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# $S^1$-equivariant Rabinowitz–Floer homology

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**Abstract.** We define the $S^1$-equivariant Rabinowitz–Floer homology of a bounding contact hypersurface $\Sigma$ in an exact symplectic manifold, and show by a geometric argument that it vanishes if $\Sigma$ is displaceable.

*Key words*: equivariant Rabinowitz–Floer homology, displaceable hypersurface.

## 1. Introduction

Consider a bounding contact hypersurface $\Sigma$ in an exact convex symplectic manifold $(M, \lambda)$. (Definitions are recalled in Section 2.) In this situation, Kai Cieliebak and the first author defined in [10] a homology group $\mathrm{RFH}(\Sigma, M)$, the Rabinowitz–Floer homology of $\Sigma$, as the Floer homology associated to the Rabinowitz action functional

$$\mathcal{A}^F \colon \mathcal{L} \times \mathbb{R} \to \mathbb{R}, \qquad (v, \eta) \mapsto -\int_{S^1} v^*\lambda - \eta \int_{S^1} F(v(t))\, dt.$$

Here, $F \colon M \to \mathbb{R}$ is a suitable function with $F^{-1}(0) = \Sigma$, and $S^1 = \mathbb{R}/\mathbb{Z}$ denotes the circle and $\mathcal{L} = C^\infty(S^1, M)$ the free loop space of $M$. Note that the Rabinowitz action functional is invariant under the circle action $\tau v(\cdot) \mapsto v(\cdot - \tau)$ obtained by rotating the loop $v$. This makes it possible to construct the equivariant Rabinowitz–Floer homology $\mathrm{RFH}^{S^1}(\Sigma, M)$ as well.

Recall that $\Sigma$ is said to be Hamiltonian displaceable if there exists a compactly supported Hamiltonian diffeomorphism that disjoins $\Sigma$ from itself. One of the most useful properties of the Rabinowitz–Floer homology of $\Sigma$ is that it vanishes if $\Sigma$ is displaceable. The main result of this note is that this fact continues to hold in the equivariant case.

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**Theorem A** *Assume that $\Sigma$ is Hamiltonian displaceable. Then* $\mathrm{RFH}^{S^1}(\Sigma, M) = \{0\}$.

We shall prove this result by a leafwise intersection argument, following [2]. A more algebraic proof of Theorem A was given in [8] in the framework of symplectic homology, and their proof should also apply to Rabinowitz–Floer homology, cf. Section 6.

This note is organized as follows. In Section 2 we recall the construction of non-equivariant Rabinowitz–Floer homology $\mathrm{RFH}(\Sigma, M)$, and in Section 3 we construct $S^1$-equivariant Rabinowitz–Floer homology $\mathrm{RFH}^{S^1}(\Sigma, M)$. The core of this note is Section 4 in which we prove Theorem A. In Section 5 we give an alternative and somewhat easier approach to the invariance of $\mathrm{RFH}^{S^1}(\Sigma, M)$. In Section 6 we briefly discuss other approaches to proving $\mathrm{RFH}^{S^1}(\Sigma, M) = 0$ for displaceable hypersurfaces.

## 2. Recollections on Rabinowitz–Floer homology

In this section we recall the construction of the (non-equivariant) Rabinowitz–Floer homology of a hypersurface $\Sigma$ of restricted contact type, following [10] and [2]. Our construction of equivariant Rabinowitz–Floer homology in the next section will be based on this construction.

Consider an exact convex symplectic manifold $(M, \lambda)$. This means that $\lambda$ is a one-form on the connected manifold $M$ such that $d\lambda$ is a symplectic form, and that $(M, d\lambda)$ is convex at infinity, i.e., there exists an exhaustion $M = \bigcup_k M_k$ of $M$ by compact subsets $M_k \subset M_{k+1}$ with smooth boundaries $\partial M_k$ such that $\lambda|_{\partial M_k}$ is a contact form. We further fix a closed connected smooth hypersurface $\Sigma$ in $M$ that is bounding and of contact type. The former means that $M \setminus \Sigma$ has two components, one compact and one non-compact, and the latter means that $\lambda|_\Sigma$ is a contact form, or equivalently that the vector field $Y_\lambda$ implicitly defined by $\iota_{Y_\lambda} d\lambda = \lambda$ is transverse to $\Sigma$.

For a smooth function $F$ on $M$, the Hamiltonian vector field $X_F$ is defined by $\iota_{X_F} d\lambda = dF$, and $\varphi_F^t$ denotes the flow of $X_F$. The Reeb flow $\varphi_R^t$ on $\Sigma$ is the flow of the vector field $R$ defined by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$.

### 2.1. The Rabinowitz action functional

A *defining Hamiltonian for* $\Sigma$ is a smooth function $F \colon M \to \mathbb{R}$ such that $\Sigma = F^{-1}(0)$, such that $dF$ has compact support, and such that $\varphi_F^t$ restricts on $\Sigma$ to the Reeb flow $\varphi_R^t$ of $(\Sigma, \lambda|_\Sigma)$. The set of defining Hamiltonians is non-empty and convex. Given a defining Hamiltonian $F$, the Rabinowitz action functional $\mathcal{A}^F \colon \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{A}^F(v, \eta) = -\int_{S^1} v^*\lambda - \eta \int_{S^1} F(v(t))\, dt. \tag{1}$$

Its critical points $(v, \eta)$ are the solutions of the problem

$$\dot{v}(t) = \eta\, X_F(v(t)), \quad 0 = \int_{S^1} F(v(t))\, dt,$$

i.e., pairs $(v, \eta)$ with $\eta \in \mathbb{R}$ and $v$ a closed curve on $\Sigma$ of the form $v(t) = \varphi_F^{\eta t}$, $t \in \mathbb{R}$. The critical points therefore correspond to closed orbits of $X_F$ on the fixed energy surface $\Sigma = F^{-1}(0)$ of *arbitrary* period $|\eta| \geqslant 0$.[1] Since $v \subset \Sigma$ and $\varphi_F^t = \varphi_R^t$ along $\Sigma$,

$$\mathcal{A}^F(v, \eta) = -\int_{S^1} v^*\lambda = -\eta,$$

that is, the critical values of $\mathcal{A}^F$ are zero and minus the periods of the closed Reeb orbits on $\Sigma$.

The action functional $\mathcal{A}^F$ is invariant under the $S^1$-action on $\mathcal{L} \times \mathbb{R}$ given by

$$\tau \cdot (v(\cdot), \eta) \mapsto (v(\cdot - \tau), \eta). \tag{2}$$

Therefore, the functional $\mathcal{A}^F$ is thus never Morse. The component $\{(p, 0) \mid p \in \Sigma\} \cong \Sigma$ of the critical set is always Morse–Bott for $\mathcal{A}^F$, see [2, Lemma 2.12]. The following assumption on $\Sigma$ is sufficient for $\mathcal{A}^F$ to be Morse–Bott:

*Every periodic orbit of the Reeb flow* $\varphi_R^t$ *is non-degenerate.* (3)

[1] Despite J. Moser's explicit statement that the action functional (1) is useless for finding periodic orbits, [19, p. 731], P. Rabinowitz in [21, p. 161 and (2.7)] used precisely this functional to prove his celebrated existence theorem for periodic orbits on starshaped hypersurfaces in $\mathbb{R}^{2n}$, thus pioneering the use of global critical point methods in Hamiltonian mechanics. In [10] and subsequent papers, the functional (1) was therefore called Rabinowitz action functional. Other good names for this functional may be "fixed energy action functional" or "Hamiltonian free period action functional", since it selects solutions on the prescribed energy level $\{H = 0\}$, allowing for arbitrary period $|\eta|$.

In other words, for a $T$-periodic orbit $\gamma$ of the Reeb flow, 1 is not in the spectrum of the linearization $T_p\varphi_R^T \colon \xi_p \to \xi_p$ at $p = \gamma(0)$, where $\xi = \ker \lambda$ denotes the contact structure of $\Sigma$. This holds if and only if for any defining Hamiltonian $F$ of $\Sigma$, for every periodic orbit of $\varphi_F^t$ on $\Sigma$ the Floquet multiplier 1 has multiplicity 2.

### 2.2. Rabinowitz–Floer homology

Rabinowitz–Floer homology $\mathrm{RFH}(\Sigma, M)$ is the Floer homology of the functional $\mathcal{A}^F$, where $F$ is any defining Hamiltonian for $\Sigma$. We assume the reader to be familiar with the construction in [10], and also refer to [2] and to the survey [3]. Here, we only point out a few aspects in the construction of $\mathrm{RFH}(\Sigma, M)$ that do not arise in the construction of usual Hamiltonian Floer homology.

**1. The chain groups.** The functional $\mathcal{A}^F$ is not Morse, but Morse–Bott. One therefore chooses an auxiliary Morse function $h \colon \operatorname{Crit} \mathcal{A}^F \to \mathbb{R}$, and generates the chain groups by the critical points of $h$. However, even though the symplectic form $d\lambda$ is exact, the generators of the Rabinowitz–Floer chain groups $\mathrm{FC}(\mathcal{A}^F, h)$ are not finite sums $\sum \xi_c c$ with $\xi_c \in \mathbb{Z}_2$ and $c \in \operatorname{Crit} h$, but possibly infinite sums $\sum \xi_c c$ that for every $\kappa \in \mathbb{R}$ satisfy the finiteness condition

$$\#\big\{c \in \operatorname{Crit} h \mid \xi_c \neq 0,\, \mathcal{A}^F(c) \geqslant \kappa\big\} < \infty.$$

This must be done so for the following reason: Assume that $c$ lies on the critical point $(v, \eta)$ of $\mathcal{A}^F$, with $\eta \neq 0$. Then $\mathcal{A}^F(v, \eta) = -\eta$. Since with $(v, \eta)$ also $(v, k\eta)$ belongs to $\operatorname{Crit} \mathcal{A}^F$ for each $k \in \mathbb{Z}$, we see that $\mathcal{A}^F$ is not bounded from below on $\operatorname{Crit} \mathcal{A}^F$. Hence there may be infinitely many critical points that appear in the image $\partial c$ of the boundary operator.

**2. The almost complex structures.** Let $\mathcal{J}_{\mathrm{con}}$ be the set of almost complex structures on $M$ that are $d\lambda$-compatible and convex at infinity. The choice of the set of almost complex structures used to define $\mathrm{RFH}(\Sigma, M)$ depends on the method that one uses to establish transversality. If one works with polyfolds, one can take a fixed $J \in \mathcal{J}_{\mathrm{con}}$. In the next paragraph we describe the boundary operator in the traditional way. For this we fix $J_* \in \mathcal{J}_{\mathrm{con}}$ and following [1] consider the set $\mathscr{J}$ of smooth $S^1 \times \mathbb{R}$-families $\mathbf{J} = \{J_t(\cdot, \eta)\} \subset \mathcal{J}_{\mathrm{con}}$ such that

$$\sup_{t,\eta} \|J_t(\cdot,\eta)\|_{C^\ell} < \infty \quad \text{for all } \ell \in \mathbb{N} \tag{4}$$

and such that there exists a constant $c > 1$ (depending on the family) for which

$$\frac{1}{c}\|J_*(x)\| \leqslant \|J_t(x,\eta)\| \leqslant c\|J_*(x)\| \quad \text{for all } x \in M \text{ and } (t,\eta) \in S^1 \times \mathbb{R}. \tag{5}$$

Here, $\|\cdot\|$ is the norm taken with respect to some background Riemannian metric on $M$.

