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On the hyperbolicity in the domain of real analytic functions and Gevrey classes

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§. 1 Introduction

We are concerned with the Cauchy problem for the following first order equation

$$(1.1)\qquad \partial_t u=\sum_{j=1}^{l} A_j(x,t)\,\partial_x u+B(x,t)\,u+f,$$

where $x=(x_1,\cdots,x_l)\in\boldsymbol{R}^l$, $t\in\boldsymbol{R}$; $u(x,t)={}^t(u_1(x,t),\cdots,u_m(x,t))$, and $A_j(1\leqq j\leqq l)$ and B are matrices of order m. All the coefficients are assumed to be real analytic in x and continuous in t.

The Cauchy-Kowalewsky theorem, more precisely the Nagumo-Ovciannikov theorem asserts that, given any real analytic initial data $\varphi(x)\in C^\omega(\mathcal{O}_x)$ and $f(x,t)\in C_t^0(C^\omega(\mathcal{O}_x))$ (continuous function of t with values in $C^\omega(\mathcal{O}_x)$), where $\mathcal{O}_x(\subset\boldsymbol{R}^l)$ is an open connected neighborhood of the origin.

We are concerned with the existence domain of u. Let $f=0$. Then its domain may depend on the initial data φ, more precisely on its radius of convergence around the origin. However, the Bony-Schapira theorem asserts that, when A_j and B are analytic in (x,t), and if the characteristic roots $\lambda_i(x,t;\xi)$ of

$$(1.2)\qquad \det\Big(\lambda I-\sum_j A_j(x,t)\,\xi_j\Big)=0$$

are all real, then there exists a neighborhood of the origin, say V, such that for any $\varphi(x)\in C^\omega(\mathcal{O}_x)$, there exists a unique solution $u(x,t)\in C^\omega(V)$. It is plausible that this result can be extended to the actual situation. Our aim is to show that

THEOREM 1. *If there exists a common existence domain V of the solution $u(x,t)$ for any real analytic initial data $\varphi(x)\in C^\omega(\mathcal{O}_x)$, then the characterustuc roots $\lambda_i(x,0;\xi)$ $(1\leqq i\leqq m)$ should be real.*

In §. 6, we shall explain what becomes Theorem 1 in the case of the class s of Gevrey $(1<s<+\infty)$. Concerning this subject, there are two articles: T. Nishitani [4], and V. Ya. Ivrii [2]. We want to show that the method used in [3] can be applied to these without significant modifications.

The purpose of this exposition is two-fold. First, as we shall see by comparing Theorem 1 with Theorem 2, there is an essential difference between the case of analytic functions and that of Gevrey class. In the work of Nishitani, we cannot recognize this difference clearly. Next, by making minimum assumptions on the coefficients, we cam make clear the nature of the problem. The work of Ivrii assumes the coefficients to be real-analytic.

The interest in these kind problems was raised by the discussions with the group of partial differential equations in the Scuola Normale Superiore di Pisa during my stay in Pisa. Especially Theorem 1 is the question posed by E. Jannelli. Let us remark that the same content was already published in [5].

§.2 Preliminaries

To avoid unessential technical complications, we consider only the following equation:

$$\partial_t u = \sum_{j=1}^{l} a_j(x, t)\, \partial_{x_j} u\,, \tag{2.1}$$

where we assume $t \mapsto a_j(x, t) \in H(\mathcal{O}'_x)$ to be continuous, where $\mathcal{O}'_x$ is a complex open neighborhood of the origin. We assume

Assumption 1. There exists an open connected neighborhood $V_x \times (-t_0, t_0) \subset \boldsymbol{R}^{l+1}$ of the origin such that, for any $\varphi(x) \in C^{\omega}(\mathcal{O}_x)$, there exists a unique solution $u(x, t) \in C^{\infty;1}_{x,t}(V_x \times (-t_0, t_0))$ of (2.1) with $u(x, 0) = \varphi(x)$.

Assumption 2. At least one of $a_j(x, t)$ is not real at the origin.

Our aim is to show that these two assumptions are not compatible.

Proposition 1. *Set $K = \overline{\mathcal{O}}_x$. Let K_ε be the complex ε-neighborhood of K. Then the mapping $\varphi(x) \mapsto u(x, t)$, from $H(K_\varepsilon)$ into $C^{\infty;1}_{x,t}(V_x \times (-t_0, t_0))$ is continuous.*

Proof. Both spaces are Fréchet, and the graph of the mapping is closed. Thus, by Banach, the mapping is continuous.

