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The metric growth of the discrete Laplacian

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Abstract. Networks play important roles in the theory of discrete potentials. Especially, the theory of Dirichlet spaces on networks has become one of the most important tools for the study of potentials on networks. In this paper, first we study some relations between the Dirichlet sums of a function and of its Laplacian. We introduce some conditions to investigate properties of several functional spaces related to Dirichlet potentials and to biharmonic functions. Our goal is to study the growth of the Laplacian related to biharmonic functions on an infinite network. As an application, we prove a Riesz Decomposition theorem for Dirichlet functions satisfying various conditions.

Key words: discrete potential theory, discrete Laplacian, Riesz Decomposition.

1. Introduction with preliminaries

Let $N = \langle X, Y, K, r\rangle$ be an infinite network which is connected and locally finite and has no self-loop. Here X is the set of nodes, Y is the set of arcs, K is a node-arc incidence matrix, and the resistance r is a strictly positive function on Y. For $x \in X$ and for $y \in Y$ we let

$$e(y) = \{x \in X : K(x, y) \neq 0\},$$
$$Y(x) = \{y \in Y : K(x, y) \neq 0\}.$$

Note that $e(y)$ consists of exactly two nodes x_+ and x_-, which satisfy $K(x_+, y) = 1$ and $K(x_-, y) = -1$.

Let $L(X)$ and $L(Y)$ be the sets of all real-valued functions on X and Y respectively and let $L_0(X)$ be the set of all $u \in L(X)$ with finite support. For every $u, v \in L(X)$, let us put

$$du(y) = -r(y)^{-1} \sum_{x\in X} K(x, y)u(x) \quad \text{(discrete derivative)},$$
$$D[u] = \sum_{y\in Y} r(y)(du(y))^2 \quad \text{(Dirichlet sum)},$$

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$$
\begin{aligned}
D[u,v] &= \sum_{y\in Y} r(y)du(y)dv(y) \quad \text{(Dirichlet mutual sum)},\\
\Delta u(x) &= \sum_{y\in Y} K(x,y)du(y) \quad \text{(Laplacian)},\\
\Delta^2 u(x) &= \Delta(\Delta u)(x) \quad \text{(bi-Laplacian)},\\
\mathbf{D}(N) &= \{u \in L(X) : D[u] < \infty\} \quad \text{(Dirichlet space)},\\
\mathbf{BD}(N) &= \{u \in \mathbf{D}(N) : u \text{ is bounded}\},\\
\mathbf{H}(N) &= \{u \in L(X) : \Delta u(x) = 0 \text{ on } X\},\\
\mathbf{HD}(N) &= \mathbf{H}(N) \cap \mathbf{D}(N),\\
\mathbf{D}^{(2)}(N) &= \{u \in L(X) : D[\Delta u] < \infty\},\\
\mathbf{H}^{(2)}(N) &= \{u \in L(X) : \Delta u \in \mathbf{H}(N)\}\\
\mathbf{QP}(N) &= \{u \in L(X) : \Delta u = -1,\ u \geq 0\},\\
\mathbf{H}^{(2)}\mathbf{D}(N) &= \mathbf{H}^{(2)}(N) \cap \mathbf{D}(N),\\
\mathbf{H}^{(2)}\mathbf{D}^{(\mathbf{2})}(N) &= \mathbf{H}^{(2)}(N) \cap \mathbf{D}^{(\mathbf{2})}(N).
\end{aligned}
$$

It is well-known that $\mathbf{D}(N)$ is a Hilbert space with respect to the norm

$$\|u\|_2 = (D[u] + u(x_0)^2)^{1/2}$$

for a fixed node $x_0 \in X$ (cf. [4, Proposition 2.1], [2, Section 9.3]). Denote by $\mathbf{D}_0(N)$ the closure of $L_0(X)$ in the space $\mathbf{D}(N)$. We say that N is of hyperbolic type if $\mathbf{D}(N) \neq \mathbf{D}_0(N)$. In case N is of hyperbolic type, for every $a \in X$ there exists a unique function $u \in L(X)$ which satisfies

$$u \in \mathbf{D}_0(N) \quad \text{and} \quad \Delta u(x) = -\varepsilon_a(x),$$

where $\varepsilon_a(x) = 0$ if $x \neq a$ and $\varepsilon_a(a) = 1$. Denote this function by $g_a(x)$ and call the Green function of N with pole at a.

For $w \in L(Y)$, define $\partial w \in X$ and $H[w]$ by

$$\partial w(x) = \sum_{y\in Y} K(x,y)w(y),$$

$$H[w] = \sum_{y \in Y} r(y) w(y)^2.$$

We introduce the following three conditions: We say that the network N satisfies condition (LD) if there exists $0 < c < \infty$ such that

$$D[\Delta f] \leq c D[f] \tag{LD}$$

for all $f \in L_0(X)$ (cf. [1]). In case network is of hyperbolic type, a function $u \in L(X)$ is said to satisfy property (P) if

$$\sum_{x \in X} g_a(x)(\Delta u(x))^2 < \infty, \tag{P}$$

and to satisfy property (SP) if

$$\sum_{x \in X} g_a(x)|\Delta u(x)| < \infty, \tag{SP}$$

where g_a denotes the Green function of N with pole at a. Properties (P) and (SP) were introduced in [3]. In Section 2 we study some sufficient conditions for (LD), in Section 4 some conditions for (P) and in Section 5 some sufficient conditions for (SP). As an application, in Section 5 we prove a Riesz Decomposition theorem for functions in $\mathbf{D}(N)$ satisfying various conditions.