**3. The boundary operator.** The boundary operator $\partial$ on $\mathrm{FC}(\mathcal{A}^F, h)$ is defined by counting gradient flow lines with cascades (see [17, Appendix A]). These flow lines consist of (partial) negative gradient flow lines of $h$ and finite energy Floer gradient flow lines of $\mathcal{A}^F$. Given a family $\mathbf{J} \in \mathscr{J}$ and two critical points $(v_-, \eta_-)$ and $(v_+, \eta_+)$ of $\mathcal{A}^F$, a Floer gradient flow line is a solution $(v, \eta) \in C^\infty(\mathbb{R} \times S^1, M \times \mathbb{R})$ of the problem

$$\left.\begin{aligned} \partial_s v(s,t) + J_t(v(s,t),\eta(s))\big(\partial_t v(s,t) - \eta(s) X_F(v(t)\big) = 0, \\ \dot\eta(s) + \int_{S^1} F(v(s,t))\,dt = 0, \end{aligned}\right\} \tag{6}$$

with asymptotic boundary conditions $(v_-, \eta_-)$ and $(v_+, \eta_+)$. The main analytical issue in defining the boundary operator $\partial$ is to prove a uniform $L^\infty$-bound on the $\eta$-component of the solutions of (6) with given boundary conditions. This is done in [10, Corollary 3.3] for $\eta$-independent $J$, and the proof goes through thanks to (5). Assumption (4) is imposed to avoid bubbling, so that the space of all solutions of (6) is $C^\infty_{\mathrm{loc}}$-compact. Transversality for the space of solutions of (6) between two critical points for a generic set of $\mathbf{J} \in \mathscr{J}$ is proven in [1, Section 4.3].

We remark that the construction of the boundary operator by gradient flow lines with cascades in [17, Appendix A] is given for Morse homology on finite-dimensional manifolds. While this construction directly carries over to the case of Floer homology, some parts of this generalisation (such as gluing) are not worked out in the literature. The same applies to the $S^1$-equivariant Rabinowitz–Floer homology described in the next section. The foundational work coming closest to the holomorphic curve set-up considered in this paper is in [5], [6] and [23, Section 10]. Another way to rigorously establish RFH and $\mathrm{RFH}^{S^1}$ is by verifying that the flow lines with cascades fit into the $M$-polyfold set-up.

**4. Invariance.** The resulting homology group $\mathrm{FH}(\mathcal{A}^F) := \ker \partial / \operatorname{im} \partial$ does not depend on the choice of a defining function $F$ for $\Sigma$. One can therefore define $\mathrm{RFH}(\Sigma, M) := \mathrm{FH}(\mathcal{A}^F)$ for any choice of $F$. Moreover, given two bounding contact hypersurfaces $\Sigma_0$ and $\Sigma_1$ that are isotopic through a family $\{\Sigma_s\}_{0 \leqslant s \leqslant 1}$ of contact hypersurfaces,

$$\mathrm{RFH}(\Sigma_0, M) \cong \mathrm{RFH}(\Sigma_1, M). \tag{7}$$

For the proof, one chooses a smooth family $F_s \colon M \to \mathbb{R}$ of defining Hamiltonians for $\Sigma_s$ such that $F_s = F_0$ for $s \leqslant 0$ and $F_s = F_1$ for $s \geqslant 1$, and uses solutions of (6) with $F$ replaced by $F_s$ to construct a chain homotopy equivalence between $\mathrm{FC}(\mathcal{A}^{F_0}, h_0)$ and $\mathrm{FC}(\mathcal{A}^{F_1}, h_1)$. The main analytical issue is again proving a bound on the $\eta$-components, which can be done as in [10, Corollary 3.4] thanks to (5).

Recall that we have worked for now under the assumption (3). This assumption on $\Sigma$ is generic in the $C^\infty$-topology. In view of (7) we can define the Rabinowitz–Floer homology $\mathrm{RFH}(\Sigma, M)$ of any bounding contact hypersurface as $\mathrm{RFH}(\Sigma', M)$ where $\Sigma'$ is a close-by hypersurface meeting assumption (3).

## 3. Construction of equivariant Rabinowitz–Floer homology

In this section we give a Borel-type construction of $S^1$-equivariant Rabinowitz–Floer homology, closely following the construction of $S^1$-equivariant symplectic homology given by Viterbo in [26, Section 5], see also [7].

### 3.1. The equivariant Rabinowitz action functional

For each integer $N \geqslant 1$ denote by $S^{2N+1}$ the odd-dimensional unit sphere in $\mathbb{C}^{N+1}$. The circle $S^1$ acts on $S^{2N+1}$ by

$$\tau \cdot (z_1, \dots, z_{N+1}) = (\tau z_1, \dots, \tau z_{N+1}).$$

The quotient of this action is complex projective space $\mathbb{C}P^N = S^{2N+1}/S^1$. Recall the action (2) of $S^1$ on the loop space $\mathcal{L}$, and let $S^1$ act on $\mathcal{L} \times \mathbb{R} \times S^{2N+1}$ by the diagonal action

$$\tau \cdot \big(v(\cdot), \eta, z\big) = \big(v(\cdot - \tau), \eta, \tau \cdot z\big). \tag{8}$$

We shall denote the circle $S^1$ with this action on $\mathcal{L} \times \mathbb{R} \times S^{2N+1}$ by $\mathbb{T}$. Denote the quotient of this action by $\mathcal{L} \times \mathbb{R} \times_{\mathbb{T}} S^{2N+1}$. The functional $\widetilde{\mathcal{A}}^{F,N;\mathbb{T}} \colon \mathcal{L} \times \mathbb{R} \times S^{2N+1} \to \mathbb{R}$ defined by

$$\widetilde{\mathcal{A}}^{F,N;\mathbb{T}}(v, \eta, z) = -\int_{S^1} v^*\lambda - \eta \int_{S^1} F\big(v(t)\big)dt \tag{9}$$

is Morse–Bott if and only if the functional $\mathcal{A}^F$ defined in (1) is Morse–Bott. Indeed, the critical set of $\widetilde{\mathcal{A}}^{F,N;\mathbb{T}}$ is the critical set of $\mathcal{A}^F$ times $S^{2N+1}$. Since the functional (9) is invariant under the action (8), we can define the *equivariant Rabinowitz action functional* $\mathcal{A}^{F,N;\mathbb{T}} \colon \mathcal{L} \times \mathbb{R} \times_{\mathbb{T}} S^{2N+1} \to \mathbb{R}$ by

$$\mathcal{A}^{F,N;\mathbb{T}}([v, \eta, z]) = -\int_{S^1} v^*\lambda - \eta \int_{S^1} F\big(v(t)\big)dt, \tag{10}$$

and since the action (8) is free, this functional is Morse–Bott under the assumption (3) on $\Sigma$.

### 3.2. Equivariant Rabinowitz–Floer homology

$\mathbb{T}$-equivariant Rabinowitz–Floer homology $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$ is the direct limit in $N$ of the Floer homology of the functional $\mathcal{A}^{F,N;\mathbb{T}}$, where $F$ is any defining Hamiltonian for $\Sigma$.

**1. The chain groups.** Fix a defining Hamiltonian $F$ for $\Sigma$ meeting assumption (3), and fix $N \in \mathbb{N}$. Then $\widetilde{\mathcal{A}}^{F,N;\mathbb{T}}$ is Morse–Bott, with critical manifolds the union of $\Sigma \times \{0\} \times S^{2N+1}$ and $C_i \times \{k\,\eta_i\} \times S^{2N+1}$, $k \in \mathbb{Z} \setminus \{0\}$, where each $C_i \times \{\eta_i\}$ is a circle of simple Reeb orbits of period $\eta_i$. Since the action of $\mathbb{T}$ on $\mathcal{L} \times \mathbb{R} \times S^{2N+1}$ is free,

$$\mathrm{Crit}\, \mathcal{A}^{F,N;\mathbb{T}} = \mathrm{Crit}\, \widetilde{\mathcal{A}}^{F,N;\mathbb{T}}/\mathbb{T} = \mathrm{Crit}\, \mathcal{A}^F \times_{\mathbb{T}} S^{2N+1}$$

is a closed manifold. Denote by $g_{S^{2N+1}}$ the round Riemannian metric on $S^{2N+1}$, and choose a Riemannian metric $g_\Sigma$ on $\Sigma$ and $S^1$-invariant Riemannian metrics $g_{C_i}$ on $C_i$. Then the Riemannian metric $g_N$ on $\mathrm{Crit}\, \widetilde{\mathcal{A}}^{F,N;\mathbb{T}}$ defined by $g_N|_{\Sigma \times \{0\} \times S^{2N+1}} = g_\Sigma \oplus g_{S^{2N+1}}$ and $g_N|_{C_i \times \{k\,\eta_i\} \times S^{2N+1}} = g_{C_i} \oplus g_{S^{2N+1}}$ is $\mathbb{T}$-invariant, and hence descends to the Riemannian metric $g_N^{\mathbb{T}}$ on $\mathrm{Crit}\, \mathcal{A}^{F,N;\mathbb{T}}$. Choose a Morse function $h_N \colon \mathrm{Crit}\, \mathcal{A}^{F,N;\mathbb{T}} \to \mathbb{R}$ such that the pair $(h_N, g_N^{\mathbb{T}})$ is Morse–Smale (that is, the stable and unstable manifolds of the negative gradient flow of $h_N$ with respect to $g_N^{\mathbb{T}}$ intersect transversally). The chain group $\mathrm{FC}(\mathcal{A}^{F,N;\mathbb{T}}, h_N)$ consists of Novikov sums $\sum \xi_c c$ with $c \in \operatorname{Crit} h_N$, as in Section 2.2.

**2. The almost complex structures.** If one works with polyfolds, one can, again, just take a fixed $J \in \mathcal{J}_{\mathrm{con}}$. Here, we again fix $J_* \in \mathcal{J}_{\mathrm{con}}$ and look at smooth $S^1 \times S^{2N+1} \times \mathbb{R}$-families $\mathbf{J} = \{J_{t,z}(\cdot, \eta)\} \subset \mathcal{J}_{\mathrm{con}}$ such that

$$\sup_{t,z,\eta} \|J_{t,z}(\cdot,\eta)\|_{C^\ell} < \infty \quad \text{for all } \ell \in \mathbb{N} \tag{11}$$

and such that there exists a constant $c > 1$ (depending on the family) for which

$$\frac{1}{c}\|J_*(x)\| \leqslant \|J_{t,z}(x,\eta)\| \leqslant c\|J_*(x)\| \qquad \text{for all } x \in M \text{ and } (t,z,\eta) \in S^1 \times S^{2N+1} \times \mathbb{R}. \tag{12}$$

Furthermore, we impose that the family $\mathbf{J}$ is $S^1$-invariant:

$$J_{t+\tau,\tau z}(\cdot,\eta) = J_{t,z}(\cdot,\eta) \qquad \text{for all } (t,z,\eta) \in S^1 \times S^{2N+1} \times \mathbb{R} \text{ and } \tau \in S^1. \tag{13}$$

The space $\mathscr{J}^{S^1}$ of all families $\mathbf{J}$ in $\mathcal{J}_{\mathrm{con}}$ satisfying (11), (12) and (13) is non-empty (since property (13) is obtained by averaging over $S^1$) and contractible.