This proposition implies in particular the following a priori estimate of u: given K (compact) $\subset V_x$, $t'_0 (0 < t'_0 < t_0)$, then three exists a constant $C(K, t'_0, \varepsilon)$ such that

$$\sup\left|u(x, t)\right| + \sum_i \sup\left|\partial_{x_i} u(x, t)\right| \leqslant C(K, t, \varepsilon) \sup_{x \in K_\varepsilon}\left|\varphi(x)\right| \tag{2.2}$$

where, in the left-hand side, sup is taken over $(x, t) \in K \times [0, t_0']$.

In the case when the coefficients depend only on t, the proof is easy. In fact Assumption 2 implies that there exists a $\xi^0 \in \boldsymbol{R}^n$, $|\xi^0| = 1$ such that

$$\mathrm{Im} \sum_j a_j(0)\, \xi_j^0 < 0 .$$

Then in the expression of the solution of (2. 1),

$$u(x, t\,;\ \xi) = e^{i\xi x} \exp\left[i \int_0^t \sum_j a_j(s)\, \xi_j ds \right],$$

if we set $\xi = \rho \xi^0$, ρ being positive parameter, we have

$$u_\rho(x, t) = e^{i\rho\xi^0 x} \exp\left[i\rho \int_0^t \sum_j a_j(s)\, \xi_j^0 ds \right].$$

First, $\quad \sup_{x \in K_\varepsilon} |u_\rho(x, 0)| \leqslant e^{\varepsilon\rho}$.

Next, by hypothesis, if t_0' is small, we have $-\mathrm{Im} \sum a_j(s)\, \xi_j^0 \geqslant \delta (>0)$ for $s \in [0, t_0']$. Thus,

$$\left| u_\rho(0, t_0') \right| \geqslant e^{\rho\delta t_0'} .$$

By taking $\varepsilon < \delta t_0'$, and making $\rho \to \infty$, we see that (2. 2), does not hold, which proves the Theorem.

Choice of the initial data. We take a series of the initial data at $t=0$ of the form,

(2. 3) $$\varphi_n(x) = (2\pi)^{-l} \int e^{ix\xi} \psi(\xi - n\xi^0)\, d\xi ,$$

where $\xi^0 \in \boldsymbol{R}^l$, $|\xi^0| = 1$, is chosen in such a way that

(2. 4) $$\mathrm{Im} \sum_j a_j(0, 0)\, \xi_j^0 = -\delta_0 < 0 ,$$

and $\psi(\xi) \geq 0$ being a continuous function with its support in $\{\xi\,;\ |\xi| \leqslant 1\}$, and $\int \psi d\xi = 1$.

Let $u_n(x, t)$ be the solution of (2. 1) corresponding to $\varphi_n(x)$. Since

$$\sup_{K_\varepsilon} \left| \varphi_n(x) \right| \leqslant e^{(n+1)\varepsilon} ,$$

(2. 2) implies,

(2. 5) $$\sup \left| u_n(x, t) \right| + \sum_i \sup \left| \partial_{x_i} u_n(x, t) \right| \leqslant C(K, t_0', \varepsilon)\, e^{(n+1)\varepsilon} .$$

Observe that $C(K, t_0', \varepsilon)$ is independent of n, and that we can take ε as small as we please.

§. 3 Microlocalization

In the C^∞-case, we used the cut off functions of the form $\alpha(\xi/n)$ where $\alpha(\xi)$ is a C^∞-function with small support, which is equal to 1 on a neighborhood of ξ^0. [3]. In actual case, this would not work well. We use here the following cut off functions due to Hörmander [1].

Let K be a compact set in $\boldsymbol{R}^l$. For each large integer N we associate $\chi_N(x)$ defined as follows: Let $r>0$ be a positive number. $\phi\in C_0^\infty$ with support in $\{x;\ |x|<1/4\}$ so that $\phi\geqslant 0$, $\int\phi dx=1$. Set

$$C_\alpha=\int|D^\alpha\phi|dx, \qquad C=\max_{|\alpha|=1}C_\alpha .$$

Let u be the characteristic function of the set of points at distance less than $1/2r$ from K, and form

$$\chi_N=u*\phi_r*\overbrace{\phi_{r/N}*\cdots*\phi_{r/N}}^{N} .$$

Here we have set

$$\phi_a(x)=a^{-l}\phi(x/a) .$$

Then $\chi_N(x)\in C_0^\infty$, equal to 1 on K, and vanishes at all points with distance greater than r from K, and we have

$$\left|\partial^{\alpha+\beta}\chi_N(x)\right|\leqslant C_\alpha r^{-|\alpha|}(CN/r)^{|\beta|} \qquad \text{if}\quad |\beta|\leqslant N .$$

