

Putting

$$\hat{P}_1 = P_1 + \frac{1}{2}FX_1, \quad \hat{P}_2 = P_2 - \frac{1}{2}X_2F, \quad \hat{X}_1 = -X_2, \quad \hat{X}_2 = X_1,$$

we have

$$\begin{aligned}\Theta_0 &= dZ - \frac{1}{3}d\hat{X}_1(\hat{P}_1 + F\hat{X}_2) - \frac{1}{3}(\hat{P}_2 + \hat{X}_1F)\,d\hat{X}_2 - \frac{1}{3}\hat{X}_1dF\hat{X}_2 \\ &\quad + \frac{2}{3}\hat{X}_1d\hat{P}_1 + \frac{2}{3}d\hat{P}_2\hat{X}_2, \\ \Theta_1 &= d\hat{P}_1 - Fd\hat{X}_2, \\ \Theta_2 &= d\hat{P}_2 - d\hat{X}_1F,\end{aligned}$$

$$\begin{aligned}d\Theta_0 &= d\hat{X}_1 \wedge d\hat{P}_1 - d\hat{P}_2 \wedge d\hat{X}_2, \\ d\Theta_1 &= -dF \wedge d\hat{X}_2, \\ d\Theta_2 &= d\hat{X}_1 \wedge dF.\end{aligned}$$

Digressing from determinating of the model equation, we now show theoretically that $(M(\mathfrak{m}), D_\mathfrak{m})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$ over some manifold Q. From the structure equation of $D_\mathfrak{m}$, we have

$$\begin{aligned}\partial D_\mathfrak{m} &= \{\Theta_0 = 0\}, \\ \mathrm{Ch}(\partial D_\mathfrak{m}) &= \big\{\Theta_0 = \Theta_1 = \Theta_2 = d\hat{X}_1 = d\hat{X}_2 = 0\big\},\end{aligned} \tag{2.6.14}$$

which are differential systems of codimension n_3 and $n_3 + n_2 + (n_1 - f)$ respectively. Here, let $n_i = \dim \mathfrak{g}_{-i}$ for $1 \le i \le 3$ and $f = \dim \mathfrak{f}$. Putting

$$F = \{\Theta_0 = d\hat{X}_1 = d\hat{X}_2 = 0\},$$

we see that F is a completely integrable differential system of codimension $n_3 + (n_1 - f)$. Let $Q = M(\mathfrak{m})/F$ be spaces of leaves of the foliation and $p : M(\mathfrak{m}) \longrightarrow Q$ the projection. Then $\operatorname{Ker} p_* = F$ is a subbundle of $\partial D_{\mathfrak{m}}$ of codimension $n_1 - f$. By applying Realization Lemma to p and $\partial D_{\mathfrak{m}}$, we see that there exists a unique map $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1 - f)$ satisfying $p = \Pi_Q \circ \psi$ and $\partial D_{\mathfrak{m}} = \psi_*^{-1}(C^1)$, where $\Pi_Q : J^1(Q, n_1 - f) \longrightarrow Q$ is the projection, and see that $\operatorname{Ker} \psi_* = \operatorname{Ch}(\partial D_{\mathfrak{m}})$. Moreover, by applying Realization Lemma to ψ and $D_{\mathfrak{m}}$, we have a map $\varphi : M(\mathfrak{m}) \longrightarrow J^1(J^1(Q, n_1 - f), n_1 - f)$ as in the lemma. Since $\operatorname{Ker} \varphi_* = \operatorname{Ker} \psi_* \cap \operatorname{Ch}(D_{\mathfrak{m}}) = \{0\}$, φ is an immersion. Since $\varphi(v) = \psi_*(D_{\mathfrak{m}}(v))$ for $v \in M(\mathfrak{m})$ and $\partial D_{\mathfrak{m}} = \psi_*^{-1}(C^1)$, we see that $\varphi(v)$ is a $(n_1 - f)$-dimensional integral element of C^1. Moreover, it follows from $F \cap D_{\mathfrak{m}} = \operatorname{Ch}(\partial D_{\mathfrak{m}})$ that $\varphi(v) \cap \operatorname{Ker} \Pi_{Q*}(\psi(v)) = \{0\}$. Thus we have $\varphi(v) \in J^2(Q, n_1 - f)$. By the definition of φ, we have $D_{\mathfrak{m}} = \varphi_*^{-1}(C^2)$. Note that $\psi = \Pi_1^2 \circ \varphi$ is not a submersion, where $\Pi_1^2 : J^2(Q, n_1 - f) \longrightarrow J^1(Q, n_1 - f)$ is the projection, since $\dim J^1(Q, n_1 - f) - \operatorname{rank} \psi = n_3(n_1 - f) - n_2 > 0$ (see (2.6.12)).

$$\begin{tikzcd}
 & J^1(Q, n_1 - f) \arrow[d, "\Pi_Q"] \\
M(\mathfrak{m}) \arrow[ur, "\psi"] \arrow[r, "p"'] & Q = M(\mathfrak{m})/F
\end{tikzcd}
\qquad
\begin{tikzcd}
 & J^1(J^1(Q, n_1 - f), n_1 - f) \arrow[d] \\
M(\mathfrak{m}) \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & J^1(Q, n_1 - f)
\end{tikzcd}$$

Now we return to the calculation of the model equation. From $F = \{\Theta_0 = d\hat{X}_1 = d\hat{X}_2 = 0\}$, we calculate

$$\Theta_0 \equiv d\Big(Z - \frac{1}{3}\hat{X}_1 F \hat{X}_2 + \frac{2}{3}\hat{X}_1 \hat{P}_1 + \frac{2}{3}\hat{P}_2 \hat{X}_2\Big) \quad (\operatorname{mod} d\hat{X}_1,\, d\hat{X}_2).$$