**3. The boundary operator.** Let $\widetilde{h}_N \colon \operatorname{Crit} \widetilde{\mathcal{A}}^{F,N;\mathbb{T}} \to \mathbb{R}$ be the lift of $h_N$. Then $\widetilde{h}_N$ is Morse–Bott, with $\mathbb{T}$-orbits as critical manifolds. Given two critical points $c^+$, $c^-$ of $h_N$, denote by $C^+$, $C^-$ the corresponding critical circles of $\widetilde{h}_N$. Given $\mathbf{J} \in \mathscr{J}^{S^1}$ consider all gradient flow lines with cascades $\widehat{\mathcal{M}}(c^+, c^-)$ from a point in $C^+$ to a point in $C^-$. Here, the (partial) Morse flow lines are (partial) negative gradient flow lines of $\widetilde{h}_N$ on $\operatorname{Crit} \widetilde{\mathcal{A}}^{F,N;\mathbb{T}}$ with respect to $g_N$, and the cascades (i.e., the Floer gradient flow lines) are finite energy solutions $(v, \eta, z) \in C^\infty(\mathbb{R} \times S^1, M \times \mathbb{R} \times S^{2N+1})$ of the problem

$$\left.\begin{aligned} \partial_s v(s,t) + J_{t,z(s)}\big(v(s,t),\eta(s)\big)\big(\partial_t v(s,t) - \eta(s) X_F\big(v(t)\big) &= 0,\\ \dot\eta(s) + \int_{S^1} F\big(v(s,t)\big)dt &= 0,\\ \dot z(s) + \nabla_{g_{S^{2N+1}}} \widetilde{h}_N(z(s)) &= 0. \end{aligned}\right\} \qquad (14)$$

Here, $\nabla_{g_{S^{2N+1}}} \widetilde{h}_N(z)$ denotes the component of $\nabla_{g_N} \widetilde{h}_N(z)$ along $T_z S^{2N+1}$. Since $g_N$ and $J$ are $\mathbb{T}$-invariant, $\mathbb{T}$ freely acts on $\widehat{\mathcal{M}}(c^+, c^-)$. The space $\widehat{\mathcal{M}}(c^+, c^-)$ therefore decomposes as

$$\widehat{\mathcal{M}}(c^+, c^-) = \coprod_{c \in C^+} \widehat{\mathcal{M}}(c, c^-)$$

where $\widehat{\mathcal{M}}(c, c^-)$ is the space of gradient flow lines with cascades from $c \in C^+$ with the last gradient flow line of $\widetilde{h}_N$ converging to an arbitrary point in $C^-$, and $\widehat{\mathcal{M}}(c^+, c^-)/\mathbb{T} \cong \widehat{\mathcal{M}}(c, c^-)$ for any $c \in C^+$. One shows as in [1, Section 4.3] that for a generic subset of families $\mathbf{J} \in \mathscr{J}^{S^1}$ the spaces $\widehat{\mathcal{M}}(c, c^-)$ are smooth manifolds.

The real numbers $s \in \mathbb{R}$ freely act by shift on each Floer gradient flow line in a gradient flow line with cascades in $\widehat{\mathcal{M}}(c^+, c^-)$. The space $\mathcal{M}(c^+, c^-) \cong \coprod_{c \in C^+} \mathcal{M}(c, c^-)$ obtained by modding out these $\mathbb{R}$-actions is compact. The main point in the proof is, again, a uniform $L^\infty$-bound on the $\eta$-component of the solutions of (14) with given boundary conditions. Such a bound is obtained exactly as in [10, Corollary 3.3], thanks to (12).

Now the boundary operator on $\mathrm{FC}(\mathcal{A}^{F,N;\mathbb{T}}, h_N)$ is defined by

$$\partial(c^+) = \sum_{c^-} \nu(c^+, c^-)\, c^-$$

where the sum runs over those $c^-$ for which $\mathcal{M}(c^+, c^-)/\mathbb{T} \cong \mathcal{M}(c, c^-)$ is 0-dimensional and where $\nu(c^+, c^-)$ is the number mod 2 of elements in this space.

**4. Invariance.** Let $\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}, h_N, J) := \ker \partial / \operatorname{im} \partial$ be the resulting homology groups. The inclusion $S^{2N+1} \to S^{2N+3}$ is $\mathbb{T}$-equivariant. In particular, $\operatorname{Crit} \mathcal{A}^{F,N;\mathbb{T}} \subset \operatorname{Crit} \mathcal{A}^{F,N+1;\mathbb{T}}$. Since $g_{S^{2N+3}}$ restricts to $g_{S^{2N+1}}$ on $S^{2N+1}$, the Riemannian metric $g_{N+1}$ restricts to $g_N$ on $\operatorname{Crit} \widetilde{\mathcal{A}}^{F,N;\mathbb{T}}$. Given a Morse function $h_N$ on $\operatorname{Crit} \mathcal{A}^{F,N;\mathbb{T}}$ as above, we choose a Morse function $h_{N+1}$ on $\operatorname{Crit} \mathcal{A}^{F,N+1;\mathbb{T}}$ such that $h_{N+1}$ extends $h_N$, such that $\operatorname{Crit} h_N \subset \operatorname{Crit} h_{N+1}$, and such that the pair $(h_{N+1}, g_{N+1}^{\mathbb{T}})$ is Morse–Smale. Further, we choose the family $J_{N+1} = J_{t,z}(\cdot,\eta)$ with $z \in S^{2N+3}$ such that it extends the family $J_N = J_{t,z}(\cdot,\eta)$ with $z \in S^{2N+1}$. The chain complex $\mathrm{FC}(\mathcal{A}^{F,N;\mathbb{T}}, h_N, J_N)$ is thus a subcomplex of $\mathrm{FC}(\mathcal{A}^{F,N+1;\mathbb{T}}, h_{N+1}, J_{N+1})$. We thus obtain a homomorphism

$$\iota_N \colon \mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}, h_N, J_N) \to \mathrm{FH}(\mathcal{A}^{F,N+1;\mathbb{T}}, h_{N+1}, J_{N+1}). \tag{15}$$

The groups $\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}, h_N, J_N)$ do not depend on the choice of $h_N$ and $J_N$, nor on the choice of $g_\Sigma$ in the definition of $g_N$, nor on the defining Hamiltonian $F$ for $\Sigma$. This is proven by Floer continuation as in [10] (see also Section 5). These continuation isomorphisms commute with the inclusion homomorphisms in (15): Given another defining Hamiltonian $F'$ and other choices $h'_N$ and $J'_N$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}, h_N, J_N) \arrow[r, "\iota_N"] \arrow[d, "\cong"] & \mathrm{FH}(\mathcal{A}^{F,N+1;\mathbb{T}}, h_{N+1}, J_{N+1}) \arrow[d, "\cong"] \\
\mathrm{FH}(\mathcal{A}^{F',N;\mathbb{T}}, h'_N, J'_N) \arrow[r, "\iota'_N"] & \mathrm{FH}(\mathcal{A}^{F',N+1;\mathbb{T}}, h'_{N+1}, J'_{N+1}).
\end{tikzcd}$$

The direct limit

$$\mathrm{RFH}^{\mathbb{T}}(\Sigma, M) := \varinjlim \mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}, h_N, J_N) \tag{16}$$

therefore only depends on $\Sigma$. In fact, $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$ is invariant under isotopies of bounding contact hypersurfaces (cf. Section 2.2).

**Remarks 3.1**

1. Our homology groups $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$ are not graded. We therefore do not need to assume that the first Chern class of $(M, d\lambda)$ vanishes on $\pi_2(M)$. Under this assumption, the groups $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$ carry a $\mathbb{Z}$-grading (with values in $1/2 + \mathbb{Z}$), cf. [10, Section 4].
2. The above construction of $S^1$-equivariant Rabinowitz–Floer homology should give the same result as the construction in [7] which uses parametrized symplectic homology, when applied to the parameter space $\mathbb{R} \times S^{2N+1}$: The difference in the construction is that our parameter space

$\mathbb{R} \times S^{2N+1}$ is not compact, and that we work with cascades instead of suitable perturbations of the Hamiltonian $F$. We expect that combining the construction in [7] with the $L^\infty$-estimates on the $\eta$-component from [10, Section 3] leads to the same groups $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$ in view of a version of the Correspondence Theorem 3.7 in [6].

A construction of an $S^1$-equivariant Rabinowitz–Floer homology that stays within the setting of $S^1$-equivariant symplectic homology was given recently in [13]. We expect that also this homology is isomorphic to $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$.

## 4. Proof of Theorem A

In this section we prove our main result: $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M) = 0$ if $\Sigma$ is displaceable. For the proof, we first recall how the analogous result is proven in the non-equivariant case. We shall apply the same method in the non-equivariant case.

### 4.1. The perturbed Rabinowitz action functional, and leafwise intersections

It has been shown in [10] that $\mathrm{RFH}(\Sigma, M)$ vanishes if $\Sigma$ is displaceable. This result has been reproved in [2] by a more geometric argument, in which the functional $\mathcal{A}^F$ is perturbed to a functional whose critical points are leafwise intersections. While the argument in [10] can be useful in problems where the leafwise intersection argument does not help (such as proving the existence of a closed characteristic on a displaceable stable hypersurface [12]), we here apply the leafwise intersection argument from [2].

A *perturbation pair* for the Rabinowitz action functional is a tuple

$$(\chi, H) \in C^\infty\big(S^1, [0,\infty)\big) \times C^\infty\big(M \times S^1, \mathbb{R}\big)$$

such that $\int_{S^1} \chi(t)\, dt = 1$. For a perturbation pair, the *perturbed Rabinowitz action functional* $\mathcal{A}^F_{\chi,H} \colon \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{A}^F_{\chi,H}(v,\eta) = -\int_{S^1} v^*\lambda - \eta \int_{S^1} \chi(t)\, F\big(v(t)\big)\, dt - \int_{S^1} H\big(v(t),t\big)\, dt. \qquad (17)$$

The critical points $(v,\eta)$ of this perturbed action functional are the solutions of the system

$$\left.\begin{aligned} \dot{v}(t) &= \eta\,\chi(t)\,X_F\big(v(t)\big) + X_H\big(v(t),t\big), \\ 0 &= \int_{S^1} \chi(t)\,F\big(v(t)\big)\,dt. \end{aligned}\right\} \tag{18}$$

As noticed in [2], it is useful to look at special perturbation pairs:

**Definition 4.1** A perturbation pair $(\chi, H)$ is called of *Moser type* if there exists $t_0 \in S^1$ such that the time support of $H$ lies in $[t_0, t_0 + 1/2]$ and the support of $\chi$ lies in $[t_0 - 1/2, t_0]$.

The energy hypersurface $\Sigma = F^{-1}(0)$ is foliated by its leaves $L_x = \{\varphi_F^t(x) \mid t \in \mathbb{R}\}$. Given a perturbation $H$ as above, a point $x \in \Sigma$ is called a *leafwise intersection point for* $H$ if $\varphi_H^1(x) \in L_x$. The following lemma was observed in [2].