Now we take as K the ball of radius $r_0/2$ with center ξ^0, and $r=r_0/2$ in the above definition. Finally, set

$$\alpha_n(\xi)=\chi_N(\xi/n) . \tag{3. 1}$$

We have

$$\left|\partial^{\alpha+\beta}\alpha_n(\xi)\right|\leqslant\frac{C'_\alpha}{n^{|\alpha|}}(C'N/n)^{|\beta|} \qquad \text{for}\ |\beta|\leqslant N, \tag{3. 2}$$

$$C'=2C/r_0,\quad C'_\alpha=C_\alpha\left(\frac{r_0}{2}\right)^{-|\alpha|} .$$

In the same way, on taking as K the ball of radius $r_0/2$ with center the origin of $\boldsymbol{R}_x^l$, and $r=r_0/2$, we set

$$\beta_n(x)=\chi_N(x) . \tag{3. 3}$$

In the same way we have

(3.4) $$\left|\partial^{\alpha+\beta}\beta_n(x)\right| \leqslant C'_\alpha (C'N)^{|\beta|} \qquad \text{for } |\beta| \leqslant N.$$

Hereafter we take N in such a way that $N \simeq \theta n$ $(0<\theta<1)$, θ being defined later.

Apply $\beta_n(x)$ to (2.1)

$$\partial_t(\beta_n u) = \sum_j a_j \partial_{x_j}(\beta_n u) - \sum_j a_j \partial_{x_j}\beta_n u .$$

Observe that supp $[\beta_n] \subset \{x;\ |x| \leqslant r_0\}$, and r_0 is small. We extend all the coefficients on $\boldsymbol{R}^l$, keeping invariant on the set $\{x;\ |x| \leqslant 2r_0\}$. For instance,

$$\tilde{a}_j(x, t) = \phi(|x|)\, a_j(x, t) + \big(1-\phi(|x|)\big)\, a_j(0, t) .$$

where $\phi(r) \in C_0^\infty$, $0 \leqslant \phi(r) \leqslant 1$, $\phi(r)=1$, for $r \leqslant 2r_0$, $\phi(r)=0$ for $r \geqslant 3r_0$. We write a_j instead of $\tilde{a}_j$.

Apply $\alpha_n(D)$ to the above equation. Then

(3.5) $$\partial_t(\alpha_n \beta_n u) = \sum_j a_j \partial_{x_j}(\alpha_n \beta_n u) + f_n ,$$

where f_n is the function arising from the actions of cut off, mamely,

(3.6) $$f_n = \sum_j\, [\alpha_n, a_j]\, \partial_{x_j}(\beta_n u) - \sum_j \alpha_n a_j \partial_{x_j}\beta_n u .$$

To estimate the error term f_n, we need to consider the equations replacing in (3.5) α_n, β_n by $\alpha_n^{(p)}$, $\beta_{n(q)}$ $(\alpha_n^{(p)}(\xi) = \partial_\xi^p \alpha_n(\xi)$, $\beta_{n(q)} = (i^{-1}\partial_x)^q \beta_n(x))$.

(3.7) $$\partial_t(\alpha_n^{(p)} \beta_{n(q)} u) = \sum_i a_j \partial_{x_j}(\alpha_n^{(p)} \beta_{n(q)} u) + f_{n,(p,q)} ,$$

(3.8) $$f_{n,(p,q)} = \sum_j\, [\alpha_n^{(p)}, a_j]\, \partial_{x_j}(\beta_{n(q)} u) - i \sum \alpha_n^{(p)} a_j (\beta_{n(q+e_j)} u) .$$

§.4 Energy inequalities

Let us recall (2.4). By virtue of the microlocalization, if we choose r_0 small, and that if we take t'_0 $(0<t'_0 \leqslant t_0)$ small, we have the following energy inequality for the solution u of (3.7).