Putting $\hat{Z} = Z - (1/3)\hat{X}_1 F \hat{X}_2 + (2/3)\hat{X}_1 \hat{P}_1 + (2/3)\hat{P}_2 \hat{X}_2$, we have achieved a normal form of $D_{\mathfrak{m}}$:

$$\Theta_0 = d\hat{Z} - d\hat{X}_1 \hat{P}_1 - \hat{P}_2 d\hat{X}_2,$$

$$\Theta_1 = d\hat{P}_1 - F d\hat{X}_2,$$

$$\Theta_2 = d\hat{P}_2 - d\hat{X}_1 F.$$

From now on, fix index ranges $1 \leq \alpha,\ \beta \leq i,\ i+1 \leq m,\ n \leq j,\ j+1 \leq s,\ t \leq k$, and $k+1 \leq a,\ b \leq l+1$. Setting

$$\hat{X}_1 = (x^a_s), \quad \hat{X}_2 = (y^m_\alpha), \quad F = (f^s_m), \quad \hat{Z} = (z^a_\alpha), \quad \hat{P}_1 = (p^s_\alpha), \quad \hat{P}_2 = (q^a_m),$$

we have

$$\begin{aligned}
\Theta_0 &= \left(dz^a_\alpha - \sum_t p^t_\alpha dx^a_t - \sum_n q^a_n dy^n_\alpha \right)_{a,\,\alpha}, \\
\Theta_1 &= \left(dp^s_\alpha - \sum_n f^s_n dy^n_\alpha \right)_{s,\,\alpha}, \\
\Theta_2 &= \left(dq^a_m - \sum_t f^t_m dx^a_t \right)_{a,\,m}.
\end{aligned}$$

Therefore we have a model equation of $(A_l, \{\alpha_i, \alpha_j, \alpha_k\})$ $(1 \leq i < j < k \leq l,\ (i,\, k) \neq (1,\, l))$ as follows:

$$\begin{cases}
\dfrac{\partial z^b_\alpha}{\partial x^a_s} = \dfrac{\partial z^a_\beta}{\partial y^m_\alpha} = 0 \quad \text{for } a \neq b,\ \alpha \neq \beta, \\[2ex]
\dfrac{\partial z^a_\alpha}{\partial x^a_s} = \dfrac{\partial z^b_\alpha}{\partial x^b_s}, \quad \dfrac{\partial z^a_\alpha}{\partial y^m_\alpha} = \dfrac{\partial z^a_\beta}{\partial y^m_\beta}, \\[2ex]
\dfrac{\partial^2 z^a_\alpha}{\partial x^a_s \partial x^a_t} = 0, \quad \dfrac{\partial^2 z^a_\alpha}{\partial y^m_\alpha \partial y^n_\alpha} = 0,
\end{cases}$$

where x^a_s, y^m_α and z^a_α are independent variables and unknown functions respectively.

Especially, a model equation of type $(A_4, \{\alpha_1, \alpha_2, \alpha_3\})$ is

$$\begin{cases}
\dfrac{\partial z_1}{\partial x_2} = \dfrac{\partial z_2}{\partial x_3}, \quad \dfrac{\partial z_1}{\partial x_3} = \dfrac{\partial z_2}{\partial x_2} = 0, \\[2ex]
\dfrac{\partial^2 z_1}{\partial x_1 \partial x_1} = \dfrac{\partial^2 z_1}{\partial x_2 \partial x_2} = \dfrac{\partial^2 z_2}{\partial x_1 \partial x_1} = 0,
\end{cases}$$

where x_1, x_2, x_3 and z_1, z_2 are independent variables and unknown functions respectively.

2.6.2 $(B_l, \{\alpha_1, \alpha_i\})$-type $(3 \le i \le l)$

$$\begin{aligned}
\Phi_3^+ &= \{\alpha_1 + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_i + \cdots + 2\alpha_l \mid 1 \le p < i\}\,, \\
\Phi_2^+ &= \{\alpha_1 + \cdots + \alpha_i + \cdots + \alpha_p \mid i \le p \le l\} \\
&\quad \cup \{\alpha_1 + \cdots + \alpha_i + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_l \mid i \le p < l\} \\
&\quad \cup \{\alpha_p + \cdots + \alpha_{q-1} + 2\alpha_q + \cdots + 2\alpha_i + \cdots + 2\alpha_l \mid 1 < p < q \le i\}\,, \\
\Phi_1^+ &= \{\alpha_1 + \cdots + \alpha_p \mid 1 \le p < i\} \\
&\quad \cup \{\alpha_p + \cdots + \alpha_i + \cdots + \alpha_q \mid 1 < p \le i \le q \le l\} \\
&\quad \cup \{\alpha_p + \cdots + \alpha_i + \cdots + \alpha_q + 2\alpha_{q+1} + \cdots + 2\alpha_l \mid 1 < p \le i \le q < l\}\,.
\end{aligned}$$

Then we see that $\Phi_{\mathfrak{f}} = \emptyset$. In fact, for any $\alpha \in \Phi_1^+$, there exists $\beta \in \Phi_2^+$ satisfying $\alpha + \beta \in \Phi_3^+$ according to the following list:

$\alpha \in \Phi_1^+$	$\beta \in \Phi_2^+$
$\alpha_1 + \cdots + \alpha_p$ $(1 \le p < i-1)$	$\alpha_{p+1} + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_l$
$\alpha_1 + \cdots + \alpha_{i-1}$	$\alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_l$
$\alpha_p + \cdots + \alpha_i + \cdots + \alpha_q$ $(1 < p \le i \le q < l)$	$\alpha_{p+1} + \cdots + \alpha_q + 2\alpha_{q+1} + \cdots + 2\alpha_l$
$\alpha_p + \cdots + \alpha_i + \cdots + \alpha_l$ $(1 < p \le i)$	$\alpha_1 + \cdots + \alpha_p + \cdots + \alpha_i + \cdots + \alpha_l$
$\alpha_p + \cdots + \alpha_i + \cdots + \alpha_q$ $+2\alpha_{q+1} + \cdots + 2\alpha_l$ $(1 < p \le i \le q < l)$	$\alpha_1 + \cdots + \alpha_p + \cdots + \alpha_i + \cdots + \alpha_q$

Therefore we have $\mathfrak{f} = \{0\}$, which implies $\mathfrak{m}$ cannot satisfy (2.5.10).