**Lemma 4.2** *If a perturbation pair is of Moser type and* $(v, \eta)$ *is a solution of* (18)*, then* $v(t_0)$ *is a leafwise intersection point for* $H$ *on* $\Sigma = F^{-1}(0)$.

### 4.2. The perturbed equivariant Rabinowitz action functional

In order to show that $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$ vanishes for displaceable $\Sigma$, we wish to apply the same method as in the non-equivariant case.

In the following $S^1$ acts diagonally on $S^1 \times S^{2N+1}$ by $\tau(\cdot, z) = (\cdot - \tau, \tau z)$, and $S^1 \times_{S^1} S^{2N+1}$ is the quotient of $S^1 \times S^{2N+1}$ under this action. A *perturbation triple* is a triple $(\psi, G, k)$ in

$$C^\infty\big(S^1 \times_{S^1} S^{2N+1}, [0,\infty)\big) \times C^\infty\big(M \times S^1 \times_{S^1} S^{2N+1}, \mathbb{R}\big) \times C^\infty\big(\mathbb{C}P^N, \mathbb{R}\big)$$

such that for every $z \in S^{2N+1}$,

$$\int_{S^1} \psi\big([t,z]\big)dt = 1, \tag{19}$$

and such that $k$ is a Morse function on $\mathbb{C}P^N$. For a perturbation triple we define the perturbed equivariant Rabinowitz action functional

$$\mathcal{A}_{\psi,G,k} := \mathcal{A}^{F,N;\mathbb{T}}_{\psi,G,k} \colon \mathcal{L} \times \mathbb{R} \times_{\mathbb{T}} S^{2N+1} \to \mathbb{R} \tag{20}$$

by

$$\begin{aligned}\mathcal{A}_{\psi,G,k}\big([v,\eta,z]\big) = &-\int_{S^1} v^*\lambda - \eta\int_{S^1}\psi\big([t,z]\big)F\big(v(t)\big)dt\\ &-\int_{S^1} G\big(v(t),[t,z]\big)dt - k\big([z]\big).\end{aligned}$$

Denote by

$$\begin{aligned}&\widetilde{\psi}\in C^\infty\big(S^1\times S^{2N+1},[0,\infty)\big),\\ &\widetilde{G}\in C^\infty(M\times S^1\times S^{2N+1},\mathbb{R}),\\ &\widetilde{k}\in C^\infty(S^{2N+1},\mathbb{R})\end{aligned}$$

the lifts of $\psi$, $G$ and $k$. We can then write the lift of $\mathcal{A}_{\psi,G,k}$ to $\mathcal{L}\times\mathbb{R}\times S^{2N+1}$ as

$$\begin{aligned}\mathcal{A}_{\widetilde{\psi},\widetilde{G},\widetilde{k}}(v,\eta,z) = &-\int_{S^1} v^*\lambda - \eta\int_{S^1}\widetilde{\psi}(t,z)F\big(v(t)\big)dt\\ &-\int_{S^1}\widetilde{G}\big(v(t),t,z\big)dt - \widetilde{k}(z).\end{aligned} \tag{21}$$

The critical points $(v,\eta,z)$ of $\mathcal{A}_{\widetilde{\psi},\widetilde{G},\widetilde{k}}$ are the solutions of the system

$$\left.\begin{aligned}\dot v(t) &= \eta\,\widetilde{\psi}(t,z)\,X_F\big(v(t)\big) + X_{\widetilde{G}}\big(v(t),t\big),\\ 0 &= \int_{S^1}\widetilde{\psi}(t,z)F\big(v(t)\big)dt,\\ 0 &= \eta\int_{S^1}F\big(v(t)\big)\partial_z\widetilde{\psi}(t,z)dt - \int_{S^1}\partial_z\widetilde{G}\big(v(t),t,z\big)dt - d\widetilde{k}(z).\end{aligned}\right\} \tag{22}$$

**Definition 4.3** A perturbation triple $(\psi,G,k)$ is called *admissible* if the following two conditions hold.

( i ) For each $z\in S^{2N+1}$ and each solution $(v,\eta)$ of equation (18) with respect to the perturbation $(\widetilde{\psi}_z,\widetilde{G}_z)$ the identity $F\big(v(t)\big)d\widetilde{\psi}_t(z)=0$ holds for all $t\in S^1$.

(ii) $|d\widetilde{G}_{x,t}(z)\,\hat z| < |d\widetilde{k}(z)\,\hat z|$ for all $z\notin \operatorname{Crit}\widetilde{k}$, $\hat z\neq 0\in T_zS^{2N+1}$ and $(x,t)\in M\times S^1$.

**Lemma 4.4** *Assume that $(\psi,G,k)$ is an admissible perturbation triple.*

*Then critical points $[v, \eta, z]$ of $\mathcal{A}_{\psi,G,k}$ have the property that $[z]$ is a critical point of $k$, and for each $z \in S^{2N+1}$ over $[z]$, the pair $(v, \eta)$ is a solution to equation* (18) *for the perturbation $(\widetilde{\psi}_z, \widetilde{G}_z)$.*

*Proof.* In view of the first two equations in (22), we see that $(v, \eta)$ is a solution of (18) for the perturbation $(\widetilde{\psi}_z, \widetilde{G}_z)$. It remains to show that $[z]$ is a critical point of $k$. In view of the last equation in (22), for every $\hat{z} \in T_z S^{2N+1}$ the equation

$$\eta \int_{S^1} F\big(v(t)\big) d\widetilde{\psi}_t(z)\, \hat{z}\, dt + \int_{S^1} d\widetilde{G}_{v,t}(z)\, \hat{z}\, dt + d\widetilde{k}(z)\, \hat{z} = 0$$

has to be met. By assertion (i) of Definition 4.3, the first term vanishes. Now assertion (ii) implies $d\widetilde{k}(z)\, \hat{z} = 0$, hence $[z]$ is a critical point of $k$. □

**Definition 4.5** Given a perturbation pair of Moser type $(\chi, H)$, we call a perturbation triple $(\psi, G, k)$ an *equivariant extension* of $(\chi, H)$ if the following conditions hold.

- (I) The perturbation triple $(\psi, G, k)$ is admissible.
- (II) For every $z \in \operatorname{Crit} \widetilde{k}$ there exists $t_z \in S^1$ such that for every $t \in S^1$ and every $x \in M$ the identities $\widetilde{G}(x, t, z) = H(x, t + t_z)$ and $\widetilde{\psi}(t, z) = \chi(t + t_z)$ hold true.

**Lemma 4.6** *For any perturbation pair $(\chi, H)$ of Moser type, there exists an equivariant extension.*

*Proof.* Choose a Morse function $k$ on $\mathbb{C}P^N$. For every $y \in \operatorname{Crit} k$ choose open neighborhoods

$$y \in U_y \subset \overline{U_y} \subset V_y \subset \overline{V_y} \subset W_y$$

with the property that $W_y$ is contractible, and for different critical points $y$ and $y'$ of $k$ the neighborhoods $W_y$ and $W_{y'}$ are disjoint. Since $W_y$ is contractible, the principal $S^1$-bundle $\pi\colon S^{2N+1} \to \mathbb{C}P^N$ can be trivialized over $W_y$. We abbreviate

$$X = \bigcup_{y \in \operatorname{Crit} k} \pi^{-1}(W_y)$$

and choose a trivialization

$$\Phi\colon X \to \pi(X) \times S^1.$$

We further choose smooth cutoff functions $\beta_1, \beta_2 \in C^\infty(\mathbb{C}P^N, [0,1])$ with the property that for every $y \in \operatorname{Crit} k$,

$$\beta_1|_{U_y} = 1, \quad \beta_2|_{V_y} = 1$$

and

$$\operatorname{supp}(\beta_1) \subset \bigcup_{y \in \operatorname{Crit} k} V_y, \qquad \operatorname{supp}(\beta_2) \subset \bigcup_{y \in \operatorname{Crit} k} W_y.$$

We further abbreviate by $p\colon X \times S^1 \to S^1$ the projection to the second factor. We now set

$$\widetilde{G}(x,t,z) = \begin{cases} \beta_1([z])\, H\big(x, t + p(\Phi(z))\big), & z \in X, \\ 0, & z \notin X. \end{cases}$$

and

$$\widetilde{\psi}(t,z) = \begin{cases} \beta_2([z])\, \chi\big(t + p(\Phi(z))\big) + 1 - \beta_2([z]), & z \in X, \\ 1, & z \notin X. \end{cases}$$

Define $G$ and $\psi$ by $G(x,[t,z]) = \widetilde{G}(x,t,z)$ and $\psi([t,z]) = \widetilde{\psi}(t,z)$. Then the perturbation triple $(\psi, G, k)$ satisfies condition (II) of an equivariant extension. Moreover, since the perturbation pair $(\chi, H)$ is of Moser type, the triple $(\psi, G, k)$ also meets condition (i) of admissibility. It does not necessarily satisfy condition (ii) of admissibility. However, we can remedy this by replacing $k$ by $Ck$ for a large enough positive constant $C$. This finishes the proof of the lemma. □

### 4.3. Proof of Theorem A

Assume that $\Sigma$ is displaceable in $M$, and choose a defining Hamiltonian $F\colon M \to \mathbb{R}$ for $\Sigma$ meeting assumption (3). In view of the definition (16) of $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$, it suffices to show that $\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}) = 0$ for each $N$. So fix $N \in \mathbb{N}$.

Choose $\chi\colon S^1 \to [0,\infty)$ with $\operatorname{supp}(\chi) \subset (0, 1/2)$ and $\int_{S^1} \chi(t)\, dt = 1$, and choose a Hamiltonian function $H\colon M\times S^1\to\mathbb{R}$ with $H(\cdot,t)=0$ for all $t\in[0,1/2]$ whose time 1-flow $\varphi_H$ displaces $\Sigma$. By Lemma 4.6, the pair $(\chi,H)$ has an equivariant extension $(\psi,G,k)$. Let $[v,\eta,z]$ be a critical point of $\mathcal{A}_{\psi,G,k}$. Choose $z\in S^{2N+1}$ over $[z]$. By Lemma 4.4 and by (II) of Definition 4.5,

$$\left.\begin{aligned} \dot v(t) &= \eta\,\chi(t+t_z)\,X_F\big(v(t)\big)+X_{H(\cdot,t+t_z)}\big(v(t),t\big),\\ 0 &= \int_{S^1}\chi(t+t_z)\,F\big(v(t)\big)dt. \end{aligned}\right\}$$

By Lemma 4.2, $v(t_z)$ is a leafwise intersection point for $H(\cdot,t+t_z)$. This is impossible because $\varphi_H$ displaces $\Sigma$. It follows that the functional $\mathcal{A}_{\psi,G,k}=\mathcal{A}^{F,N;\mathbb{T}}_{\psi,G,k}$ has no critical points. The Floer homology $\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}_{\psi,G,k})$ is defined along the lines of Section 3.2, see Section 5. Since $\mathcal{A}^{F,N;\mathbb{T}}_{\psi,G,k}$ has no critical points, the Floer complex of $\mathcal{A}^{F,N;\mathbb{T}}_{\psi,G,k}$ is trivial, and hence $\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}_{\psi,G,k})=0$. Theorem A thus follows from the invariance $\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}_{\psi,G,k})\cong\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}})$, which is proven in the next section.