2. Relations between $D[u]$ and $D[\Delta u]$

As in [1], we define the arc-arc incidence function $b(y, y')$ by

$$b(y, y') = \sum_{x \in X} K(x, y) K(x, y').$$

For $w \in L(Y)$, we define $B_r w \in L(Y)$ by

$$B_r w(y) = r(y)^{-1} \sum_{y' \in Y} b(y, y') w(y') = r(y)^{-1} \sum_{y' \in Y(e(y))} b(y, y') w(y'),$$

where we mean $Y(e(y)) = \bigcup_{x \in e(y)} Y(x)$. For $u \in L(X)$ and $f \in L_0(X)$, the following relations will be used frequently:

$$\Delta u = \partial du, \quad d\Delta u = -B_r du, \quad D[\Delta u] = H[B_r du],$$

$$D[u,f] = -\sum_{x\in X} u(x)\Delta f(x) = -\sum_{x\in X} f(x)\Delta u(x).$$

Taking $u = 1$ in the formula for $D[u,f]$, we see for $f \in L_0(X)$

$$\sum_{x\in X} \Delta f(x) = 0.$$

We show some sufficient conditions of condition (LD).

Lemma 2.1 *Denote by* $\gamma(y)$ *the number of arcs in* $Y(e(y))$. *For* $u \in L(X)$, *the following inequality holds*:

$$D[\Delta u] \le 4\sum_{y\in Y} r(y)^{-1}\gamma(y) \sum_{y'\in Y(e(y))} du(y')^2.$$

Proof. Since $|b(y,y')| \le 2$, we have

$$|B_r w(y)| \le 2r(y)^{-1} \sum_{y'\in Y(e(y))} |w(y')|,$$

so that by the Cauchy-Schwarz Inequality

$$|B_r w(y)|^2 \le 4r(y)^{-2}\left(\sum_{y'\in Y(e(y))} w(y')\right)^2 \le 4r(y)^{-2}\gamma(y) \sum_{y'\in Y(e(y))} w(y')^2.$$

Letting $w = du$ we have

$$\begin{aligned} D[\Delta u] = H[B_r w] &= \sum_{y\in Y} r(y)|B_r w(y)|^2 \\ &\le 4\sum_{y\in Y} r(y)^{-1}\gamma(y) \sum_{y'\in Y(e(y))} w(y')^2. \end{aligned}$$ □

Theorem 2.1 *Assume that there exist* $r_0 > 0$ *and* $\gamma_0 < \infty$ *such that* $r(y) \ge r_0$ *and* $\gamma(y) \le \gamma_0$ *for all* $y \in Y$. *For* $u \in L(X)$ *the following inequality holds*:

$$D[\Delta u] \le \frac{4\gamma_0^2}{r_0} \sum_{y\in Y} du(y)^2 \le \frac{4\gamma_0^2}{r_0^2} D[u].$$

Especially N satisfies condition (LD).

Proof. Lemma 2.1 shows that

$$\begin{aligned} D[\Delta u] &\le 4\sum_{y\in Y} r(y)^{-1}\gamma(y) \sum_{y'\in Y(e(y))} du(y')^2 \le \frac{4\gamma_0}{r_0} \sum_{y\in Y}\sum_{y'\in Y(e(y))} du(y')^2 \\ &\le \frac{4\gamma_0^2}{r_0} \sum_{y\in Y} du(y)^2 \le \frac{4\gamma_0^2}{r_0^2} \sum_{y\in Y} r(y) du(y)^2 = \frac{4\gamma_0^2}{r_0^2} D[u] \end{aligned}$$

as required. □

Corollary 2.1 *Assume that $\gamma(y) \le \gamma_0 < \infty$ and $r(y) \equiv 1$ on Y. Then $D[\Delta u] \le 4\gamma_0^2 D[u]$ for every $u \in L(X)$.*

Remark 2.1 Denote by $\nu(x)$ the number of arcs which meet x, i.e., $\nu(x) = \#Y(x) = \sum_{y\in Y} |K(x,y)|$. Then $\gamma(y) \le \nu(a) + \nu(b)$, where $e(y) = \{a, b\}$. If $\nu(x) \le \nu_0$ for $x \in X$, then $\gamma_0 \le 2\nu_0$. Corollary 2.1 implies that $D[\Delta u] \le 16\nu_0^2 D[u]$.

We proved in [1, Proposition 6.1]

Proposition 2.1 *Assume that $\nu(x) \le \nu_0$ for all $x \in X$ and $r(y) \equiv 1$ on Y. Then $D[\Delta u] \le 8\nu_0^2 D[u]$ for every $u \in L(X)$.*

Theorem 2.2 *Let δ be the number defined by*

$$\delta = \sum_{y\in Y}\sum_{y'\in Y} r(y)^{-1} |b(y,y')|^2 r(y')^{-1}.$$

Then $D[\Delta u] \le \delta D[u]$. Especially N satisfies condition (LD) *if $\delta < \infty$.*

Proof. By Cauchy-Schwarz's inequality, we have

$$\begin{aligned} (B_r w(y))^2 &\le r(y)^{-2} \Big(\sum_{y'\in Y} |b(y,y')|^2 r(y')^{-1}\Big)\Big(\sum_{y'\in Y} r(y') w(y')^2\Big) \\ &= r(y)^{-2}\Big(\sum_{y'\in Y} |b(y,y')|^2 r(y')^{-1}\Big) H[w]. \end{aligned}$$

Letting $w = du$ we have

$$\begin{aligned} D[\Delta u] = H[B_r w] &= \sum_{y \in Y} r(y)(B_r w(y))^2 \\ &\leq \left(\sum_{y \in Y} \sum_{y' \in Y} r(y)^{-1} |b(y,y')|^2 r(y')^{-1} \right) H[w] = \delta D[u]. \end{aligned}$$

□

Corollary 2.2 $D[\Delta u] \leq 4\big(\sum_{y \in Y} r(y)^{-1}\big)^2 D[u]$.