Now we will describe a model equation of the PD-manifold of type $(B_l, \{\alpha_1, \alpha_i\})$ $(3 \le i \le l)$. Let $n = 2l+1$ and let E_k be the identity matrix of size k. We have

$$\mathfrak{o}(n, \mathbb{C}) = \left\{ X \in \mathfrak{gl}(n, \mathbb{C}) \mid {}^t X J + J X = 0 \right\},$$

where

$$J = \left(\begin{array}{c|c|c|c|c} & & & & 1 \\ \hline & & & E_{i-1} & \\ \hline & & E_{n-2i} & & \\ \hline & E_{i-1} & & & \\ \hline 1 & & & & \end{array}\right).$$

Then we have the following matrix representation of $\mathfrak{o}(n,\mathbb{C}) = \bigoplus_{p=-3}^{3} \mathfrak{g}_p$ of type $(B_l, \{\alpha_1, \alpha_i\})$:

$$\mathfrak{g}_{-3} = \left\{ \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline Z & 0 & 0 & 0 & 0 \\ \hline 0 & -{}^tZ & 0 & 0 & 0 \end{array}\right) \,\middle|\, Z \in M(i-1,1) \right\},$$

$$\mathfrak{g}_{-2} = \left\{ \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline P_1 & 0 & 0 & 0 & 0 \\ \hline 0 & P_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^tP_1 & 0 & 0 \end{array}\right) \,\middle|\, \begin{array}{l} P_1 \in M(n-2i,1), \\ \\ P_2 \in \mathfrak{o}(i-1,\mathbb{C}) \end{array} \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline X_1 & 0 & 0 & 0 & 0 \\ \hline 0 & X_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^tX_2 & 0 & 0 \\ \hline 0 & 0 & 0 & -{}^tX_1 & 0 \end{array}\right) \,\middle|\, \begin{array}{l} X_1 \in M(i-1,1), \\ \\ X_2 \in M(n-2i,i-1) \end{array} \right\},$$

$$\mathfrak{g}_{0} = \left\{ \left(\begin{array}{c|c|c|c|c} L_1 & 0 & 0 & 0 & 0 \\ \hline 0 & L_2 & 0 & 0 & 0 \\ \hline 0 & 0 & L_3 & 0 & 0 \\ \hline 0 & 0 & 0 & -{}^tL_2 & 0 \\ \hline 0 & 0 & 0 & 0 & -{}^tL_1 \end{array}\right) \,\middle|\, \begin{array}{l} L_1 \in \mathbb{C},\ L_2 \in M(i-1,i-1), \\ \\ L_3 \in \mathfrak{o}(n-2i,\mathbb{C}) \end{array} \right\},$$

$$\mathfrak{g}_i = \left\{ {}^tX \mid X \in \mathfrak{g}_{-i} \right\} \quad \text{for } 1 \leq i \leq 3.$$

With respect to the matrix representation of $\mathfrak{o}(n,\mathbb{C})$, we may write the projection $u_{-p} : \mathfrak{m} \longrightarrow \mathfrak{g}_{-p}$ as

$$u_{-3} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline Z & 0 & 0 & 0 & 0 \\ \hline 0 & -{}^tZ & 0 & 0 & 0 \end{array}\right), \qquad u_{-2} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline P_1 & 0 & 0 & 0 & 0 \\ \hline 0 & P_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^tP_1 & 0 & 0 \end{array}\right) \ (P_2 = -{}^tP_2),$$

$$u_{-1} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline X_1 & 0 & 0 & 0 & 0 \\ \hline 0 & X_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^tX_2 & 0 & 0 \\ \hline 0 & 0 & 0 & -{}^tX_1 & 0 \end{array}\right).$$

Substituting them for (2.6.13), we have

$$\eta_{-3} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \Theta_0 & 0 & 0 & 0 & 0 \\ \hline 0 & -{}^t\Theta_0 & 0 & 0 & 0 \end{array}\right), \qquad \eta_{-2} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \Theta_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \Theta_2 & 0 & 0 & 0 \\ \hline 0 & 0 & -{}^t\Theta_1 & 0 & 0 \end{array}\right),$$

and

$$D_{\mathfrak{m}} = \{\Theta_0 = \Theta_1 = \Theta_2 = 0\},$$

where

$$\begin{aligned} \Theta_0 &= dZ - \frac{1}{3}P_2 dX_1 - \frac{1}{3}d{}^tX_2 P_1 + \frac{2}{3}{}^tX_2 dP_1 + \frac{2}{3}dP_2 X_1 \\ &\quad - \frac{1}{6}{}^tX_2(X_2 dX_1 - dX_2 X_1) + \frac{1}{6}({}^tX_2 dX_2 - d{}^tX_2 X_2)X_1, \\ \Theta_1 &= dP_1 - \frac{1}{2}X_2 dX_1 + \frac{1}{2}dX_2 X_1, \\ \Theta_2 &= dP_2 + \frac{1}{2}{}^tX_2 dX_2 - \frac{1}{2}d{}^tX_2 X_2. \end{aligned}$$