## 5. Invariance

The goal of this section is to prove

**Proposition 5.1** $\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}}_{\psi,G,k})\cong\mathrm{FH}(\mathcal{A}^{F,N;\mathbb{T}})$.

This isomorphism can be proven along the lines of the proof of Corollary 3.4 in [10]. In this section we give a different proof.

We start with reviewing two continuation methods for showing invariance of a Floer-type homology. For simplicity, we describe these methods in the setting of Morse homology and Morse–Bott homology on a non-compact manifold $M$. For $i=0,1$ let $f_i\colon M\to\mathbb{R}$ be smooth Morse functions with compact critical sets $\mathrm{Crit}\,f_i$.

**Method 1** Assume that there is a smooth family $\{f_s\}_{0\leqslant s\leqslant 1}$ of Morse functions $f_s\colon M\to\mathbb{R}$ such that the critical sets $\mathrm{Crit}\,f_s$ are all isotopic. More precisely, assume that there is a diffeomorphism

$$\Psi\colon\ \mathrm{Crit}\,f_0\times[0,1]\to\coprod_{0\leqslant s\leqslant 1}\mathrm{Crit}\,f_s\times\{s\},\qquad \big(x,s\big)\mapsto\big(x_s,s\big).$$

For a Riemannian metric $j_s$ on $M$ and for $x_s, y_s \in \operatorname{Crit} f_s$ denote by $\widehat{\mathcal{M}}_{f_s,j_s}(x_s, y_s)$ the set of negative gradient flow lines from $x_s$ to $y_s$, and by $\mathcal{M}_{f_s,j_s}(x_s, y_s) := \widehat{\mathcal{M}}_{f_s,j_s}(x_s, y_s)/\mathbb{R}$ the space of unparametrized gradient flow lines.

For $s = 0, 1$ choose a Riemannian metric $j_s$ on $M$ such that the pair $(f_s, j_s)$ is Morse–Smale. Then one can define the Morse homology of $f_s$ by counting elements of $\mathcal{M}_{f_s,j_s}(x_s, y_s)$ for $x_s, y_s \in \operatorname{Crit} f_s$ with $\operatorname{ind}(x_s) = \operatorname{ind}(y_s) + 1$, $s = 0, 1$. For a generic smooth path of Riemannian metrics $\{j_s\}$ from $j_0$ to $j_1$ and for $x, y \in \operatorname{Crit} f_0$ with $\operatorname{ind}(x) = \operatorname{ind}(y) + 1$, the union of moduli spaces

$$\mathcal{M}_{\{f,j\}}(\{x\}, \{y\}) = \bigcup_{0 \leqslant s \leqslant 1} \mathcal{M}_{f_s,j_s}\big(x_s, y_s\big) \times \{s\}$$

is then a 1-dimensional smooth manifold with boundary that is "transverse at 0 and 1", i.e., for $s = 0, 1$ the points in $\mathcal{M}_{f_s,j_s}\big(x_s, y_s\big) \times \{s\}$ belong to the boundary of $\mathcal{M}_{\{f,j\}}(\{x\}, \{y\})$, see Figure 1. If one can show that the sets $\widehat{\mathcal{M}}_{f_s,j_s}\big(x_s, y_s\big)$, $0 \leqslant s \leqslant 1$, are uniformly bounded, it follows that the Morse homologies of $f_0$ and of $f_1$ are isomorphic.

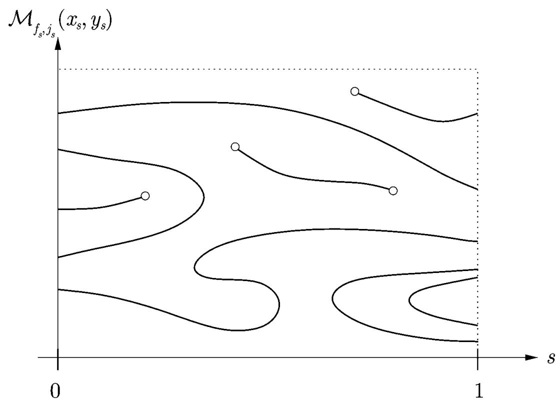



Figure 1. The union of moduli spaces $\bigcup_{0 \leqslant s \leqslant 1} \mathcal{M}_{f_s,j_s}\big(x_s, y_s\big) \times \{s\}$.

Indeed, $\mathcal{M}_{\{f,j\}}(\{x\}, \{y\})$ is the union $\mathcal{M}_1 \coprod \mathcal{M}_2$ of two types of components: The components of $\mathcal{M}_1$ are compact intervals with boundary over 0 and 1, and the components of $\mathcal{M}_2$ are half-open intervals (with boundary over 0 or 1) or open intervals. If $\mathcal{M}_2$ is empty, then the coefficients

$\nu(x_i, y_i^k) = \#\mathcal{M}_{f_i,j_i}(x_i, y_i^k) \mod 2$ in the boundary operator

$$\partial_i\, x_i = \sum_k \nu(x_i, y_i^k)\, y_i^k$$

are the same for $i = 0, 1$. The components of $\mathcal{M}_2$ may change the coefficients $\nu(x_i, y_i^k)$, but they do not alter the Morse homology. Indeed, the contribution of the components of $\mathcal{M}_2$ to the boundary operator can be computed explicitely, and from this one can write down an explicit chain homotopy equivalence between the Morse chain complexes of $(f_0, j_0)$ and $(f_1, j_1)$, see [15, Lemmata 3.5 and 3.6]. We illustrate this by an example:

Suppose $\operatorname{Crit} f_s$ has three critical points, $a_s, b_s$ of index 1 and $c_s$ of index 0. Suppose that at $s = 0$ there is exactly one gradient flow line $\gamma_0$, from $b_0$ to $c_0$. Then the Morse homology is generated by $a_0$:

$$\mathrm{MH}(f_0, j_0) = \mathrm{MH}_1(f_0, j_0) = \mathbb{Z}_2\langle a_0 \rangle.$$

Assume now that at some time $s^* \in (0,1)$ a gradient flow line $\gamma_{ab}$ from $a_{s^*}$ to $b_{s^*}$ appears. This flow line is not generic, and immediately disappears. The flow line $\gamma_{ab}$ affects the two families of moduli spaces $\mathcal{M}_{f_s,j_s}(b_s, c_s)$ and $\mathcal{M}_{f_s,j_s}(a_s, c_s)$ as follows: The moduli spaces $\mathcal{M}_{f_s,j_s}(b_s, c_s)$ are not affected: Before time $s^*$ this space contains exactly one gradient flow line $\gamma_s$, which persists beyond time $s^*$.

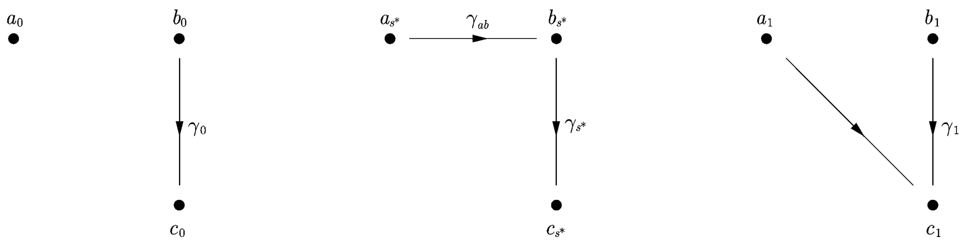
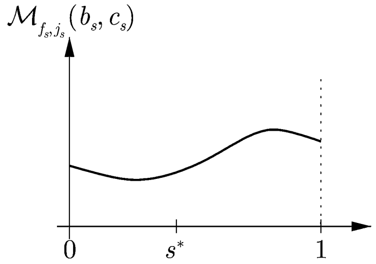
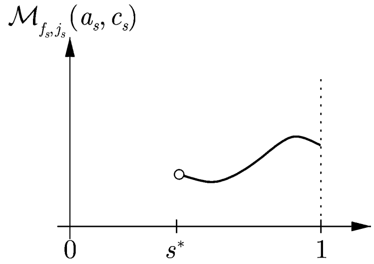



Figure 2. The gradient flow lines at $s = 0$, $s = s^*$, $s = 1$.

The moduli spaces $\mathcal{M}_{f_s,j_s}(a_s, c_s)$ were empty for $s < s^*$. At time $s^*$ there is a broken gradient flow line from $a_{s^*}$ to $c_{s^*}$, namely $\gamma_{ab}$ followed by the gradient flow line $\gamma_{s^*}$ from $b_{s^*}$ to $c_{s^*}$. These two flow lines can be glued together to a unique gradient flow line from $a_s$ to $c_s$. Hence $\nu(a_s, c_s)$ changes at $s^*$ from 0 to 1. For $s > s^*$ we now have one gradient flow line from $a_s$ to $c_s$ and one from $b_s$ to $c_s$. But this change does not affect the




Morse homology: $c_s$ is still in the image of the boundary operator $\partial_s$, and while now neither $a_s$ nor $b_s$ are in the kernel, $a_s - b_s$ is in the kernel of $\partial_s$. Hence we still have

$$\mathrm{MH}(f_1, j_1) = \mathrm{MH}_1(f_1, j_1) = \mathbb{Z}_2\langle a_1 - b_1\rangle.$$

A bifurcation as above, that creates a component in $\mathcal{M}_2$, is called a slide bifurcation, or a handle slide, since such a bifurcation acts on the corresponding handle decomposition of $M$ by sliding one handle over another. The other type of bifurcation that appears in a generic isotopy between Morse functions are birth bifurcations and death bifurcations, namely the birth of two critical points or the cancellation of two critical points. Such bifurcations do not arise in the situation at hand.

Below we shall apply this method in a Morse–Bott set-up: Assume there is a smooth family $\{f_s\}_{0\leqslant s\leqslant 1}$ of Morse–Bott functions $f_s\colon M \to \mathbb{R}$ with compact critical sets $\operatorname{Crit} f_s$ and a diffeomorphism

$$\Psi\colon\ \operatorname{Crit} f_0 \times [0,1] \to \coprod_{0\leqslant s\leqslant 1} \operatorname{Crit} f_s \times \{s\}, \qquad (x,s) \mapsto (x_s, s).$$

Choose a Morse function $h_0$ on $\operatorname{Crit} f_0$. Then the functions

$$h_s(x_s) := h_0(x)$$

are Morse functions on $\operatorname{Crit} f_s$, and the sets $\operatorname{Crit} h_s$ are isotopic. For a Riemannian metric $g_s$ on $\operatorname{Crit} f_s$, for a Riemannian metric $j_s$ on $M$ and for $x_s, y_s \in \operatorname{Crit} h_s$ denote by $\widehat{\mathcal{M}}_{f_s,j_s,h_s,g_s}(x_s,y_s)$ the set of negative gradient flow lines with cascades from $x_s$ to $y_s$, and by $\mathcal{M}_{f_s,j_s,h_s,g_s}(x_s,y_s)$ the space of unparametrized gradient flow lines with cascades.