Proof. It suffices to note that $|b(y,y')| \leq 2$. □

Remark 2.2 There is no relation between $D[\Delta u]$ and $D[u]$ in general. We show this by using the following network:

Proposition 2.2 *Let $X = \{x_n\}_{n \geq 0}$ and $Y = \{y_n\}_{n \geq 1}$. Let $K(x_n, y_n) = 1$, $K(x_{n-1}, y_n) = -1$ for $n \geq 1$ and $K(x,y) = 0$ for any other pairs. We call this network a linear network. Assume that $\inf_n r(y_n) = 0$. Then condition* (LD) *does not hold for $\langle X, Y, K, r \rangle$.*

Proof. Let

$$u_n(x_k) = \begin{cases} 1 & \text{if } k < n; \\ 0 & \text{if } k \geq n. \end{cases}$$

Then

$$\begin{aligned} du_n(y_n) &= -r(y_n)^{-1}(u_n(x_n) - u_n(x_{n-1})) = r(y_n)^{-1}, \\ du_n(y_k) &= -r(y_k)^{-1}(u_n(x_k) - u_n(x_{k-1})) = 0 \quad \text{for } k \neq n; \\ \Delta u_n(x_{n-1}) &= du_n(y_{n-1}) - du_n(y_n) = -r(y_n)^{-1}, \\ \Delta u_n(x_n) &= du_n(y_n) - du_n(y_{n+1}) = r(y_n)^{-1}; \\ d\Delta u_n(y_n) &= -r(y_n)^{-1}(\Delta u_n(x_n) - \Delta u_n(x_{n-1})) = -2r(y_n)^{-2}. \end{aligned}$$

Therefore

$$\begin{aligned} D[u_n] &= r(y_n) \times (r(y_n)^{-1})^2 = r(y_n)^{-1}, \\ D[\Delta u_n] &\geq r(y_n) \times (-2r(y_n)^{-2})^2 = 4r(y_n)^{-3}. \end{aligned}$$

We have

$$\inf_n \frac{D[u_n]}{D[\Delta u_n]} \le \inf_n \frac{r(y_n)^2}{4} = 0$$

as required. □

Corollary 2.3 *A linear network satisfies condition* (LD) *if and only if* $\inf_n r(y_n) > 0$.

Proof. Theorem 2.1 and Proposition 2.2 show the assertion. □

A network with $\inf_{y\in Y} r(y) > 0$ does not satisfy condition (LD) in general.

Example 2.1 Let $X = \{x_n\}_{n\ge 0} \cup \{z_{n,j}\}_{n\ge 1, 1\le j\le n}$ and $Y = \{y^+_{n,j}, y^-_{n,j}\}_{n\ge 1, 1\le j\le n}$. Let $K(x_n, y^+_{n,j}) = 1$, $K(z_{n,j}, y^+_{n,j}) = -1$, $K(x_{n-1}, y^-_{n,j}) = -1$, $K(z_{n,j}, y^-_{n,j}) = 1$ for $n \ge 1$ and $1 \le j \le n$ and $K(x,y) = 0$ for any other pairs. Let $r(y^+_{n,j}) \equiv r(y^-_{n,j}) \equiv 1$. Let

$$u_n(x_k) = \begin{cases} 2 & \text{if } k < n; \\ 0 & \text{if } k \ge n, \end{cases} \qquad u_n(z_{k,j}) = \begin{cases} 2 & \text{if } k < n; \\ 1 & \text{if } k = n; \\ 0 & \text{if } k > n. \end{cases}$$

Then

$$du_n(y^-_{k,j}) = -(u_n(z_{k,j}) - u_n(x_{k-1})) = \begin{cases} 1 & \text{if } k = n; \\ 0 & \text{if } k \ne n, \end{cases}$$

$$du_n(y^+_{k,j}) = -(u_n(x_k) - u_n(z_{k,j})) = \begin{cases} 1 & \text{if } k = n; \\ 0 & \text{if } k \ne n, \end{cases}$$

$$\Delta u_n(x_k) = \sum_{j=1}^{k} du_n(y^+_{k,j}) - \sum_{j=1}^{k+1} du_n(y^-_{k+1,j}) = \begin{cases} -n & \text{if } k = n-1; \\ n & \text{if } k = n; \\ 0 & \text{otherwise,} \end{cases}$$

$$\Delta u_n(z_{k,j}) = du_n(y^-_{k,j}) - du_n(y^+_{k,j}) = 0,$$

$$d\Delta u_n(y_{k,j}^-) = -(\Delta u_n(z_{k,j}) - \Delta u_n(x_{k-1})) = \begin{cases} -n & \text{if } k = n; \\ n & \text{if } k = n+1; \\ 0 & \text{otherwise,} \end{cases}$$