The exterior derivative of Θ_0 is

$$d\Theta_0 = -d\left(P_2 + \frac{1}{2}{}^tX_2X_2\right) \wedge dX_1 + d^tX_2 \wedge d\left(P_1 + \frac{1}{2}X_2X_1\right).$$

Putting

$$\hat{P}_1 = P_1 + \frac{1}{2}X_2X_1, \qquad \hat{P}_2 = P_2 + \frac{1}{2}{}^tX_2X_2,$$

we have

$$\begin{aligned}
\Theta_0 &= dZ - \frac{1}{3}(\hat{P}_2 + {}^tX_2X_2)\,dX_1 - \frac{1}{3}d^tX_2(\hat{P}_1 + X_2X_1) - \frac{1}{3}{}^tX_2dX_2X_1 \\
&\quad + \frac{2}{3}{}^tX_2d\hat{P}_1 + \frac{2}{3}d\hat{P}_2X_1, \\
\Theta_1 &= d\hat{P}_1 - X_2dX_1, \\
\Theta_2 &= d\hat{P}_2 - d^tX_2X_2,
\end{aligned}$$

$$\begin{aligned}
d\Theta_0 &= -d\hat{P}_2 \wedge dX_1 + d^tX_2 \wedge d\hat{P}_1, \\
d\Theta_1 &= -dX_2 \wedge dX_1, \\
d\Theta_2 &= d^tX_2 \wedge dX_2.
\end{aligned}$$

Digressing from determinating of the model equation, we now show theoretically that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$ over some manifold Q. From the structure equation of $D_{\mathfrak{m}}$, we have

$$\begin{aligned}
\partial D_{\mathfrak{m}} &= \{\Theta_0 = 0\}, \\
\mathrm{Ch}(\partial D_{\mathfrak{m}}) &= \left\{\Theta_0 = \Theta_1 = \Theta_2 = dX_1 = d^tX_2 = 0\right\} = \{0\},
\end{aligned} \tag{2.6.15}$$

which are differential systems of codimension n_3 and $n_3 + n_2 + n_1$. Here, let $n_i = \dim \mathfrak{g}_{-i}$ for $1 \leq i \leq 3$. Putting

$$F = \left\{\Theta_0 = dX_1 = d^tX_2 = 0\right\} = \{\Theta_0 = dX_1 = dX_2 = 0\},$$

we see that F is a completely integrable differential system of codimension $n_3 + n_1$. Let $Q = M(\mathfrak{m})/F$ be spaces of leaves of the foliation and $p : M(\mathfrak{m}) \longrightarrow Q$ the projection. Then $\mathrm{Ker}\, p_* = F$ is a subbundle of $\partial D_{\mathfrak{m}}$ of codimension n_1. By applying Realization Lemma to p and $\partial D_{\mathfrak{m}}$, we see that there exists a unique map $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ satisfying $p = \Pi_Q \circ \psi$

and $\partial D_{\mathfrak{m}} = \psi_*^{-1}(C^1)$, where $\Pi_Q : J^1(Q, n_1) \longrightarrow Q$ is the projection, and see that $\operatorname{Ker}\psi_* = F \cap \operatorname{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Moreover, by applying Realization Lemma to ψ and $D_{\mathfrak{m}}$, we have a map $\varphi : M(\mathfrak{m}) \longrightarrow J^1(J^1(Q, n_1), n_1)$ as in the lemma. Since $\operatorname{Ker}\varphi_* = \operatorname{Ker}\psi_* \cap \operatorname{Ch}(D_{\mathfrak{m}}) = \{0\}$, φ is an immersion. Since $\varphi(v) = \psi_*(D_{\mathfrak{m}}(v))$ for $v \in M(\mathfrak{m})$ and $\psi_*^{-1}(C^1) = \partial D_{\mathfrak{m}}$, we see that $\varphi(v)$ is a n_1-dimensional integral element of C^1. Moreover, it follows from $F \cap D_{\mathfrak{m}} = \{0\}$ that $\varphi(v) \cap \operatorname{Ker}\Pi_{Q*}(\psi(v)) = \{0\}$. Thus we have $\varphi(v) \in J^2(Q, n_1)$. By the definition of φ, we have $D_{\mathfrak{m}} = \varphi_*^{-1}(C^2)$. Note that $\dim J^1(Q, n_1) - \dim M(\mathfrak{m}) = n_3 n_1 - n_2 > 0$, namely ψ is not submersion.

Now we return to the calculation of the model equation. From $F = \{\Theta_0 = dX_1 = dX_2 = 0\} = \{\Theta_0 = dX_1 = d^tX_2 = 0\}$, we calculate

$$\Theta_0 \equiv d\left(Z - \frac{1}{3}{}^tX_2X_2X_1 + \frac{2}{3}{}^tX_2\hat{P}_1 + \frac{2}{3}\hat{P}_2X_1\right) \quad (\mathrm{mod}\ dX_1,\ d^tX_2).$$

Putting $\hat{Z} = Z - (1/3){}^tX_2X_2X_1 + (2/3){}^tX_2\hat{P}_1 + (2/3)\hat{P}_2X_1$, we have achieved a normal form of $D_{\mathfrak{m}}$:

$$\begin{aligned} \Theta_0 &= d\hat{Z} - d^tX_2\hat{P}_1 - \hat{P}_2 dX_1, \\ \Theta_1 &= d\hat{P}_1 - X_2 dX_1, \\ \Theta_2 &= d\hat{P}_2 - d^tX_2X_2. \end{aligned}$$

Since $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $J^2(Q, n_1)$, we should think n_1-dimensional integral elements and manifolds where dX_1 and dX_2 never vanish. However, by $d\Theta_1 = -dX_2 \wedge dX_1$, there are no such integral elements and manifolds.