For $s = 0,1$ choose a Riemannian metric $g_s$ on $\operatorname{Crit} f_s$ such that the pair $(h_s, g_s)$ is Morse–Smale. For generic Riemannian metrics $j_s$ on $M$ one can then define the Morse–Bott homology of the quadruples $(f_s, j_s, h_s, g_s)$, $s = 0, 1$, by counting elements of the 0-dimensional components $\mathcal{M}_{f_s,j_s,h_s,g_s}(x_s, y_s)$, see [17, Appendix A]. For a generic smooth path of Riemannian metrics $\{g_s\}$ on $\operatorname{Crit} f_s$ and for a generic smooth path of Riemannian metrics $\{j_s\}$ on $M$ from $(j_0, g_0)$ to $(j_1, g_1)$, we have that for each pair $x, y \in \operatorname{Crit} h_0$ for which $\mathcal{M}_{f_0,j_0,h_0,g_0}(x, y)$ is 0-dimensional, the union of moduli spaces

$$\mathcal{M}_{\{f,j,h,g\}}(\{x\}, \{y\}) = \big\{(u, s) \mid u \in \mathcal{M}_{f_s,j_s,h_s,g_s}\big(x_s, y_s\big);\ 0 \leqslant s \leqslant 1\big\},$$

is a 1-dimensional smooth manifold with boundary that is "transverse at 0 and 1". If one can show that the sets $\widehat{\mathcal{M}}_{f_s,j_s,g_s,h_s}\big(x_s, y_s\big)$, $0 \leqslant s \leqslant 1$, are uniformly bounded, it follows that the Morse homologies of $f_0$ and of $f_1$ are isomorphic.

**Method 2** (Floer continuation) Choose a smooth monotone function $\beta\colon \mathbb{R} \to [0, 1]$ with $\beta(s) = 0$ for $s \leqslant 0$ and $\beta(s) = 1$ for $s \geqslant 1$. For $s \in \mathbb{R}$ define the function

$$f_s = (1 - \beta(s))f_0 + \beta(s)f_1.$$

For $x \in \operatorname{Crit} f_0$ and $y \in \operatorname{Crit} f_1$ and for a smooth family of Riemannian metrics $\{g_s\}$ with $g_s = g_0$ for $s \leqslant 0$ and $g_s = g_1$ for $s \geqslant 1$ consider the gradient equation with asymptotic boundary conditions

$$\begin{cases} \dot u(s) = -\nabla_{g_s} f_s(u(s)), \quad s \in \mathbb{R}; \\ \lim\limits_{s\to-\infty} u(s) = x, \quad \lim\limits_{s\to\infty} u(s) = y. \end{cases} \tag{23}$$

For a generic choice of the path $\{g_s\}$ and for $x \in \operatorname{Crit} f_0$ and $y \in \operatorname{Crit} f_1$ with $\operatorname{ind}(x) = \operatorname{ind}(y)$, the space of solutions to (23) is a smooth 0-dimensional manifold. If one can show that this space is bounded, then it is finite. Counting these solutions then defines a chain homomorphism between the Morse chain complexes of $f_0$ and $f_1$, that induces an isomorphism between the Morse homologies of $f_0$ and $f_1$.

Similarly, given triples $(j_s, h_s, g_s)$ for $s = 0, 1$ with $(h_s, g_s)$ Morse–Smale pairs and $j_s$ generic, Floer continuation can be used to show that the Morse homologies of $(f_0, j_0, h_0, g_0)$ and $(f_1, j_1, h_1, g_1)$ are isomorphic, see [17, Theorem A.17].

**Historical Remark** Floer used Method 1 in [15] to prove invariance of his homology for Lagrangian intersections. (He also dealt with isolated bifurcations of the critical sets, namely birth and death bifurcations, by first putting them into normal form and then constructing a chain map between the complex before and after the bifurcation that induces an isomorphism in homology.) Such a bifurcation analysis was later also used in [14], [18], [25]. The powerful and flexible Method 2 was invented by Floer only later in [16]. $\diamondsuit$

Proposition 5.1 can be proven by Method 2, by adapting the proof of Corollary 3.4 in [10]. We leave the minor modifications to the interested reader. Here we give a different argument, that takes into account the structure of the functional $\mathcal{A}^{F,N;\mathbb{T}}_{\psi,G,k}$, and uses Method 1 once and Method 2 twice.

Consider the four functionals on $\mathcal{L} \times \mathbb{R} \times_{\mathbb{T}} S^{2N+1}$,

$$
\begin{aligned}
\mathcal{A}_0([v,\eta,z]) &= -\int_{S^1} v^*\lambda - \eta \int_{S^1} F(v(t))dt, \\
\mathcal{A}_1([v,\eta,z]) &= -\int_{S^1} v^*\lambda - \eta \int_{S^1} F(v(t))dt \qquad\qquad\qquad\qquad - \widetilde{k}(z), \\
\mathcal{A}_2([v,\eta,z]) &= -\int_{S^1} v^*\lambda - \eta \int_{S^1} \widetilde{\psi}(t,z)F(v(t))dt \qquad\qquad\qquad - \widetilde{k}(z), \\
\mathcal{A}_3([v,\eta,z]) &= -\int_{S^1} v^*\lambda - \eta \int_{S^1} \widetilde{\psi}(t,z)F(v(t))dt - \int_{S^1} \widetilde{G}(v(t),t,z)dt - \widetilde{k}(z).
\end{aligned}
$$

The functionals $\mathcal{A}_0$ and $\mathcal{A}_1$ are Morse–Bott by our assumption (3) on $\Sigma$ and since $k$ is Morse, while $\mathcal{A}_2$ is Morse–Bott by Lemma 5.2 below. The functional $\mathcal{A}_3$ is Morse–Bott because it has no critical points. Hence the four lifted functionals $\widetilde{\mathcal{A}}_i \colon \mathcal{L} \times \mathbb{R} \times S^{2N+1} \to \mathbb{R}$ are also Morse–Bott.

The Floer homology $\mathrm{FH}(\mathcal{A}_0) = \mathrm{FH}(\mathcal{A}_0, h_0, J_0)$ was defined in Section 3.2, and the Floer homology $\mathrm{FH}(\mathcal{A}_i)$ for $i = 1, 2, 3$ is defined in the same way: One chooses a Morse function $h_i$ and a Riemannian metric $g_i$ on $\mathrm{Crit}\,\mathcal{A}_i$ such that $(h_i, g_i)$ is a Morse–Smale pair, lifts them to the Morse–Bott function $\widetilde{h}_i$ and the $\mathbb{T}$-invariant metric $\widetilde{g}_i$ on $\mathrm{Crit}\,\widetilde{\mathcal{A}}_i$, and defines the boundary of a critical point $c^+$ of $h_i$ by counting rigid $\mathbb{T}$-families of unparametrized negative gradient flow lines with cascades in $\mathcal{M}(c^+, c^-)$ between critical $\mathbb{T}$-orbits $C^+$ and $C^-$ of $\widetilde{h}_i$, with respect to the $\mathbb{T}$-invariant Riemannian metric $\widetilde{g}_i$ on $\operatorname{Crit} \widetilde{\mathcal{A}}_i$ and a generic family $J_{t,z}(\cdot, \eta)$ in $\mathscr{J}^{S^1}$.

It follows from Method 2 that $\mathrm{FH}(\mathcal{A}_0) \cong \mathrm{FH}(\mathcal{A}_1)$ and that $\mathrm{FH}(\mathcal{A}_2) \cong \mathrm{FH}(\mathcal{A}_3)$. This is easy for the passage $\mathcal{A}_0 \rightsquigarrow \mathcal{A}_1$: The summand $\widetilde{k}(z)$ is bounded with all its derivatives. The claim thus follows from the $L^\infty$-bound on each space $\widehat{\mathcal{M}}(c^+, c^-)$ of gradient flow lines with cascades between a pair of critical circles of $\widetilde{h}_0$ given in the proof of Corollary 3.3 in [10]. For the passage $\mathcal{A}_2 \rightsquigarrow \mathcal{A}_3$, invariance follows as in [2, Section 2], by either choosing $\widetilde{G}$ sufficiently small in $L^\infty$ (which we are free to do) or by decomposing the isotopy $\mathcal{A}_2 \rightsquigarrow \mathcal{A}_3$ into many small isotopies.

The isomorphism $\mathrm{FH}(\mathcal{A}_1) \cong \mathrm{FH}(\mathcal{A}_2)$ can also be shown by applying Method 2 to the parts of a sufficiently fine decomposition of the isotopy $\mathcal{A}_1 \rightsquigarrow \mathcal{A}_2$ (see the proof of Corollary 3.4 in [10]). This argument is somewhat harder, since $\eta$ appears in front of the summand that is altered. We circumvent this difficulty by applying Method 1. Choose a smooth monotone function $\beta \colon [1,2] \to [0,1]$ with $\beta(s) = 0$ for $s$ near 1 and $\beta(s) = 1$ for $s$ near 2. For $s \in [1,2]$ set

$$\widetilde{\psi}_s(t,z) = \big(1 - \beta(s)\big) \cdot 1 + \beta(s) \cdot \widetilde{\psi}(t,z) = 1 + \beta(s)\big(\widetilde{\psi}(t,z) - 1\big).$$

Then $\int_{S^1} \widetilde{\psi}_s(t,z)dt = 1$ for all $s$. Consider the family of functionals

$$\widetilde{\mathcal{A}}_s(v,\eta,z) := -\int_{S^1} v^*\lambda - \eta \int_{S^1} \widetilde{\psi}_s(t,z)\, F(v(t))\, dt + \widetilde{k}(z), \qquad 1 \leqslant s \leqslant 2.$$

Then $\widetilde{\mathcal{A}}_s = \widetilde{\mathcal{A}}_1$ for $s$ near 1 and $\widetilde{\mathcal{A}}_s = \widetilde{\mathcal{A}}_2$ for $s$ near 2.

The critical manifolds $\operatorname{Crit} \widetilde{\mathcal{A}}_s$ are in canonical bijection with $\operatorname{Crit} \widetilde{\mathcal{A}}_1$. Indeed, looking at (22) with $\widetilde{G} = 0$ and $\widetilde{\psi}$ replaced by $\widetilde{\psi}_s$, we see that they all contain $\Sigma \times \{0\} \times \operatorname{Crit} \widetilde{k}$. Moreover, given $z \in \operatorname{Crit} \widetilde{k}$, and with

$$s_z(t) := \int_0^t \widetilde{\psi}_s(\tau, z)\, d\tau,$$

the periodic orbit $(v(t), \eta, z)$ of $X_F$ with period $|\eta|$ corresponds to the reparametrized orbit $(v(s_z(t)), \eta, z)$ of $\widetilde{\psi}_s(t,z)\, X_F$ with period $|\eta|$. (The orbit $v(s_z(t))$ also has period $|\eta|$ because $s_z(1) = 1$.) More formally, the reparametrization map

$$\widetilde{\Psi}\colon \operatorname{Crit}\widetilde{\mathcal{A}}_1 \times [1,2] \to \coprod_{1\leqslant s\leqslant 2} \operatorname{Crit}\widetilde{\mathcal{A}}_s \times \{s\},$$
$$\big((v(\cdot),\eta,z),s\big) \mapsto \big((v(s_z(\cdot)),\eta,z),s\big)$$

is a diffeomorphism.