$$d\Delta u_n(y_{k,j}^+) = -(\Delta u_n(x_k) - \Delta u_n(z_{k,j})) = \begin{cases} n & \text{if } k = n-1; \\ -n & \text{if } k = n; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$D[u_n] = \sum_{k=1}^{\infty}\sum_{j=1}^{k}\big(du_n(y_{k,j}^-)^2 + du_n(y_{k,j}^+)^2\big) = \sum_{j=1}^{n}(1^2+1^2) = 2n,$$

$$\begin{aligned} D[\Delta u_n] &= \sum_{k=1}^{\infty}\sum_{j=1}^{k}\big(d\Delta u_n(y_{k,j}^-)^2 + d\Delta u_n(y_{k,j}^+)^2\big) \\ &= \sum_{j=1}^{n-1} d\Delta u_n(y_{n-1,j}^+)^2 + \sum_{j=1}^{n}\big(d\Delta u_n(y_{n,j}^-)^2 + d\Delta u_n(y_{n,j}^+)^2\big) \\ &\quad + \sum_{j=1}^{n+1} d\Delta u_n(y_{n+1,j}^-)^2 \\ &= (n-1)n^2 + n\big((-n)^2 + (-n)^2\big) + (n+1)n^2 = 4n^3. \end{aligned}$$

We have

$$\frac{D[u_n]}{D[\Delta u_n]} = \frac{2n}{4n^3} \to 0$$

as $n \to \infty$.

3. Condition (LD)

We proved in [1, Lemma 6.1]

Lemma 3.1 *Condition* (LD) *implies* $\Delta(\mathbf{D}_0(N)) \subset \mathbf{D}_0(N)$.

Theorem 3.1 *Assume condition* (LD). *Then* $\mathbf{D}(N) \subset \mathbf{D}^{(2)}(N)$.

Proof. Let $u \in \mathbf{D}(N)$. There exist $v \in \mathbf{D}_0(N)$ and $h \in \mathbf{HD}(N)$ such that $u = v + h$ by Royden's decomposition (cf. [5, Theorem 4.1], [2, Exercise 9.6(f)]). Since $\Delta u = \Delta v + \Delta h = \Delta v$, we have $D[\Delta u] = D[\Delta v] < \infty$ by Lemma 3.1. □

Lemma 3.2 *Assume condition* (LD)*. Then there exists $c > 0$ such that $D[\Delta u] \leq cD[u]$ for all $u \in \mathbf{D}(N)$.*

Proof. By Royden's decomposition we find $v \in \mathbf{D}_0(N)$ and $h \in \mathbf{HD}(N)$ such that $u = v+h$. Let $\{f_n\}_n$ be a sequence in $L_0(X)$ such that $\|v-f_n\|_2 \to 0$ as $n \to \infty$. We have $D[f_n, h] = -\sum_{x\in X} f_n(x)\Delta h(x) = 0$ and

$$|D[v, h] - D[f_n, h]| = |D[v - f_n, h]| \leq D[v - f_n]^{1/2} D[h]^{1/2} \to 0$$

as $n \to \infty$. These imply $D[v, h] = 0$, and that

$$D[u] = D[v] + D[v, h] + D[h] = D[v] + D[h] \geq D[v].$$

Since $\{f_n(x)\}_n$ converges pointwise to $v(x)$, we see that $\{\Delta f_n(x)\}_n$ converges pointwise to $\Delta v(x)$, so that by Fatou's lemma

$$D[\Delta v] \leq \liminf_{n\to\infty} D[\Delta f_n].$$

Condition (LD) implies that $D[\Delta f_n] \leq cD[f_n]$ for all n. We have

$$D[\Delta u] = D[\Delta v] \leq \liminf_{n\to\infty} D[\Delta f_n] \leq c \liminf_{n\to\infty} D[f_n] = cD[v] \leq cD[u]$$

as required. □

Proposition 3.1 *Assume condition* (LD)*. Then $D[h, \Delta v] = 0$ for $h \in \mathbf{H}^{(2)}\mathbf{D}(N)$ and $v \in \mathbf{D}_0(N)$.*

Proof. Let $\{f_n\}_n$ be a sequence in $L_0(X)$ such that $\|v - f_n\|_2 \to 0$ as $n \to \infty$. Since $\Delta f_n \in L_0(X)$, we have

$$\begin{aligned} D[h, \Delta f_n] &= -\sum_{x\in X} (\Delta h(x))(\Delta f_n(x)) \\ &= D[\Delta h, f_n] = -\sum_{x\in X} (\Delta^2 h(x)) f_n(x) = 0. \end{aligned}$$

Lemma 3.2 shows that

$$D[h, \Delta v] = D[h, \Delta v - \Delta f_n] \leq D[h]^{1/2} D[\Delta(v - f_n)]^{1/2}$$
$$\leq c^{1/2} D[h]^{1/2} D[v - f_n]^{1/2} \to 0$$

as $n \to \infty$, so that $D[h, \Delta v] = 0$. $\square$

Proposition 3.2 *Let $h \in \mathbf{H}^{(2)}\mathbf{D}^{(2)}(N) = \mathbf{H}^{(2)}(N) \cap \mathbf{D}^{(2)}(N)$. If condition* (LD) *is fulfilled, then $D[\Delta h, \Delta v] = 0$ for every $v \in \mathbf{D}_0(N)$.*