For $t \in [0, t'_0]$, it holds $(p \geqslant 0, q \geqslant 0)$

(4.1) $$\frac{d}{dt}\|\alpha_n^{(p)} \beta_{n(q)} u\| \geqslant \delta' n \|\alpha_n^{(p)} \beta_{n(q)} u\| - \|f_{n,(p,q)}\| ,$$

where δ' $(<\delta)$ is a positive constant independent of (n, p, q).

§. 5 Estimates of the error terms

Put $u=u_n$ in (4. 1). Recall that $u_n(x,t)$ is the solution of (2. 1) corresponding to $\varphi_n(x)$. Our aim is to show the following. Define

$$S_n(t)=\sum_{0\le|p+q|\le N} M^{|p+q|}n^{-|q|}\|\alpha_n^{(p)}\beta_{n(q)}u_n\|\,. \tag{5. 1}$$

Then, if we take M large, we have

$$\sum_{0\leqslant|p+q|\leqslant N} M^{|p+q|}n^{-|q|}\|f_{n,(p,q)}\|<\frac{1}{2}\,\delta' n\,S_n(t)+o(1)\,. \tag{5. 2}$$

To estimate $\|f_{n,(p,q)}\|$, we divide it in two parts:

I) $\|f_{n,(p,q)}\|_{\{x;|x|\leqslant r_0\}}$,

II) $\|f_{n,(p,q)}\|_{\{x;|x|\geqslant 2r_0\}}$.

Let us recall that $\beta_n(x)$ has its supports in $\{x;\ |x|\leqslant r_0\}$, and moreover, on the set $\{x;\ |x|\leqslant 2r_0\}$, $a_j(x,t)$ are real analytic. Namely,

$$\left|a_{j(\nu)}(x,t)\right|_{\{x;|x|\leqslant 2r_0\}}\leqslant AC_0^{|\nu|}\nu\,!\,.$$

Without loss of generality, are can assume.

$$C'\geqslant C_0\,,$$

where C' is the constant in (3. 2) and (3. 4). Look at I). The asymptotic expansion becomes

$$[\alpha_n^{(p)},\,a_j]\sim\sum_{|\nu|\ge 1}\nu\,!^{-1}a_{j(\nu)}\alpha_n^{(p+\nu)}\,.$$

Using the above estimate

$$\begin{aligned}&\|\nu\,!^{-1}a_{j(\nu)}\alpha_n^{(p+\nu)}\partial_{x_j}(\beta_{n(q)}u_n)\|_{\{x;|x|\leqslant 2r_0\}}\\ &\quad\leqslant AC_0^{|\nu|}\|\alpha_n^{(p+\nu)}\partial_{x_j}(\beta_{n(q)}u_n)\|\\ &\quad\leqslant(1+r_0)nAC'^{|\nu|}\|\alpha_n^{(p+\nu)}\beta_{n(q)}u_n\|\,.\end{aligned}$$

We take the asymptotic expansion up to

$$|\nu|+|p+q|=N\,. \tag{5. 3}$$

Thus, we have

$$\begin{cases} n^{-|q|}M^{|p+q|}\|f_{n,(p,q)}\|_{\{x;|x|\leqslant 2r_0\}}\\ \leqslant l(1+r_0)\,n\,A\Big[\displaystyle\sum_{1\le|\nu|\le N-|p+q|}(C'/M)^{|\nu|}M^{|p+q+\nu|}n^{-|q|}\|\alpha_n^{(p+\nu)}\beta_{n(q)}u_n\|\\ +\displaystyle\frac{1}{M}\sum_{j=1}^{l}\sum_{0\le|\nu|\le N-|p+q|}(C'/M)^{|\nu|}M^{|p+q+\nu|+1}n^{-|q|-1}\|\alpha_n^{(p+\nu)}\beta_{n(q+e_j)}u_n\|\Big]\\ +(\text{remainder term})\,. \end{cases} \tag{5. 4}$$

Summing up the above expression, we see that the coefficient of $M^{|p+q|}n^{-|q|}\times\alpha\|\alpha_n^{(p)}\beta_{n(q)}u_n\|$ is majorized by

$$n\left\{l(1+r_0)A\sum_{j\geq 1}(j+1)^{l-1}\left(\frac{C'}{M}\right)^j+\frac{l(1+r_0)A}{M}\sum_{j\geq 0}(j+1)^{l-1}\left(\frac{C'}{M}\right)^j\right\}.$$

Now we fix $M(>C')$ is such a way that

$$\{\cdots\cdots\}\leq\frac{1}{2}\delta', \tag{5.5}$$

where δ' is the constant appearing in (4.1).