**Lemma 5.2** *For each $s \in [1,2]$ the critical set $\operatorname{Crit}\widetilde{\mathcal{A}}_s$ is a Morse–Bott submanifold of $\widetilde{\mathcal{A}}_s$.*

Before giving the proof, we use the lemma to prove Proposition 5.1. All the functionals $\widetilde{\mathcal{A}}_s$ and all the sets $\operatorname{Crit}\widetilde{\mathcal{A}}_s$ are $\mathbb{T}$-invariant. Choose a Morse function $h_1$ on $\operatorname{Crit}\mathcal{A}_1$. Then the functions

$$h_s([v(s_z(\cdot)),\eta,z]) := h_1([v(\cdot),\eta,z])$$

are Morse functions on $\operatorname{Crit}\mathcal{A}_s$, and the sets $\operatorname{Crit} h_s$ are isotopic.

For a Riemannian metric $g_s$ on $\operatorname{Crit}\mathcal{A}_s$, for a family $\mathbf{J}_s := (J_{t,z}(\cdot,\eta))_s$ in $\mathscr{J}^{S^1}$ and for $c_s^+, c_s^- \in \operatorname{Crit} h_s$, denote by $\widehat{\mathcal{M}}_{\mathcal{A}_s,\mathbf{J}_s,h_s,g_s}(c_s^+,c_s^-)$ the set of negative gradient flow lines with cascades from $c_s^+$ to $c_s^-$, and by $\mathcal{M}_{\mathcal{A}_s,\mathbf{J}_s,h_s,g_s}(c_s^+,c_s^-)$ the space of unparametrized $\mathbb{T}$-families of gradient flow lines with cascades, as constructed in Section 3.2.3.

For $s = 1,2$ choose $g_s$ and $\mathbf{J}_s$ as in the definition of the Floer homologies $\mathrm{FH}(\mathcal{A}_s)$: $g_s$ is a Riemannian metric on $\operatorname{Crit}\mathcal{A}_s$ such that $(h_s,g_s)$ is a Morse–Smale pair, and $\mathbf{J}_s$ is a generic family in $\mathscr{J}^{S^1}$. Then for a generic smooth path of Riemannian metrics $\{g_s\}$ on $\operatorname{Crit}\mathcal{A}_s$ and for a generic smooth path of families $\{\mathbf{J}_s\}$ in $\mathscr{J}^{S^1}$ from $(g_1,\mathbf{J}_1)$ to $(g_2,\mathbf{J}_2)$, we have that for each pair $c^+, c^- \in \operatorname{Crit} h_1$ for which $\mathcal{M}_{\mathcal{A}_1,\mathbf{J}_1,h_1,g_1}(c^+,c^-)$ is 0-dimensional, the union of moduli spaces

$$\mathcal{M}_{\mathcal{A},\mathbf{J},h,g}(\{c^+\},\{c^-\}) = \bigcup_{1\leqslant s\leqslant 2} \mathcal{M}_{\mathcal{A}_s,\mathbf{J}_s,h_s,g_s}\big(c_s^+,c_s^-\big) \times \{s\}$$

is a 1-dimensional smooth manifold that is "transverse at 0 and 1". Notice that the map $\widetilde{\Psi}$ is action-preserving: $\widetilde{\mathcal{A}}_s(x_s) = \widetilde{\mathcal{A}}_1(x)$. The space $\widehat{\mathcal{M}}_{\mathcal{A}_1,\mathbf{J}_1,h_1,g_1}(c^+,c^-)$ is $L^\infty$-bounded, and in fact there is a uniform $L^\infty$-bound on the spaces $\widehat{\mathcal{M}}_{\mathcal{A}_s,\mathbf{J}_s,h_s,g_s}\big(c_s^+,c_s^-\big)$, $1 \leqslant s \leqslant 2$, see the proof of

Corollary 3.3 in [10]. It follows that $\mathrm{FH}(\mathcal{A}_1) \cong \mathrm{FH}(\mathcal{A}_2)$.

*Proof of Lemma 5.2.* We use the method in Appendix A.1 of [2]. Fix $s$, and fix a critical point $(v_0, \eta_0, z_0) \in \mathcal{L} \times \mathbb{R} \times S^{2N+1}$. We decompose $\widetilde{\mathcal{A}}_s$ as

$$\widetilde{\mathcal{A}}_s(v, \eta, z) = \mathcal{A}_0(v) + \eta_0 \, \mathcal{F}_{\Delta\psi}(v, z) + (\eta_0 - \eta) \, \mathcal{F}_\psi(v, z) + \widetilde{k}(z)$$

where

$$\begin{aligned} \mathcal{A}_0(v) &:= -\int_{S^1} v^* \lambda - \eta_0 \int_{S^1} \widetilde{\psi}_s(t, z_0) F(v(t)) \, dt, \\ \mathcal{F}_{\Delta\psi}(v, z) &:= \int_{S^1} \big(\widetilde{\psi}_s(t, z_0) - \widetilde{\psi}_s(t, z)\big) F(v(t)) \, dt, \\ \mathcal{F}_\psi(v, z) &:= \int_{S^1} \widetilde{\psi}_s(t, z) F(v(t)) \, dt. \end{aligned}$$

In order to compute the Hessian of $\widetilde{\mathcal{A}}_s$ at $(v_0, \eta_0, z_0)$, we apply "a change of coordinates": Consider the twisted loop space

$$\mathcal{L}_{\eta_0 F} := \big\{ w \in C^\infty([0,1], M) \mid w(0) = \phi^1_{\eta_0 F}(w(1)) \big\}$$

and the diffeomorphism $\Phi_{\eta_0 F} \colon \mathcal{L}_{\eta_0 F} \to \mathcal{L} = C^\infty(S^1, M)$ given by

$$\Phi_{\eta_0 F}(w)(t) = \phi^t_{\eta_0 F_s}(w(t))$$

where we abbreviated $F_s(\cdot) := \widetilde{\psi}_s(t, z_0) F(\cdot)$. Then the path $w_0 = \Phi^{-1}_{\eta_0 F} \circ v_0 = v_0(0) \in \Sigma$ is constant. Hence tangent vectors $\hat{w}(t)$ at $w_0$ are curves in the linear space $T_{w_0} M$ with

$$\hat{w}(1) = d\phi^{-1}_{\eta_0 F_s}(w_0) \, \hat{w}(0). \tag{24}$$

We are going to compute the kernel of the Hessian of the pulled-back functional

$$\mathcal{A}^\Phi_s := (\Phi_{\eta_0 F} \times \mathrm{id}_{\mathbb{R}} \times \mathrm{id}_{S^{2N+1}})^* \widetilde{\mathcal{A}}_s \colon \mathcal{L}_{\eta_0 F} \times \mathbb{R} \times S^{2N+1} \to \mathbb{R}$$

at the critical point $(w_0, \eta_0, z_0)$. First notice that $\Phi^*_{\eta_0 F} d\mathcal{A}_0(w)[\hat{w}] = \int_0^1 \omega(\frac{d}{dt} w, \hat{w}) dt$ for any $w \in \mathcal{L}_{\eta_0 F}$ and $\hat{w} \in T_w \mathcal{L}_{\eta_0 F}$, and that

$$(\Phi_{\eta_0 F}\times \mathrm{id}_{S^{2N+1}})^*\mathcal{F}_{\Delta\psi}(w,z) = \int_0^1 \big(\widetilde{\psi}_s(t,z_0)-\widetilde{\psi}_s(t,z)\big)F(w(t))\,dt,$$

$$(\Phi_{\eta_0 F}\times \mathrm{id}_{S^{2N+1}})^*\mathcal{F}_{\psi}(w,z) = \int_0^1 \widetilde{\psi}_s(t,z)\,F(w(t))\,dt$$

(since $F$ is preserved under $\phi^t_{\eta_0 F_s}$). The differential of $\mathcal{A}^\Phi_s$ therefore is

$$\begin{aligned}
&d\mathcal{A}^\Phi_s(w,\eta,z)[\hat{w},\hat{\eta},\hat{z}]\\
&\quad= \int_0^1 \omega\Big(\frac{d}{dt}w,\hat{w}\Big)dt\\
&\qquad+\eta_0\int_0^1 \Big\{\big(\widetilde{\psi}_s(t,z_0)-\widetilde{\psi}_s(t,z)\big)dF(w(t))\,\hat{w}(t)-\partial_z\widetilde{\psi}_s(t,z)\,\hat{z}\,F(w(t))\Big\}dt\\
&\qquad-\hat{\eta}\int_0^1 \widetilde{\psi}_s(t,z)\,F(w(t))\,dt\\
&\qquad+(\eta_0-\eta)\int_0^1 \widetilde{\psi}_s(t,z)\,dF(w(t))\,\hat{w}(t)+\partial_z\widetilde{\psi}_s(t,z)\,\hat{z}\,F(w(t))\,dt\\
&\qquad+d\widetilde{k}(z)\hat{z}.
\end{aligned}$$

At the critical point $x_0 := (w_0,\eta_0,z_0)$ the Hessian of $\mathcal{A}^\Phi_s$ applied to $\xi_i := (\hat{w}_i,\hat{\eta}_i,\hat{z}_i)$ therefore is

$$\begin{aligned}
&\mathrm{Hess}\,\mathcal{A}^\Phi_s(x_0)[\xi_1,\xi_2]\\
&\quad= \int_0^1 \omega\Big(\frac{d}{dt}\hat{w}_1,\hat{w}_2\Big)dt\\
&\qquad-\eta_0\int_0^1 \Big\{\partial_z\widetilde{\psi}_s(t,z_0)\,\hat{z}_1\,dF(w_0)\,\hat{w}_2(t)+\partial_z\widetilde{\psi}_s(t,z_0)\,\hat{z}_2\,dF(w_0)\,\hat{w}_1(t)\Big\}dt\\
&\qquad-\hat{\eta}_1\int_0^1 \widetilde{\psi}_s(t,z_0)\,dF(w_0)\,\hat{w}_2(t)\,dt\\
&\qquad-\hat{\eta}_2\int_0^1 \widetilde{\psi}_s(t,z_0)\,dF(w_0)\,\hat{w}_1(t)\,dt\\
&\qquad+\mathrm{Hess}\,\widetilde{k}(z)(\hat{z}_1,\hat{z}_2)
\end{aligned}$$

where we have used that $F(w_0) = 0$. A tangent vector $(\hat{w}, \hat{\eta}, \hat{z})$ therefore belongs to the kernel of Hess $\mathcal{A}_s^{\Phi}(w_0, \eta_0, z_0)$ if and only if

$$0 = \frac{d}{dt}\hat{w}(t) - \hat{\eta}\,\widetilde{\psi}_s(t, z_0)\,X_F(w_0) - \eta_0\,\partial_z\widetilde{\psi}_s(t, z_0)\,\hat{z}\,X_F(w_0), \tag{25}$$

$$0 = \int_0^1 \widetilde{\psi}_s(t, z_0)\,dF(w_0)\,\hat{w}(t)\,dt, \tag{26}$$

$$0 = -\eta_0 \int_0^1 dF(w_0)\,\hat{w}(t)\,\partial_z\widetilde{\psi}_s(t, z_0)(\cdot)\,dt + \text{Hess}\,\widetilde{k}(z_0)(\hat{z}, \cdot). \tag{27}$$

Denote by $H_{z_0} = \{\tau z_0 \mid \tau \in S^1\}$ the Hopf circle in $S^{2N+1}$ through $z_0$.