Proof. We find $\{f_n\}_n \subset L_0(X)$ such that $\|v - f_n\|_2 \to 0$. Then $D[\Delta h, \Delta f_n] = D[\Delta^2 h, f_n] = 0$. Using Lemma 3.1 we know that $D[\Delta h, \Delta v]$ is well-defined and, using Lemma 3.2, that

$$D[\Delta h, \Delta v] \leq D[\Delta h]^{1/2} D[\Delta(v - f_n)]^{1/2} \leq c^{1/2} D[\Delta h]^{1/2} D[v - f_n]^{1/2} \to 0$$

as $n \to \infty$. $\square$

Proposition 3.3 *Assume that N is of hyperbolic type and that condition* (LD) *is fulfilled. Then $\mathbf{D}_0(N) \cap \mathbf{H}^{(2)}(N) = \{0\}$.*

Proof. Let $u \in \mathbf{D}_0(N)$ and $\Delta^2 u = 0$. Then $\Delta u \in \mathbf{D}_0(N)$ by Lemma 3.1, so that $\Delta u \in \mathbf{HD}(N)$. Since N is of hyperbolic type, we have $\mathbf{D}_0(N) \cap \mathbf{HD}(N) = \{0\}$ (see [5, Lemma 1.3]), so that $\Delta u = 0$. Namely $u \in \mathbf{D}_0(N) \cap \mathbf{HD}(N)$, and hence $u = 0$. $\square$

Corollary 3.1 *Assume that N is of hyperbolic type and that condition* (LD) *is fulfilled. Then $\mathbf{H}^{(2)}\mathbf{D}(N) = \mathbf{HD}(N)$.*

Proof. Clearly, $\mathbf{HD}(N) \subset \mathbf{H}^{(2)}\mathbf{D}(N)$. Let $u \in \mathbf{H}^{(2)}\mathbf{D}(N)$. There exist $h \in \mathbf{HD}(N)$ and $v \in \mathbf{D}_0(N)$ such that $u = h + v$. Then $\Delta^2 v = \Delta^2 u = 0$. Proposition 3.3 shows that $v = 0$, and hence $u = h \in \mathbf{HD}(N)$. $\square$

Let $\{N_n = \langle X_n, Y_n, K_n, r_n \rangle\}_n$ be an exhaustion of N and $a \in X$. The Green function $g_a^{(n)}$ of N_n with pole at a is defined by

$$\Delta g_a^{(n)}(x) = -\varepsilon_a(x) \ (x \in X_n), \quad g_a^{(n)}(x) = 0 \ (x \in X \setminus X_n).$$

It is well-known that $g_a^{(n)}(x) \leq g_a^{(n+1)}(x)$ on X and $g_a^{(n)}(x) \to g_a(x)$ as $n \to \infty$ for each $x \in X$ (cf. [5, Remark 3.1]). Noting $D[g_a] = g_a(a)$ and

$$D[g_a^{(n)}] = -\sum_{x\in X} g_a^{(n)}(x), \quad D[g_a^{(n)} - g_a] = g_a(a) - g_a^{(n)}(a),$$

we have

Lemma 3.3 $\lim_{n\to\infty} D[g_a^{(n)}] = D[g_a]$ *and* $\lim_{n\to\infty} D[g_a^{(n)} - g_a] = 0$

Theorem 3.2 *Assume that N is of hyperbolic type and satisfies condition* (LD)*. If $u \in \mathbf{D}(N)$, then $\sum_{x\in X}(\Delta u(x))^2 < \infty$.*

Proof. Let $\{N_n = \langle X_n, Y_n, K_n, r_n\rangle\}_n$ be an exhaustion of N and let $g_z^{(n)}$ be the Green function of N_n with pole at $z \in X_n$. We set $g_z^{(n)}(x) = 0$ for $x \in X \setminus X_n$ and put

$$f_n(x) = \sum_{z\in X_n} g_z^{(n)}(x)\Delta u(z).$$

Then

$$f_n(x) = 0 \quad \text{on } X \setminus X_n \quad \text{and} \quad \Delta f_n(x) = -\Delta u(x) \quad \text{on } X_n.$$

Especially $\{\Delta f_n(x)\}_n$ converges pointwise to $-\Delta u(x)$. By our assumption, there exists a constant $c > 0$ such that $D[\Delta f_n] \le cD[f_n]$ for all n.

Let $h_n = u + f_n$. Then h_n is harmonic on X_n and

$$\begin{aligned} D[u] &= D[h_n - f_n] = D[f_n] + D[h_n] - 2D[f_n, h_n] \\ &= D[f_n] + D[h_n] + 2\sum_{x\in X_n} f_n(x)\Delta h_n(x) = D[f_n] + D[h_n], \end{aligned}$$

so that $D[f_n] \le D[u]$. We have

$$\begin{aligned} \sum_{x\in X}(\Delta f_n(x))^2 &= -D[f_n, \Delta f_n] \le D[f_n]^{1/2} D[\Delta f_n]^{1/2} \\ &\le c^{1/2} D[f_n] \le c^{1/2} D[u]. \end{aligned}$$

By Fatou's Lemma, we have

$$\sum_{x\in X}(\Delta u(x))^2 \le \liminf_{n\to\infty} \sum_{x\in X}(\Delta f_n(x))^2 \le c^{1/2} D[u] < \infty. \qquad \square$$

4. The metric growth of the Laplacian

The metric growth of the Laplacian of u is said to be so slow in [3] if u satisfies property (P). As a discrete analog of [3, Theorem VII.1.1], we have

Theorem 4.1 *Let N be of hyperbolic type and $u \in \mathbf{D}(N)$. If $\Delta u \in \mathbf{BD}(N)$, then u satisfies property* (P).