Now we generalize the above discussion as follows:

Proposition 2.10 *Let $\mathfrak{m}$ be a fundamental graded Lie algebra of third kind and $(M(\mathfrak{m}), D_{\mathfrak{m}})$ the standard differential system of type $\mathfrak{m}$. Assume $\operatorname{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Then $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$ and furthermore $(M(\mathfrak{m}), D_{\mathfrak{m}})$ has no n_1-dimensional integral elements and manifolds, where $n_1 = \dim \mathfrak{g}_{-1}$.*

Proof. Let η_{-p} be the $\mathfrak{g}_{-p}$-component of the Maurer-Cartan form of $M(\mathfrak{m})$. From (2.6.13) in Section 2.6.1, we have the structure equation of $D_{\mathfrak{m}}$:

$$\begin{cases} d\eta_{-3} = -[\eta_{-1},\, \eta_{-2}], \\ d\eta_{-2} = -\dfrac{1}{2}[\eta_{-1},\, \eta_{-1}], \end{cases} \tag{2.6.16}$$

Therefore we have $\partial D_{\mathfrak{m}} = \{\eta_{-3} = 0\}$. Let $F = \{\eta_{-3} = \eta_{-1} = 0\}$. It follows from the above structure equation that F is completely integrable. Let $Q = M(\mathfrak{m})/F$ be the space of leaves of the foliation and $p : M(\mathfrak{m}) \longrightarrow Q$ the projection. By applying Realization Lemma to p and $\partial D_{\mathfrak{m}}$, we have a unique map $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ such that $p = \Pi_Q \circ \psi$ and $\partial D_{\mathfrak{m}} = \psi_*^{-1}(C^1)$, where $\Pi_Q : J^1(Q, n_1) \longrightarrow Q$ is the projection. Then we have $\operatorname{Ker}\psi_* = F \cap \operatorname{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. Moreover, by applying Realization Lemma to ψ and $D_{\mathfrak{m}}$, we have a map $\varphi : M(\mathfrak{m}) \longrightarrow J^1(J^1(Q, n_1), n_1)$ as in the lemma. Then φ is an immersion since $\operatorname{Ker}\varphi_* = \operatorname{Ker}\psi_* \cap \operatorname{Ch}(D_{\mathfrak{m}}) = \{0\}$. Since $\partial D_{\mathfrak{m}} = \psi_*^{-1}(C^1)$, $\varphi(v)$ is a n_1-dimensional integral element of C^1 for $v \in M(\mathfrak{m})$. It follows from $F \cap D_{\mathfrak{m}} = \{0\}$ that $\varphi(v) \cap \operatorname{Ker}{\Pi_Q}_*(\psi(v)) = \{0\}$ for $v \in M(\mathfrak{m})$, which implies $\varphi(v) \in J^2(Q, n_1)$. By definition of φ, we see that $D_{\mathfrak{m}} = \varphi_*^{-1}(C^2)$. Thus we have found that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into $(J^2(Q, n_1), C^2)$. Regarding $M(\mathfrak{m})$ as a submanifold of $J^2(Q, n_1)$, we consider n_1-dimensional integral elements ν with the independence condition $\eta_{-1}|_\nu \neq 0$. However, it follows from $d\eta_{-2} = -\frac{1}{2}[\eta_{-1},\, \eta_{-1}]$ that $D_{\mathfrak{m}}$ has no such integral elements. $\square$

$$\begin{tikzcd} & J^1(Q, n_1) \arrow[d, "\Pi_Q"] \\ M(\mathfrak{m}) \arrow[ur, "\psi"] \arrow[r, "p"'] & Q = M(\mathfrak{m})/F \end{tikzcd} \qquad \begin{tikzcd} & J^1(J^1(Q, n_1), n_1) \arrow[d] \\ M(\mathfrak{m}) \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & J^1(Q, n_1) \end{tikzcd}$$

2.6.3 $(D_l, \{\alpha_1, \alpha_i\})$-type $(3 \leq i \leq l-2)$

$$\begin{aligned} \Phi_3^+ = \{&\alpha_1 + \cdots + \alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_i \\ &+ \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid 1 < p \leq i\}, \\ \Phi_2^+ = \{&\alpha_1 + \cdots + \alpha_i + \cdots + \alpha_p \mid i \leq p \leq l\} \\ &\cup \{\alpha_1 + \cdots + \alpha_i + \cdots + \alpha_{l-2} + \alpha_l\} \\ &\cup \{\alpha_1 + \cdots + \alpha_i + \cdots + \alpha_p + 2\alpha_{p+1} \\ &\qquad + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid i \leq p \leq l-3\} \end{aligned}$$

$$
\begin{aligned}
&\cup \{\alpha_p + \cdots + \alpha_{q-1} + 2\alpha_q + \cdots + 2\alpha_i \\
&\qquad + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid 1 < p < q \leq i\}, \\
\Phi_1^+ &= \{\alpha_1 + \cdots + \alpha_p \mid 1 \leq p < i\} \\
&\cup \{\alpha_p + \cdots + \alpha_i + \cdots + \alpha_q \mid 1 < p \leq i \leq q \leq l\} \\
&\cup \{\alpha_p + \cdots + \alpha_i + \cdots + \alpha_{l-2} + \alpha_l \mid 1 < p \leq i\} \\
&\cup \big\{\alpha_p + \cdots + \alpha_i + \cdots + \alpha_{q-1} + 2\alpha_q \\
&\qquad + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid 1 < p \leq i < q \leq l-2\big\},
\end{aligned}
$$