Assume first that $\eta_0 = 0$. Then $(v_0, \eta_0, z_0)$ belongs to the critical component $\Sigma \times \{0\} \times H_{z_0}$ of "constant in $\Sigma$ loops". Since $\eta_0 = 0$, (27) yields $\hat{z} \in T_{z_0}H_{z_0}$, and integrating (25) yields

$$\hat{w}(1) = \hat{w}(0) + \hat{\eta}\,X_F(w_0)$$

(since $s_{z_0}(1) = 1$). Since in this case $\Phi_{\eta_0 F} \colon \mathcal{L} \to \mathcal{L}$ is the identity mapping, $\hat{w}(1) = \hat{w}(0)$, and so $\hat{\eta} = 0$. By now, (25) reads $\frac{d}{dt}\hat{w}(t) = 0$, that is, $\hat{w}(t) \equiv \hat{w}(0) \in T_{w_0}M$ is constant. Finally, (26) shows that $\hat{w}(0) \in T_{w_0}\Sigma$. The kernel of the Hessian of $\widetilde{\mathcal{A}}_s$ at $(v_0, \eta_0, z_0) = (w_0, 0, z_0)$ is thus identified with $T_{w_0}\Sigma \times T_{z_0}H_{z_0}$.

Assume now that $\eta_0 \neq 0$. Then $S_{v_0} := \{v_0(\cdot - \tau) \mid \tau \in S^1\}$ is an embedded circle in $\mathcal{L}$. Hence the critical component of $(v_0, \eta_0, z_0)$ is the torus $S_{v_0} \times \{\eta_0\} \times H_{z_0}$. It is clear that the kernel of the Hessian of $\widetilde{\mathcal{A}}_s$ at $(v_0, \eta_0, z_0)$ has dimension at least two, and we must show that the dimension is two. By assumption (3), 1 has multiplicity 2 in the spectrum of $d\phi_{\eta_0 F}^{-1}(w_0)$. Since $\phi_{\eta_0 F_s} = \phi_{\eta_0 F}$, the same holds true for $L_s := d\phi_{\eta_0 F_s}^{-1}(w_0)$. Recall that $s_z(t) = \int_0^t \widetilde{\psi}_s(\tau, z)\,d\tau$. Integrating (25) we get

$$\hat{w}(t) = \hat{w}(0) + \hat{\eta}\,s_{z_0}(t)\,X_F(w_0) + \eta_0\,\partial_z\big|_{z_0} s_z(t)\,\hat{z}\,X_F(w_0). \tag{28}$$

In particular (since $s_z(1) = 1$ for all $z$), and by (24),

$$\hat{w}(1) = \hat{w}(0) + \hat{\eta}\,X_F(w_0) = L_s\,\hat{w}(0). \tag{29}$$

Consider the sub-vector space $V$ of $T_{w_0}M$ spanned by $\hat{w}(0)$ and $X_F(w_0)$.

Assume that $V$ is 2-dimensional. Then (29) and the fact that 1 has multiplicity 2 in the spectrum of $L_s$ show that $V$ is the whole 1-eigenspace of $L_s$. In particular, $V$ is symplectic. On the other hand, since $dF(X_F) = 0$, equations (28) and (26) show that

$$\begin{aligned} dF(w_0)\,\hat{w}(0) &= \int_0^1 \widetilde{\psi}_s(t,z_0)\,dF(w_0)\,\hat{w}(0)\,dt \\ &= \int_0^1 \widetilde{\psi}_s(t,z_0)\,dF(w_0)\,\hat{w}(t)\,dt = 0 \end{aligned}$$

and hence $\hat{w}(0) \in T_{w_0}\Sigma$. Since $X_F(w_0)$ generates the kernel of $\omega|_{T_{w_0}\Sigma}$, this contradicts $V$ being symplectic.

It follows that $\hat{w}(0) = r\,X_F(w_0)$ for some $r \in \mathbb{R}$. In particular, $L_s\,\hat{w}(0) = \hat{w}(0)$. The second equation in (29) thus shows that $\hat{\eta} = 0$. Since $\hat{w}(0) \in V = \operatorname{span}(X_F(w_0))$, equation (28) shows that $\hat{w}(t) \in V$ for all $t$. Therefore (27) gives $\hat{z} \in \ker \operatorname{Hess} \widetilde{k}(z_0) = T_{z_0}H_{z_0}$. We conclude with (28) that the kernel of Hess $\mathcal{A}_s^{\Phi}(w_0,\eta_0,z_0)$ is

$$\begin{aligned} &\bigl\{(\hat{w}(t),0,\hat{z}) \mid \hat{z} \in T_{z_0}H_{z_0}\bigr\} \\ &\quad = \Bigl\{\Bigl(r + \eta_0\,\partial_z\big|_{z_0} s_z(t)\,\hat{z}\Bigr)\,X_F(w_0),0,\hat{z}\Bigr) \mid r \in \mathbb{R},\ \hat{z} \in T_{z_0}H_{z_0}\Bigr\}. \end{aligned}$$

Hence $\dim\ker\operatorname{Hess}\mathcal{A}_s^{\Phi}(w_0,\eta_0,z_0) = \dim\ker\operatorname{Hess}\widetilde{\mathcal{A}}_s(v_0,\eta_0,z_0) = 2$. $\square$

## 6. Other approaches

In this note we have defined $\mathbb{T}$-equivariant Rabinowitz–Floer homology $\mathrm{RFH}^{\mathbb{T}}(\Sigma,M)$ via the Borel construction and Floer homology with cascades, and we have proven the vanishing of $\mathrm{RFH}^{\mathbb{T}}(\Sigma,M)$ for displaceable $\Sigma$ by a leave-wise intersection argument. There are several other approaches to construct a $\mathbb{T}$-equivariant Rabinowitz–Floer homology (two are mentioned in Remark 3.1, and one more is outlined in 3. below), all of which are expected to give the same result. And there are different ways to prove the vanishing of $\mathrm{RFH}^{\mathbb{T}}(\Sigma,M)$ or of these other versions for displaceable $\Sigma$. In particular, the arguments in 1. and 2. below imply the vanishing of the version defined in [13], see items (4) and (3) on page 70 of [13]. Let $V$ be the bounded component of $M \setminus \Sigma$, and denote by $\mathrm{SH}_*(V)$ its symplectic homology and by $\mathrm{SH}_*^{\mathbb{T}}(V)$ its equivariant symplectic homology.

**1. Vanishing of $\mathbf{RFH}^{\mathbb{T}}(\boldsymbol{\Sigma}, \boldsymbol{M})$ via vanishing of $\mathbf{SH}^{\mathbb{T}}(\boldsymbol{V})$.** There should be a $\mathbb{T}$-equivariant version of the long exact sequence

$$\cdots \longrightarrow \mathrm{SH}^{-*}(V) \longrightarrow \mathrm{SH}_*(V) \longrightarrow \mathrm{RFH}_*(\Sigma, M) \longrightarrow \mathrm{SH}^{-*+1}(V) \to \cdots$$

from [11]. The vanishing of $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$ for displaceable $\Sigma$ would then follow from the vanishing of $\mathrm{SH}^{\mathbb{T}}(V)$ proven in [8].

**2. Vanishing of $\mathbf{RFH}^{\mathbb{T}}(\boldsymbol{\Sigma}, \boldsymbol{M})$ via vanishing of $\mathbf{RFH}(\boldsymbol{\Sigma}, \boldsymbol{M})$.** It is shown in [8, Theorem 1.2] that

$$\mathrm{SH}(V) = 0 \Longleftrightarrow \mathrm{SH}^{\mathbb{T}}(V) = 0.$$

While the implication $\Longleftarrow$ follows from the Gysin exact sequence in [7], the implication $\Longrightarrow$ follows from the fact that $\mathrm{SH}^{\mathbb{T}}(V)$ is the limit of a spectral sequence whose second page is the tensor product of the homology of the classifying space $BS^1$ and of $\mathrm{SH}(V)$, [8, Section 2.2]. It is expected that these two algebraic constructions can be adapted to Rabinowitz–Floer homology (cf. [8, p. 6]). Then

$$\mathrm{RFH}(\Sigma, M) = 0 \iff \mathrm{RFH}^{\mathbb{T}}(\Sigma, M) = 0.$$

In particular, the vanishing of $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$ for displaceable $\Sigma$ would then follow from the vanishing of $\mathrm{RFH}(\Sigma, M)$ proven in [10]. Together with the equivalence from [22, Theorem 13.3] we could then conclude the equivalences

$$\mathrm{RFH}^{\mathbb{T}}(\Sigma, M) = 0 \Longleftrightarrow \mathrm{RFH}(\Sigma, M) = 0 \Longleftrightarrow \mathrm{SH}(V) = 0 \Longleftrightarrow \mathrm{SH}^{\mathbb{T}}(V) = 0.$$

**3. Chekanov's construction of $\boldsymbol{S^1}$-equivariant Floer homology.** In the Borel-construction, approximations $S^{2N+1}$ of the classifying space $S^\infty = ES^1$ are somewhat clumsily added to the loop space as direct summands. In Chekanov's version of $S^1$-equivariant Floer homology, $S^{2N+1}$ does not appear as a space, but is incorporated into the boundary operator: In the setting of Morse theory for a function $f \colon M \to \mathbb{R}$ on a compact $S^1$-manifold $M$, with action $S^1 \times M \to M$, $(s, x) \mapsto s\,x$, one proceeds as follows. Given times $t_1 < \cdots < t_N \in \mathbb{R}$ and angles $s_1, \ldots, s_N \in S^1$ one considers the functions

$$f_t(x) = \begin{cases} f(x) & \text{if } t < t_1, \\ f(s_1 x) & \text{if } t_1 \leqslant t < t_2, \\ f((s_2 + s_1)x) & \text{if } t_2 \leqslant t < t_3, \\ \quad\vdots & \\ f((s_N + \cdots + s_1)x) & \text{if } t_N \leqslant t, \end{cases} \tag{30}$$

and counts gradient "$N$-jump flow lines" of the vector field $-\nabla f_t$. A neat way to see that a point $(t_1, \ldots, t_N, s_1, \ldots, s_N)$ corresponds to a point in $S^{2N-1}$ is through the join construction, [8, Section 2.5].

This construction of equivariant Morse and Floer homology was explained in several lectures by Chekanov [9], and worked out by Noetzel [20], though never written up. The construction and an isomorphism to the Borel construction is worked out for Morse homology in [4] based on [9], and for Floer homology in [8] building on [24, Section 8b]. The construction and the isomorphism in [8] can be adapted to Rabinowitz–Floer homology, yielding a homology $\mathrm{RFH}^{\mathbb{T}}_{\mathrm{jump}}(\Sigma, M)$ isomorphic to $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$. The vanishing of $\mathrm{RFH}^{\mathbb{T}}(\Sigma, M)$ for displaceable $\Sigma$ then follows from the vanishing of $\mathrm{RFH}^{\mathbb{T}}_{\mathrm{jump}}(\Sigma, M)$, which in turn follows as for the non-equivariant $\mathrm{RFH}(\Sigma, M)$ by a leafwise intersection argument, because the chain groups of $\mathrm{RFH}^{\mathbb{T}}_{\mathrm{jump}}(\Sigma, M)$ are exactly the chain groups of $\mathrm{RFH}(\Sigma, M)$.

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