Proof. Let $a \in X$ and $\{N_n = \langle X_n, Y_n, K_n, r_n \rangle\}_n$ be an exhaustion of N such that $a \in X_1$ and let $g_a^{(n)}$ be the Green function of N_n with pole at a. We set $g_a^{(n)}(x) = 0$ for $x \in X \setminus X_n$. Then $g_a^{(n)} \in L_0(X)$ and $D[g_a^{(n)} - g_a] \to 0$ as $n \to \infty$. We put

$$\varphi_n(x) = -g_a^{(n)}(x)\Delta u(x).$$

By our assumption, there exists $c_1 > 0$ such that $|\Delta u(x)| \le c_1$ on X. We shall show that

$$|d\varphi_n(y)| \le M(|dg_a^{(n)}(y)| + |d\Delta u(y)|)$$

where $M = \max(c_1, g_a(a))$. Let $e(y) = \{x_1, x_2\}$. Since $g_a^{(n)}(x) \le g_a^{(n)}(a) \le g_a(a)$,

$$\begin{aligned}
|r(y)d\varphi_n(y)| &= |g_a^{(n)}(x_1)\Delta u(x_1) - g_a^{(n)}(x_2)\Delta u(x_2)| \\
&\le |g_a^{(n)}(x_1)||\Delta u(x_1) - \Delta u(x_2)| + |g_a^{(n)}(x_1) - g_a^{(n)}(x_2)||\Delta u(x_2)| \\
&\le g_a(a)|r(y)d\Delta u(y)| + c_1|r(y)dg_a^{(n)}(y)| \\
&\le Mr(y)(|dg_a^{(n)}(y)| + |d\Delta u(y)|).
\end{aligned}$$

We have

$$\begin{aligned}
D[\varphi_n] &\le \sum_{y \in Y} r(y)\big(M(|dg_a^{(n)}(y)| + |d\Delta u(y)|)\big)^2 \\
&\le 2M^2\big(D[g_a^{(n)}] + D[\Delta u]\big) \to 2M^2\big(D[g_a] + D[\Delta u]\big).
\end{aligned}$$

There is a constant $c_2 > 0$ such that $D[\varphi_n] \le c_2$. Since $\varphi_n \in L_0(X)$, we have

$$\sum_{x\in X} g_a^{(n)}(x)(\Delta u(x))^2 = -\sum_{x\in X}\varphi_n(x)\Delta u(x) = D[\varphi_n, u]$$
$$\leq D[\varphi_n]^{1/2} D[u]^{1/2} \leq c_2^{1/2} D[u]^{1/2}.$$

By Fatou's lemma

$$\sum_{x\in X} g_a(x)(\Delta u(x))^2 \leq \liminf_{n\to\infty} \sum_{x\in X} g_a^{(n)}(x)(\Delta u(x))^2 \leq c_2^{1/2} D[u]^{1/2},$$

that is, u satisfies property (P). □

We prepare

Lemma 4.1 *The following relation holds for* $f_1, f_2 \in L(X)$:

$$\Delta(f_1 f_2)(x) = (\Delta f_1(x)) f_2(x) + f_1(x)(\Delta f_2(x)) + \sum_{y\in Y(x)} r(y)\, df_1(y)\, df_2(y).$$

Proof. For $x \in X$ and $y \in Y(x)$ we let $\mathrm{n}_x(y)$ be a unique node with

$$K(x,y)K(\mathrm{n}_x(y), y) = -1.$$

Then

$$\begin{aligned} du(y) &= -r(y)^{-1}(K(x,y)u(x) + K(\mathrm{n}_x(y),y)u(\mathrm{n}_x(y))) \\ &= -r(y)^{-1}K(x,y)(u(x) - u(\mathrm{n}_x(y))), \end{aligned} \tag{1}$$

$$\Delta u(x) = \sum_{y\in Y(x)} K(x,y)\,du(y) = -\sum_{y\in Y(x)} r(y)^{-1}(u(x) - u(\mathrm{n}_x(y))). \tag{2}$$

We have

$$\begin{aligned} &\Delta(f_1 f_2)(x) - (\Delta f_1(x)) f_2(x) - f_1(x)(\Delta f_2(x)) \\ &= -\sum_{y\in Y(x)} r(y)^{-1}\big(f_1(x)f_2(x) - f_1(\mathrm{n}_x(y))f_2(\mathrm{n}_x(y))\big) \\ &\quad + \sum_{y\in Y(x)} r(y)^{-1}\big(f_1(x) - f_1(\mathrm{n}_x(y))\big) f_2(x) \\ &\quad + f_1(x)\sum_{y\in Y(x)} r(y)^{-1}\big(f_2(x) - f_2(\mathrm{n}_x(y))\big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{y\in Y(x)} r(y)^{-1}\big(f_1(x)f_2(x) + f_1(\mathrm{n}_x(y))f_2(\mathrm{n}_x(y)) \\
&\qquad - f_1(\mathrm{n}_x(y))f_2(x) - f_1(x)f_2(\mathrm{n}_x(y))\big) \\
&= \sum_{y\in Y(x)} r(y)^{-1}\big(f_1(x) - f_1(\mathrm{n}_x(y))\big)\big(f_2(x) - f_2(\mathrm{n}_x(y))\big),
\end{aligned}
$$

using (1),

$$
\begin{aligned}
&= \sum_{y\in Y(x)} r(y)^{-1}\big(-r(y)K(x,y)df_1(y)\big)\big(-r(y)K(x,y)df_2(y)\big) \\
&= \sum_{y\in Y(x)} r(y)df_1(y)df_2(y).
\end{aligned}
$$