Then we see that $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.4 $(D_l, \{\alpha_1, \alpha_{l-1}, \alpha_l\})$-type

$$
\begin{aligned}
\Phi_3^+ &= \{\alpha_1 + \cdots + \alpha_{l-1} + \alpha_l\} \\
&\cup \{\alpha_1 + \cdots + \alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid 1 < p \leq l-2\}, \\
\Phi_2^+ &= \{\alpha_1 + \cdots + \alpha_{l-1}\} \cup \{\alpha_1 + \cdots + \alpha_{l-2} + \alpha_l\} \\
&\cup \{\alpha_p + \cdots + \alpha_{l-1} + \alpha_l \mid 1 < p \leq l-2\} \\
&\cup \big\{\alpha_p + \cdots + \alpha_{q-1} + 2\alpha_q \\
&\qquad + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \mid 1 < p < q < l-1\big\} \\
\Phi_1^+ &= \{\alpha_1 + \cdots + \alpha_p \mid 1 \leq p < l-1\} \\
&\cup \{\alpha_p + \cdots + \alpha_{l-1} \mid 1 < p \leq l-1\} \\
&\cup \{\alpha_p + \cdots + \alpha_{l-2} + \alpha_l \mid 1 < p \leq l-2\} \cup \{\alpha_l\},
\end{aligned}
$$

Then we see that $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_{\mathfrak{m}}$

has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.5 $(E_6, \{\alpha_4\})$-type

Let $\begin{smallmatrix} c_1 & c_3 & c_4 & c_5 & c_6 \\ & & c_2 & & \end{smallmatrix}$ denote the root $c_1\alpha_1 + \cdots + c_6\alpha_6$ of E_6.

Φ_3^+ consists of the following roots:

$$\begin{array}{cc} 1\,2\,3\,2\,1 & 1\,2\,3\,2\,1 \\ 1 & 2 \end{array}$$

Φ_2^+ consists of the following roots:

$$\begin{array}{ccccccc} 0\,1\,2\,1\,0 & 1\,1\,2\,1\,0 & 0\,1\,2\,1\,1 & 1\,2\,2\,1\,0 & 1\,1\,2\,1\,1 & 0\,1\,2\,2\,1 & 1\,2\,2\,1\,1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1\,1\,2\,2\,1 & 1\,2\,2\,2\,1 & & & & & \\ 1 & 1 & & & & & \end{array}$$

Φ_1^+ consists of the following roots:

$$\begin{array}{ccccccc} 0\,0\,1\,0\,0 & 0\,1\,1\,0\,0 & 0\,0\,1\,1\,0 & 0\,0\,1\,0\,0 & 1\,1\,1\,0\,0 & 0\,1\,1\,1\,0 & 0\,1\,1\,0\,0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0\,0\,1\,1\,1 & 0\,0\,1\,1\,0 & 1\,1\,1\,1\,0 & 1\,1\,1\,0\,0 & 0\,1\,1\,1\,1 & 0\,1\,1\,1\,0 & 0\,0\,1\,1\,1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1\,1\,1\,1\,1 & 1\,1\,1\,1\,0 & 0\,1\,1\,1\,1 & 1\,1\,1\,1\,1 & & & \\ 0 & 1 & 1 & 1 & & & \end{array}$$

Then we see that $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.6 $(E_6, \{\alpha_1, \alpha_5\})$-type

Φ_3^+ consists of the following roots:

$$\begin{array}{cccc} 1\,1\,2\,2\,1 & 1\,2\,2\,2\,1 & 1\,2\,3\,2\,1 & 1\,2\,3\,2\,1 \\ 1 & 1 & 1 & 2 \end{array}$$

Φ_2^+ consists of the following roots:

$$\begin{array}{ccccccc}
1\,1\,1\,1\,0 & 1\,1\,1\,1\,1 & 1\,1\,1\,1\,0 & 1\,1\,1\,1\,1 & 1\,1\,2\,1\,0 & 1\,2\,2\,1\,0 & 1\,1\,2\,1\,1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0\,1\,2\,2\,1 & 1\,2\,2\,1\,1 & & & & & \\
1 & 1 & & & & &
\end{array}$$

Φ_1^+ consists of the following roots:

$$\begin{array}{ccccccc}
1\,0\,0\,0\,0 & 0\,0\,0\,1\,0 & 1\,1\,0\,0\,0 & 0\,0\,1\,1\,0 & 0\,0\,0\,1\,1 & 1\,1\,1\,0\,0 & 0\,1\,1\,1\,0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0\,0\,1\,1\,1 & 0\,0\,1\,1\,0 & 1\,1\,1\,0\,0 & 0\,1\,1\,1\,1 & 0\,1\,1\,1\,0 & 0\,0\,1\,1\,1 & 0\,1\,1\,1\,1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0\,1\,2\,1\,0 & 0\,1\,2\,1\,1 & & & & & \\
1 & 1 & & & & &
\end{array}$$

Then we see that $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.7 $(E_7, \{\alpha_3\})$-type

Let $\left(\begin{smallmatrix} c_1 & c_3 & c_4 & c_5 & c_6 & c_7 \\ & & c_2 & & & \end{smallmatrix}\right)$ denote the root $c_1\alpha_1 + \cdots + c_7\alpha_7$ of E_7,