Next, we estimate the remainder term. It would be enough to estimate the quantity $M^{|p+q|}n^{-|q|}C'^{|\nu|}\|\alpha_n^{(p+\nu)}\beta_{n(q)}u_n\|$, for $|p+q|+|\nu|=N$. This is estimated by

$$(MC'N/n)^N\|u_n\|_{\{x;|x|\leq 2r_0\}}.$$

We take integer N in such a way that

$$N\simeq\frac{1}{MC'e}n\equiv\theta n,\quad\left(\theta=\frac{1}{MC'e}\right). \tag{5.6}$$

More precisely, the nearest integer to θn.

Since $(MC'N/n)^N\simeq e^{-\theta n}$, in view of (2.5) (a priori estimates of u_n), the above quantity is estimated by

$$C(K,t_0',\varepsilon)\,e^{-\theta n}e^{\varepsilon(n+1)}.$$

So, if we choose ε less than θ, mere precisely if

$$\varepsilon<\frac{1}{MC'e},\quad\varepsilon<\frac{1}{2}\delta' t_0', \tag{5.7}$$

the remainder term becomes negligible when $n\to\infty$.

Now we pass to the estimates of the term $M^{|p+q|}n^{-|q|}\|f_{n,(p,q)}\|_{\{x;|x|\geq 2r_0\}}$. In the domain $\{x;|x|\geq 2r_0\}$, the above asymptotic expansion does not work, because we cannot assume $a_j(x,t)$ to be real analytic in $\boldsymbol{R}^n$. However, in virtue of the pseudo-local property of $\alpha_n(D)$, we see easily these terms are of $o(1)$.

In fact,

$$[\alpha_n^{(p)},a_j]\,\partial_{x_j}(\beta_{(q)}u_n)=\alpha_n^{(p)}a_j\partial_{x_j}(\beta_{(q)}u_n)-a_j\alpha_n^{(p)}\partial_{x_j}(\beta_{(q)}u_n).$$

Let

$$\tilde{\alpha}_n^{(p)}(x)=(2\pi)^{-l}\int e^{ix\xi}\alpha_n^{(p)}(\xi)\,d\xi.$$

Then $\alpha_n^{(p)}(D)v=\tilde{\alpha}_n^{(p)}(x)*v$. Now

$$|x|^{2k}\tilde{\alpha}_n^{(p)}(x)=(2\pi)^{-l}\int e^{ix\xi}(-\Delta)^k\alpha_n^{(p)}(\xi)\,d\xi\,.$$

Suppose $|p|\leqslant N$, then we have

$$\left|\tilde{\alpha}_n^{(p)}(x)\right|\leqslant \text{const.}\ l^k(C'N/n)^{|p|+2k}/|x|^{2k}$$

where k is an arbitrary integer satisfying $|p|+2k\leqslant N$. In that estimate, if $N-|p|\leqslant l$ (dimension of the space), we replace the right-hand side by const. $(C'N/n)^N/|x|^{l+1}$. This implies

LEMMA. *(pseudo-local property of* $\alpha_n^{(p)}$*)*

If $v(x)\in L^2$ *with support in* $\{x;\ |x|\leqslant r_0\}$*, then*

$$||\alpha_n^{(p)}v||_{\{x;|x|\geqslant 2r_0\}}\leq \text{cost.}\ l^k(C'N/n)^{|p|+2k}/r_0^{2k}||v||\,,$$

where k *is an arbitary constant satisfying* $|p|+2k\leq N$*, and* const. *is independent of* (k,p).

From this lemma, and in view of the form of $[\alpha_n^{(p)},a_j]\partial_{x_j}(\beta_{(q)}u_n)$, we have

PROPOSITION.

$$M^{|p+q|}n^{-|q|}||f_{n,(p,q)}||_{\{x;|x|\geqslant 2r_0\}}\leqq \text{const.}\ (MC'N/n)^N|u_n|_{1,\{x;|x|\leqslant r_0\}}\ ,$$

if $M\geqslant\sqrt{l}/r_0$.

This implies, since the right-hand side is estimated by

$$\text{const.}\ e^{-(\theta-\varepsilon)}n\,,$$

we have completed the proof of (5. 2).