This completes the proof. □

As a discrete analog of [3, Theorem VII.1.2], we shall prove

Theorem 4.2 *Assume that N is of hyperbolic type. If $u \in \mathbf{H}^{(2)}\mathbf{D}^{(2)}(N) \cap \mathbf{BD}(N)$, then u satisfies property* (P).

Proof. Let $\{N_n = \langle X_n, Y_n, K_n, r_n\rangle\}_n$ be an exhaustion of N and let $g_z^{(n)}$ be the Green function of N_n with pole at $z \in X_n$. We set $g_z^{(n)}(x) = 0$ for $x \in X \setminus X_n$. Now we consider $f_n \in L(X)$ defined by

$$f_n(x) = \sum_{z\in X_n} g_z^{(n)}(x)\Delta u(z).$$

Then

$$f_n(x) = 0 \quad \text{on } X \setminus X_n \quad \text{and} \quad \Delta f_n(x) = -\Delta u(x) \quad \text{on } X_n.$$

By our assumption, there exists $c > 0$ such that $|u(x)| \le c$ on X. We shall show that $|f_n(x)| \le 2c$ on X. We may assume $x \in X_n$. Define $\varphi(z) = g_z^{(n)}(x) = g_x^{(n)}(z)$ for $z \in X_n$ and $\varphi(z) = 0$ for $z \in X \setminus X_n$. Notice that $\Delta\varphi(z) = -\varepsilon_x(z)$ for $z \in X_n$. For $z \in \partial X_n$ (the outer boundary of X_n), we use (2) and obtain

$$\Delta\varphi(z) = -\sum_{y\in Y(z)} r(y)^{-1}(\varphi(z)-\varphi(\mathrm{n}_z(y))) = \sum_{\substack{y\in Y(z);\\ \mathrm{n}_z(y)\in X_n}} r(y)^{-1} g_x^{(n)}(\mathrm{n}_z(y)) \geq 0.$$

Since $\varphi \in L_0(X)$, we have

$$0 = \sum_{z\in X} \Delta\varphi(z) = -1 + \sum_{z\in\partial X_n} \Delta\varphi(z)$$

and

$$f_n(x) = \sum_{z\in X} \varphi(z)\Delta u(z) = \sum_{z\in X} u(z)\Delta\varphi(z) = -u(x) + \sum_{z\in\partial X_n} u(z)\Delta\varphi(z).$$

These implies

$$|f_n(x)| \leq |u(x)| + \sum_{z\in\partial X_n} |u(z)\Delta\varphi(z)] \leq c + c\sum_{z\in\partial X_n} \Delta\varphi(z) = 2c.$$

Applying Lemma 4.1 to $g_a^{(n)}(z)$ and $\Delta u(z)$ and using $u \in \mathbf{H}^{(2)}(N)$, we have

$$\begin{aligned}\Delta(g_a^{(n)}\Delta u)(z) &= \Delta g_a^{(n)}(z)\Delta u(z) + g_a^{(n)}(z)\Delta^2 u(z)\\ &\qquad + \sum_{y\in Y(z)} r(y) dg_a^{(n)}(y) d\Delta u(y)\\ &= \Delta g_a^{(n)}(z)\Delta u(z) + \sum_{y\in Y(z)} r(y) dg_a^{(n)}(y) d\Delta u(y).\end{aligned}$$

Thus we have

$$\begin{aligned}&\sum_{z\in X} g_a^{(n)}(z)(\Delta u(z))^2 = \sum_{z\in X_n} g_a^{(n)}(z)(\Delta u(z))^2\\ &\quad = -\sum_{z\in X_n} g_a^{(n)}(z)\Delta f_n(z)\Delta u(z) = -\sum_{z\in X} \Delta f_n(z) g_a^{(n)}(z)\Delta u(z)\\ &\quad = -\sum_{z\in X} f_n(z)\Delta(g_a^{(n)}\Delta u)(z) = -\sum_{z\in X_n} f_n(z)\Delta(g_a^{(n)}\Delta u)(z)\\ &\quad = -\sum_{z\in X_n} f_n(z)\Delta g_a^{(n)}(z)\Delta u(z) - \sum_{z\in X_n}\sum_{y\in Y(z)} f_n(z) r(y) dg_a^{(n)}(y) d\Delta u(y)\end{aligned}$$

$$= f_n(a)\Delta u(a) - \sum_{z\in X_n}\sum_{y\in Y(z)} f_n(z)r(y)dg_a^{(n)}(y)d\Delta u(y)$$

$$\leq |f_n(a)\Delta u(a)| + \sum_{z\in X_n}\sum_{y\in Y(z)} |f_n(z)r(y)dg_a^{(n)}(y)d\Delta u(y)|$$

$$\leq 2c|\Delta u(a)| + 4c\sum_{y\in Y} r(y)|dg_a^{(n)}(y)||d\Delta u(y)|$$

$$\leq 2c|\Delta u(a)| + 4cD[g_a^{(n)}]^{1/2}D[\Delta u]^{1/2}.$$

Since $D[g_a^{(n)}] \to D[g_a]$ as $n \to \infty$, we have

$$\sum_{z\in X} g_a(z)[\Delta u(z)]^2 \leq \liminf_{n\to\infty}\sum_{z\in X} g_a^{(n)}(z)(\Delta u(z))^2$$
$$\leq 2c|\Delta u(a)| + 4cD[g_a]^{1/2}D[\Delta u]^{1/2} < \infty.$$