Φ_3^+ consists of the following roots:

$$\begin{array}{cc}
1\,3\,4\,3\,2\,1 & 2\,3\,4\,3\,2\,1 \\
2 & 2
\end{array}$$

Φ_2^+ consists of the following roots:

$$\begin{array}{cccccc}
1\,2\,2\,1\,0\,0 & 1\,2\,2\,1\,1\,0 & 1\,2\,2\,2\,1\,0 & 1\,2\,3\,2\,1\,0 & 1\,2\,3\,2\,1\,0 & 1\,2\,2\,1\,1\,1 \\
1 & 1 & 1 & 1 & 2 & 1 \\
1\,2\,2\,2\,1\,1 & 1\,2\,2\,2\,2\,1 & 1\,2\,3\,2\,1\,1 & 1\,2\,3\,2\,2\,1 & 1\,2\,3\,2\,1\,1 & 1\,2\,3\,3\,2\,1 \\
1 & 1 & 1 & 1 & 2 & 1 \\
1\,2\,3\,2\,2\,1 & 1\,2\,3\,3\,2\,1 & 1\,2\,4\,3\,2\,1 & & & \\
2 & 2 & 2 & & &
\end{array}$$

Φ_1^+ consists of the following roots:

0 1 0 0 0 0	1 1 0 0 0 0	0 1 1 0 0 0	1 1 1 0 0 0	0 1 1 1 0 0	0 1 1 0 0 0
0	0	0	0	0	1
1 1 1 1 0 0	1 1 1 0 0 0	0 1 1 1 1 0	0 1 1 1 0 0	1 1 1 1 1 0	1 1 1 1 0 0
0	1	0	1	0	1
0 1 1 1 1 1	0 1 1 1 1 0	1 1 1 1 1 1	1 1 1 1 1 0	0 1 1 1 1 1	1 1 1 1 1 1
0	1	0	1	1	1
0 1 2 1 0 0	1 1 2 1 0 0	0 1 2 1 1 0	1 1 2 1 1 0	0 1 2 2 1 0	1 1 2 2 1 0
1	1	1	1	1	1
0 1 2 1 1 1	1 1 2 1 1 1	0 1 2 2 1 1	1 1 2 2 1 1	0 1 2 2 2 1	1 1 2 2 2 1
1	1	1	1	1	1

Then we see that $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.8 $(E_7, \{\alpha_5\})$-type

Φ_3^+ consists of the following roots:

1 2 3 3 2 1	1 2 3 3 2 1	1 2 4 3 2 1	1 3 4 3 2 1	2 3 4 3 2 1
1	2	2	2	2

Φ_2^+ consists of the following roots:

0 1 2 2 1 0	1 1 2 2 1 0	1 2 2 2 1 0	1 2 3 2 1 0	1 2 3 2 1 0	0 1 2 2 1 1
1	1	1	1	2	1
1 1 2 2 1 1	0 1 2 2 2 1	1 2 2 2 1 1	1 1 2 2 2 1	1 2 2 2 2 1	1 2 3 2 1 1
1	1	1	1	1	1
1 2 3 2 2 1	1 2 3 2 1 1	1 2 3 2 2 1			
1	2	2			

Φ_1^+ consists of the following roots:

$$\begin{array}{cccccc}
000100 & 001100 & 000110 & 011100 & 001100 & 001110 \\
0 & 0 & 0 & 0 & 1 & 0 \\
000111 & 111100 & 011110 & 011100 & 001110 & 001111 \\
0 & 0 & 0 & 1 & 1 & 0 \\
111110 & 111100 & 011111 & 011110 & 001111 & 111111 \\
0 & 1 & 0 & 1 & 1 & 0 \\
111110 & 011111 & 111111 & 012100 & 112100 & 012110 \\
1 & 1 & 1 & 1 & 1 & 1 \\
122100 & 112110 & 122110 & 012111 & 112111 & 122111 \\
1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Then we have $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.9 $(E_7, \{\alpha_2, \alpha_7\})$-type

Φ_3^+ consists of the following roots:

$$\begin{array}{cccccc}
123211 & 123221 & 123321 & 124321 & 134321 & 234321 \\
2 & 2 & 2 & 2 & 2 & 2
\end{array}$$

Φ_2^+ consists of the following roots:

$$\begin{array}{cccccc}
001111 & 011111 & 111111 & 123210 & 012111 & 112111 \\
1 & 1 & 1 & 2 & 1 & 1 \\
012211 & 122111 & 112211 & 012221 & 122211 & 112221 \\
1 & 1 & 1 & 1 & 1 & 1 \\
122221 & 123211 & 123221 & 123321 & & \\
1 & 1 & 1 & 1 & &
\end{array}$$

Φ_1^+ consists of the following roots:

0 0 0 0 0 1	0 0 0 0 0 0	0 0 1 0 0 0	0 0 0 0 1 1	0 1 1 0 0 0	0 0 1 1 0 0
0	1	1	0	1	1
0 0 0 1 1 1	1 1 1 0 0 0	0 1 1 1 0 0	0 0 1 1 1 0	0 0 1 1 1 1	1 1 1 1 0 0
0	1	1	1	0	1
0 1 1 1 1 1	0 1 1 1 1 0	1 1 1 1 1 1	1 1 1 1 1 0	0 1 2 1 0 0	1 1 2 1 0 0
0	1	0	1	1	1
0 1 2 1 1 0	1 2 2 1 0 0	1 1 2 1 1 0	0 1 2 2 1 0	1 2 2 1 1 0	1 1 2 2 1 0
1	1	1	1	1	1
1 2 2 2 1 0	1 2 3 2 1 0				
1	1				

Then we see that $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_\mathfrak{m})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_\mathfrak{m}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_\mathfrak{m})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_\mathfrak{m}$ has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.10 $(E_8, \{\alpha_2\})$-type

Let $\begin{smallmatrix} c_1 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ & & c_2 & & & & \end{smallmatrix}$ denote the root $c_1\alpha_1 + \cdots + c_8\alpha_8$ of E_8.