Now it is easy to see that

$$S_n(0)\geqq||\alpha_n\beta_n\varphi_n||\geqslant c_0(>0)\,.$$

Thus, we have,

$$S_n(t)\geqslant c_0'\,e^{\frac{1}{2}\delta' nt}\qquad(0<c_0'<c_0) \tag{5. 8}$$

On the other hand,

$$\begin{cases} S_n(t)=\displaystyle\sum_{0\leq|p+q|\leq N}M^{|p+q|}n^{-|q|}||\alpha_n^{(p)}\beta_{n(q)}\,u_n|| \\ \leqslant A\displaystyle\sum_{(p,q)}M^{|p+q|}\,(C'N/n)^{|p+q|}|u_n|_{0,\{x;|x|\leqslant r_0\}} \\ \leqslant A'e^{\varepsilon n}\displaystyle\sum_{(p,q)}e^{-|p+q|}\leqslant A''e^{\varepsilon n}\,. \end{cases} \tag{5. 9}$$

(5. 8) and (5. 9) are not compatible at $t=t_0'$, because we have chosen ε in such a way that $\varepsilon<\frac{1}{2}\delta' t_0'$.

§. 6 Theorem in the case of Gebrey class

We can extend the above arguments to the case of Gevrey class. Let $\mathcal{O}_x\subset \boldsymbol{R}^l$ be a relatively compact open set. We say that $f(x)\in C^\infty(\mathcal{O}_x)$ belongs to $\gamma^{(s)}(\mathcal{O}_x)$, $(s\geqslant 1)$, if there exist positive constants C and K such that

$$\forall\alpha\geqslant 0 \qquad \sup_{x\in\mathcal{O}}\left|\partial^\alpha f(x)\right|\leqslant C\alpha\,!^s K^{|\alpha|}\,.$$

Next, we say that, $f(x)\in\gamma_A^{(s)}(\mathcal{O}_x)$, if $f(x)\in\gamma^{(s)}(\mathcal{O}_x)$, and moreover $\|f\|_{s,A}=\sup_{\nu\geqslant 0}\sup_{x\in\mathcal{O}_x}|\partial^\nu f(x)|/\nu\,!^s A^{|\nu|}<+\infty$.

Of course, if $A<A'$, $\gamma_A^{(s)}\subset\gamma_{A'}^{(s)}$, and

$$\gamma^{(s)}(\mathcal{O}_x)=\bigcup_{A>0}\gamma_A^{(s)}(\mathcal{O}_x)\,.$$

DEFINITION. *We say that* (1. 1) *with* $f=0$ *is well-posed in* $\gamma^{(s)}$, *or in short* $\gamma^{(s)}$*-well-posed, at the origin, if for every* $u_0(x)\in\gamma^{(s)}(\mathcal{O}_x)$, *there exists a unique solution* $u(x,t)\in C^{\infty;1}_{x,t}(V)$ *in a neighborhood* V *of the origin, where* $\mathcal{O}_x$ *is a fixed open neighborhood of the origin, and* V *may depend on* u_0.

We can prove the following theorem, which is the same as in the C^∞-case:

THEOREM 2. *Suppose* $s>1$. *The coefficients of* (1. 1) *are assumed to be in* $\gamma^{(s)}$ *in* x, *and continuous in* t. *Then, in order that* (1. 1) *to be* $\gamma^{(s)}$*-well-posed at the origin, it is necessary that all the eigenvalues at the origin* λ_i $(0,0;\xi)$ *be real.*

REMARK 1. Theorem 2 can be stated as follows. If one of eigenvalues is not real at the origin, then we are at the following situation: i) the uniqueness of the Cauchy problem does not hold, ii) the uniqueness holds, however there exists a least an initial data $u_0(x)\in\gamma^{(s)}(\mathcal{O}_x)$ such that the corresponding solution does not exist in any neighborhood of the origin.

REMARK 2. Theorem 2 has been proved by Nishitani under the assumption that the coefficients belong to $\gamma^{(s')}$ with $s'<s$, and, it seems to us, this is an essential assumption to carry out the arguments. It is important to prove the theorem under the assumption that the coefficients belong to $\gamma^{(s)}$, as function of x, in view of the several results on the sufficiency.

We are going to explain how to modify the above arguments to derive theorem 2.