This shows that u satisfies property (P). □

Theorem 4.3 *Assume that N is of hyperbolic type and satisfies condition* (LD). *Then every $u \in \mathbf{D}(N)$ satisfies property* (P).

Proof. Since $g_a(x) \leq g_a(a) < \infty$, Theorem 3.2 implies

$$\sum_{x\in X} g_a(x)(\Delta u(x))^2 \leq g_a(a)\sum_{x\in X}(\Delta u(x))^2 < \infty. \qquad \square$$

5. Riesz Decomposition

To study property (SP) introduced in Section 1, we recall the following set of functions on N:

$$\mathbf{QP}(N) = \{u \in L(X) : \Delta u = -1,\ u \geq 0\}.$$

We denote by $\mathbf{O}_{QP}$ the set of all locally finite infinite networks N such that the set $\mathbf{QP}(N)$ is the empty set.

We proved in [6, Theorem 3.1]

Lemma 5.1 *$N \notin \mathbf{O}_{QP}$ if and only if $\sum_{x\in X} g_a(x) < \infty$ for all $a \in X$.*

Notice that if $N \notin \mathbf{O}_{QP}$, then N is of hyperbolic type, so that there exists the Green function $g_a(x)$ of N with pole at a.

Proposition 5.1 *Let $N \notin \mathbf{O}_{QP}$. Then property* (P) *implies property* (SP) *for every $u \in \mathbf{D}(N)$.*

Proof. Since u satisfies property (P), we see by Lemma 5.1

$$\sum_{x\in X} g_a(x)|\Delta u(x)| \le \left(\sum_{x\in X} g_a(x)\right)^{1/2} \left(\sum_{x\in X} g_a(x)(\Delta u(x))^2\right)^{1/2} < \infty$$

for every $a \in X$. □

We have

Theorem 5.1 *Let $N \notin \mathbf{O}_{QP}$. If $u \in \mathbf{D}(N)$ satisfies property* (SP), *then u has the following Riesz Decomposition: There exists $h \in \mathbf{HD}(N)$ such that*

$$u(x) = h(x) + \sum_{z\in X} g_z(x)(-\Delta u(z))$$

on X.

Proof. By our assumption, the Green potential

$$v(x) := \sum_{z\in X} g_z(x)\Delta u(z) \in L(X)$$

of a signed measure Δu is well-defined. Let $\{N_n = \langle X_n, Y_n, K_n, r_n\rangle\}_n$ be an exhaustion of N and let $g_z^{(n)}$ be the Green function of N_n with pole at $z \in X_n$. We set $g_z^{(n)}(x) = 0$ for $x \in X \setminus X_n$ and define f_n and h_n by

$$f_n(x) = \sum_{z\in X_n} g_z^{(n)}(x)\Delta u(z), \quad h_n = u + f_n.$$

Notice that h_n is harmonic on X_n and

$$D[h_n, f_n] = -\sum_{x\in X} (\Delta h_n(x)) f_n(x) = 0,$$

so that $D[u] = D[h_n] + D[f_n]$. We see by Lebesgue's dominated convergence theorem that $\{f_n(x)\}_n$ converges pointwise to a function $v(x)$ for every

$x \in X$. Since $\{D[f_n]\}_n$ is bounded, we see that $v \in \mathbf{D}_0(N)$ (cf. [7, Theorem 4.1]). Let h be the pointwise limit of h_n. Then $h = u + v \in \mathbf{D}(N)$ and h is harmonic on X, i.e., $h \in \mathbf{HD}(N)$. We have

$$u(x) = h(x) - v(x) = h(x) + \sum_{z \in X} g_z(x)(-\Delta u(z))$$

for $x \in X$. □

Remark 5.1 In the above theorem, we see that $h = 0$ if $u \in \mathbf{D}_0(N)$.

Corollary 5.1 *Let $N \notin \mathbf{O}_{QP}$. Then $u \in \mathbf{D}(N)$ admits the following Riesz Decomposition*:

$$u(x) = h(x) + \sum_{z \in X} g_x(z)(-\Delta u(z)), \quad h \in \mathbf{HD}(N)$$

if any one of the following conditions is satisfied:

(1) $\Delta u \in \mathbf{BD}(N)$;
(2) *u is bounded and $u \in \mathbf{H}^{(2)}\mathbf{D}^{(\mathbf{2})}(N)$*;
(3) *condition* (LD) *is satisfied.*

Proof. Since $N \notin \mathbf{O}_{QP}$, N is of hyperbolic type. One of Theorems 4.1, 4.2 and 4.3 can be applied and implies that u satisfies property (P). Proposition 5.1 shows that u satisfies property (SP). By Theorem 5.1 we have the assertion. □

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