Φ_3^+ consists of the following roots:

1 3 5 4 3 2 1	2 3 5 4 3 2 1	2 4 5 4 3 2 1	2 4 6 4 3 2 1	2 4 6 5 3 2 1	2 4 6 5 4 2 1	2 4 6 5 4 3 1
3	3	3	3	3	3	3
2 4 6 5 4 3 2						
3						

Φ_2^+ consists of the following roots:

1 2 3 2 1 0 0	1 2 3 2 1 1 0	1 2 3 2 2 1 0	1 2 3 3 2 1 0	1 2 4 3 2 1 0	1 3 4 3 2 1 0	2 3 4 3 2 1 0
2	2	2	2	2	2	2
1 2 3 2 1 1 1	1 2 3 2 2 1 1	1 2 3 3 2 1 1	1 2 3 2 2 2 1	1 2 4 3 2 1 1	1 2 3 3 2 2 1	1 3 4 3 2 1 1
2	2	2	2	2	2	2
1 2 4 3 2 2 1	1 2 3 3 3 2 1	2 3 4 3 2 1 1	1 3 4 3 2 2 1	1 2 4 3 3 2 1	2 3 4 3 2 2 1	1 3 4 3 3 2 1
2	2	2	2	2	2	2
1 2 4 4 3 2 1	2 3 4 3 3 2 1	1 3 4 4 3 2 1	1 3 5 4 3 2 1	2 3 4 4 3 2 1	2 3 5 4 3 2 1	2 4 5 4 3 2 1
2	2	2	2	2	2	2

Φ_1^+ consists of the following roots:

0000000	0010000	0110000	0011000	1110000	0111000	0011100
1	1	1	1	1	1	1
1111000	0111100	0011110	1111100	0111110	0011111	1111110
1	1	1	1	1	1	1
0111111	1111111	0121000	1121000	0121100	1221000	1121100
1	1	1	1	1	1	1
0122100	1221100	1122100	1222100	1232100	0121110	1121110
1	1	1	1	1	1	1
0122110	1221110	1122110	0122210	1222110	1122210	1222210
1	1	1	1	1	1	1
1232110	1232210	1233210	0121111	0122111	1121111	0122211
1	1	1	1	1	1	1
1221111	1122111	1222111	1122211	0122221	1232111	1222211
1	1	1	1	1	1	1
1122221	1232211	1222221	1233211	1232221	1233221	1233321
1	1	1	1	1	1	1

Then we see that $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.11 $(E_8, \{\alpha_7\})$-type

Φ_3^+ consists of the following roots:

2465431	2465432
3	3

Φ_2^+ consists of the following roots:

0122221	1122221	1222221	1232221	1232221	1233221	1233221
1	1	1	1	2	1	2
1233321	1243221	1233321	1343221	1243321	2343221	1343321
1	2	2	2	2	2	2
1244321	2343321	1344321	1354321	2344321	1354321	2354321
2	2	2	2	2	3	2
2354321	2454321	2454321	2464321	2465321	2465421	
3	2	3	3	3	3	

Φ_1^+ consists of the following roots:

0000010 0	0000110 0	0000011 0	0001110 0	0000111 0	0011110 0	0001111 0
0111110 0	0011111 0	0011110 1	1111110 0	0111111 0	0111110 1	0011111 1
1111111 0	1111110 1	0111111 1	1111111 1	0121110 1	1121110 1	0122110 1
1221110 1	1122110 1	0122210 1	1222110 1	1122210 1	1222210 1	1232110 1
1232210 1	1232110 2	1233210 1	1232210 2	1233210 2	1243210 2	1343210 2
2343210 2	0121111 1	0122111 1	1121111 1	0122211 1	1221111 1	1122111 1
1222111 1	1122211 1	1232111 1	1222211 1	1232111 2	1232211 1	1232211 1
1233211 1	1233211 2	1243211 2	1343211 2	2343211 2		

Then we see that $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.12 $(F_4, \{\alpha_2\})$-type

Let $(c_1\ c_2\ c_3\ c_4)$ denote the root $c_1\alpha_1 + \cdots + c_4\alpha_4$ of F_4.

Φ_3^+ consists of the following roots:

1342 2342

Φ_2^+ consists of the following roots:

1220 1221 1231 1222 1232 1242

Φ_1^+ consists of the following roots:

$$\begin{array}{cccccc} 0100 & 1100 & 0110 & 1110 & 0111 & 1111 \\ 0120 & 1120 & 0121 & 1121 & 0122 & 1122 \end{array}$$

Then we see that $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10).

Let $(M(\mathfrak{m}), D_{\mathfrak{m}})$ be the standard differential system of type $\mathfrak{m}$. Since $\mathfrak{f} = \{0\}$, we have $\mathrm{Ch}(\partial D_{\mathfrak{m}}) = \{0\}$. It follows from Proposition 2.10 that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is immersed into a 2-jet space $(J^2(Q, n_1), C^2)$, however $D_{\mathfrak{m}}$ has no n_1-dimensional integrable elements and manifolds. By counting the number of roots in each Φ_i^+, we see that $\psi : M(\mathfrak{m}) \longrightarrow J^1(Q, n_1)$ as in Proposition 2.10 is not submersion.

2.6.13 $(G_2, \{\alpha_1\})$-type

We have $\Phi_3^+ = \{3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2\}$, $\Phi_2^+ = \{2\alpha_1 + \alpha_2\}$, $\Phi_1^+ = \{\alpha_1, \alpha_1 + \alpha_2\}$. Then $\mathfrak{f} = \{0\}$, which implies that $\mathfrak{m}$ cannot satisfy (2.5.10). For more detail, we refer to [5], [16], and [21].

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