First of all, observe that $\gamma_A^{(s)}(\mathcal{O}_x)$, with the norm $||f||_{s,A}$ is a Banach space. Then, if we assume $\gamma^{(s)}$-well-posedness at the origin, then by Category theorem, every time A is fixed, there exists a common existence domain D_A of the solutions $u(x, t)$ for all $u_0 \in \gamma_A^{(s)}(\mathcal{O}_x)$. Thus by Banach,

(6. 1) the mapping $u_0(x) \in \gamma_A^{(s)}(\mathcal{O}_x) \mapsto C_{x,t}^{1,1}(D_A)$ is continuous.

Again we argue to (2. 1) to simplify the arguments. In the case when the coefficients a_j depend only on t, the argument is simple. In fact, let $\xi^0 \in \boldsymbol{R}^n$, $|\xi^0|=1$, then

$$||e^{i\rho\xi^0 x}||_{s,A} \leqslant \exp(\varepsilon\rho^{1/s}) \text{ when } \rho \in \boldsymbol{R}_+^1 \to +\infty,$$

where ε=const. $A^{-1/s}$.

Now, as explained before, there exists a positive constant $\delta(>0)$ such that

$$\left|u_\rho(0, t_0')\right| \geqslant e^{\rho\delta t_0'}.$$

Of course we suppose $\{0\} \times [0, t_0'] \subset D_A$. Since $s>1$, (6. 1) never holds, which proves Theorem 2. In the general case, we take $\varphi_n(x)$ defined by (2. 3) as the series of initial data. Then

$$(6.\ 2) \qquad \begin{cases} ||\varphi_n(x)||_{s,A} \leqslant \exp(\varepsilon n^{1/s}). \\ \varepsilon = \text{const. } A^{-1/s}. \end{cases}$$

In general, we should take A large to make ε small (In the case when the coefficients belong to $\gamma^{(s')}$, $s'<s$, there is no need of such a consideration). Then, since the size of D_A might become small when A is taken large, we should take the support of cut off function $\beta_n(x)$ small. In appearance, this seems to be a serious difficulty, which is an essential difference between the analytic case and the Gevrey case. However, as we shall show in the next section, a little careful examination of the arguments in the analytic case enables us to overcome this difficulty.

§. 7 Proof of Theorem 2

1°) We define $\alpha_n(\xi) = \chi_N(\xi/n)$, where $N \simeq \theta n^{1/s}$, θ is to be defined later. $\beta_n(x) = \chi_{N'}(x)$, $N' \simeq \theta' n$, θ' is defined later.

2°) In the analytical case, we considered all the terms of the form

$$M^{|p+q|} n^{-|q|} ||\alpha_n^{(p)} \beta_{n(q)} u_n||, \ |p+q| \leqslant N,$$

here we consider all the terms of the form

$$M^{|p|}|p|\,!^{s-1}n^{-|q|}\|\alpha_n^{(p)}\beta_{n(q)}u_n\|,\ |p|\leqslant N, \qquad |q|\leqslant N'.$$

In fact, in the estimates of commutators with $\alpha_n^{(p)}$, we are concerned with the terms $\nu\,!^{-1}a_{j(\nu)}\alpha_n^{(p+\nu)}\beta_{n(q)}u_n$. Since

$$\left|\partial^\nu a_j(x,t)\right|\leqslant A'\nu\,!^s C_0^{|\nu|},$$

these terms are estimated by $A'\nu\,!^{s-1}C_0^{|\nu|}\,\|\alpha_n^{(p+\nu)}\beta_{n(q)}u_n\|$.

3°) We defined r_0 as the size of cut off functions $\alpha_n(\xi)$, $\beta_n(x)$ which guarantees the energy inequality. In the analytic case, we decided the siezs of two functions in a almost symmetric way. In the actual case, we decide the sizes of these two functions in different ways. First we decide the size of $\alpha_n(\xi)$ by r_0. supp $[\beta_n]$ should be taken small according the size of D_A. Of course, we can assume supp $[\beta_n]\subset\{x;\ |x|\leq r_0\}$. In this sense, r_0 may be considered independent of A.

4°) M is defined by (5.5) with $M\geqslant\sqrt{l}/r_0$.

5°) N is defined by

$$N\simeq\left(\frac{1}{MC'e}\right)^{1/s}n^{\frac{1}{s}}\,;\ \theta=\left(\frac{1}{MC'e}\right)^{1/s}. \tag{7.1}$$

Then, the remainder term is estimated by

$$M^{|p|}|p|\,!^{s-1}\nu\,!^{s-1}C_0^{|\nu|}\|\alpha_n^{(p+\nu)}\beta_{n(q)}u_n\|n^{-|q|},$$

where $|p|+|\nu|\simeq N$. This is estimated by

$$\begin{aligned}&M^{|p+\nu|}|p+\nu|\,!^{s-1}|\alpha_u^{(p+\nu)}|_0\|\beta_{n(q)}u_n\|n^{-|q|}.\\&\leqslant N\,!^{s-1}(MC'N/n)^N\|\beta_{n(q)}u_n\|n^{-|q|}\\&\leqslant(MC'N^s/n)^N\|\beta_{n(q)}u_n\|n^{-|q|}\\&\leqslant e^{-N}\|\beta_{n(q)}u_n\|n^{-|q|}=\exp(-\theta n^{1/s})\|\beta_{n(q)}u_n\|n^{-|q|}.\end{aligned}$$

6°) θ is thus determined, we take A large in such a way that (see (6.2))

$$\varepsilon<\theta. \tag{7.2}$$

7°) When D_A is defined, we define $\beta_n(x)$. We can assume supp $[\beta_n]\subset\{x;\ |x|\leq r_0\}$ and that if we choose $t_0'(>0)$ small, we have

$$\text{supp}\,[\beta_n(x)]\times[0,t_0']\subset D_A.$$

By the definition $\beta_n(x)=\chi_{N'}(x)$, we have

$$\left|\partial^\alpha\beta_n(x)\right|\leqslant(C''N')^{|\alpha|},\qquad\text{for}\qquad|\alpha|\leqslant N',$$

Where C'' can be considered as the number which is determined only by the size of $\beta_n(x)$. So we define

$$\theta' = 1/C''e, \qquad N' = \theta' n.$$

Thus,

$$\left|\beta_{n(q)}(x)\right| n^{-|q|} \leqslant \left(\frac{C''N'}{n}\right)^{|q|} \qquad \text{for } |q| \leqslant N'.$$

Observe that the right-hand side $\leqslant 1$, and that for $|q| = N'$,

$$(C''N'/n)^{|q|} = \exp(-\theta' n).$$

Now, taking into account (6. 1) and (6. 2), we know that $\sup |u_n(x, t)|$ on $(x, t) \in \operatorname{supp}[\beta_n] \times [0, t_0']$ is estimated by const. $\exp(\varepsilon n^{1/s})$. Then, by virtue of (7. 2), the remainder terms are negligible in view of the results of 5°) and 7°). Moreover, we have

$$S_n(t) = \sum_{\substack{|p| \leq N \\ |q| \leq N'}} M^{|p|} |p|\,!^{s-1} n^{-|q|} \| \alpha_n^{(p)} \beta_{n(q)} u_n \| \leqslant A'' \exp(\varepsilon n^{1/s}).$$

On the other hand, we have

$$S_n(t) \geqslant C_0' \exp\left(\frac{1}{2} \delta' n t\right), \; t \in [0, t_0'].$$

These two estimates are not compatible when $n \to \infty$, since $s > 1$.

§. 8 Remark

More refined arguments will prove the following fact. Let

$$P(x, t; D_x, D_t) = P_m + P_{m-1} + \cdots,$$

with the principal symbol

$$P_m(x, t; \xi, \tau) = \prod_{j=1}^{s} (\tau - \lambda_j)^2 \prod_{j=s+1}^{m-s} (\tau - \lambda_j),$$

where $\lambda_j(x, t, \xi)$ $(1 \leqslant j \leqslant m-s)$ are real and distinct. The coefficients are supposed in $\gamma^{(s)}$ $(s \geqq 1)$. We know that if the Levi condition is satisfied, then the Cauchy problem is C^∞-well posed.

Assume now that the Levi condition is violated at the origin, namely for some i $(1 \leqslant i \leqslant s)$ and for some $\xi \in R^n$, it holds

$$P'_{m-1}\big(0, 0; \xi, \lambda_i(0, 0; \xi)\big) \neq 0,$$

where $P'_{m-1}(x, t; \xi, \tau)$ is the subprincipal symbol of P. We can prove the following

THEOREM 3. *In order that the Cauchy problem for P be $\gamma^{(s)}$-well posed in a neighborhood of the origin, it is necessary that* $s \leqq 2$.

We remark that this theorem is already proved in [2] under the assumption that the coefficients of P are real analytic. A forthcoming paper will give the proof with some related results.

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