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Aberrant CR structures

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0. Introduction. Statement of results

Throughout this work Ω denotes a C^{∞} manifold, countable at infinity, of dimension $2n+1$ ($n\geq 1$). What we call here an *abstract CR structure* (to be precise one should add "of codimension one") is the datum of a C^{∞} vector subbundle $\mathscr{C}$ of the complex tangent bundle $CT\Omega$ (henceforth called *the CR bundle*) submitted to the following three conditions:

(0.1) $[\mathscr{C}, \mathscr{C}]\subset\mathscr{C}$, *i.e., the commutation bracket of any two smooth sections of* $\mathscr{C}$ *over an open subset of* Ω *is a section of* $\mathscr{C}$ *over that same subset*;

(0.2) $\mathscr{C}\cap\overline{\mathscr{C}}=\{0\}$ ($\overline{\mathscr{C}}$ is "the complex conjugate" of $\mathscr{C}$);

(0.3) *the fibre dimension over* $\boldsymbol{C}$ *of* $\mathscr{C}$ *is equal to* n.

Call $\mathscr{C}'$ the orthogonal of $\mathscr{C}$ in the complex cotangent bundle $CT^{*}\Omega$ for the duality between tangent and cotangent vectors. Note that (0.2) is equivalent to

(0.4) $$CT^{*}\Omega=\mathscr{C}'+\overline{\mathscr{C}}'\ .$$

Let Ω' be any open subset of Ω. A C^{1} function (resp., a distribution) f in Ω' is called a CR function (resp., a CR distribution) if $Lf=0$ whatever the smooth section L of $\mathscr{C}$ over Ω'. The differentials of the C^{1} CR functions are continuous sections of $\mathscr{C}'$. The CR structure $\mathscr{C}$ is said to be *locally*

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integrable if at any point p of Ω there are $n+1$ germs of C^∞ CR functions whose differentials at p are linearly independent (and thus make up a linear basis of $\mathscr{C}'_p$).

Let U be an open subset of Ω in which there are n linearly independent C^∞ sections of $\mathscr{C}$, $L_1, \cdots, L_n$; they generate $\mathscr{C}$ at every point of U. Take any smooth real vector field L_0 in U such that

$$L_0,\ L_1, \cdots, L_n,\ \bar{L}_1, \cdots, \bar{L}_n\,,$$

make up a basis of $\boldsymbol{C}T_p\Omega$ for every $p \in U$. For every pair of indices j, $k=1, \cdots, n$, there is a complex number $c_{jk}(p)$ such that, at the point p,

$$\frac{1}{\sqrt{-1}}[L_j, \bar{L}_k] - c_{jk}(p)\, L_0 \in \mathscr{C} + \overline{\mathscr{C}}\,.$$

It is customary to call

(0.5) $$\mathscr{L}(p) = \left(c_{jk}(p)\right)_{1 \le j,k \le n}$$

the *Levi matrix* of the system $\boldsymbol{L}=(L_1, \cdots, L_n)$ at the point $p \in U$. Note that (0.5) is a self-adjoint $n \times n$ matrix with complex entries. The associated quadratic form $\mathscr{L}(p)v \cdot \bar{v}/2$ ($v \in \boldsymbol{C}^n$) is called the *Levi form* of the system $\boldsymbol{L}$. Actually it not only depends on the choice of $L_1, \cdots, L_n$ but also on that of L_0. However, when true, the following is an intrinsic property of the CR structure $\mathscr{C}$:

(0.6) *At every point of Ω the Levi form* (of some — of any — system $L_0, L_1, \cdots, L_n$ defined in the neighborhood of that point) *is non-degenerate and has exactly $n-1$ eigenvalues of one sign and one of the opposite sign.*

In the present paper we shall solely deal with CR structures that satisfy Condition (0.6).

Let us underline the fact that, when $n=1$, in which case $\dim \Omega = 3$, Condition (0.6) simply means that *the Levi constant* ($=1\times 1$ matrix) *is nowhere zero*, or equivalently, that

(0.7) *L, $\bar{L}$, $[L, \bar{L}]$ are linearly independent.*

In [4] L. Nirenberg gave the first example of a CR structure on $\boldsymbol{R}^3$ satisfying (0.7) such that any germ of CR function at the origin, of class C^1, is constant. Nirenberg's example is a perturbation of the Lewy structure, which agrees with the latter to infinite order at the origin. The Lewy structure on $\boldsymbol{R}^3$ is the one defined by the vector field

$$L=\frac{\partial}{\partial \bar{z}}-iz\frac{\partial}{\partial u}$$

(coordinates in $\boldsymbol{R}^3$ are x, y, u, and $z=x+iy$).

In [2] the present authors showed that, if (0.6) holds (now for any $n\geq 1$), an otherwise arbitrary CR structure can be perturbed in such a way as to obtain a new CR structure, agreeing with the original one to infinite order at a given point p_0, and which is not locally integrable at p_0 (i.e., there is no neighborhood of p_0 in which the new structure is locally integrable).

DEFINITION 0.1. *We say that two CR structures in Ω, $\mathscr{C}^{(j)}$ $(j=1,2)$ agree to infinite order at a point p of Ω, if there is an open neighborhood U of p in Ω, and for each $j=1,2$, a basis $L_1^{(j)},\cdots,L_n^{(j)}$ of $\mathscr{C}^{(j)}$ in U such that*

(0.8) *for every $k=1,\cdots,n$, $L_k^{(1)}-L_k^{(2)}$ vanish to infinite order at p.*

The reader will easily check that the condition in Def. 0.1 is equivalent to the following property:

(0.9) *given any germ of C^∞ section $L^{(1)}$ of $\mathscr{C}^{(1)}$ at p there is a germ of C^∞ section $L^{(2)}$ of $\mathscr{C}^{(2)}$ at p such that $L^{(1)}-L^{(2)}$ vanishes to infinite order at p.*

The first result proved in the present work improves the corresponding result in [2]:

THEOREM I. *Let the CR structure $\mathscr{C}$ on Ω satisfy Condition* (0.6).

Then, given any point p_0 of Ω, there is a CR *structure $\tilde{\mathscr{C}}(p_0)$ on Ω, also satisfying* (0.6), *agreeing with $\mathscr{C}$ to infinite order at p_0, and such that the following is true:*

(0.10) *The differential at p_0 of every germ at p_0 of CR function* (in the sense of $\tilde{\mathscr{C}}(p_0)$), *of class C^1, vanishes.*

The proof of Th. I (Sections 1 to 4) is by construction. The modified structure $\tilde{\mathscr{C}}(p_0)$ coincides with the original one, $\mathscr{C}$, in the complement of an arbitrarily small neighborhood of p_0.

Our second result applies rather to linear bases, over some open subset of Ω, of the CR bundle $\mathscr{C}$. We show that they can be approximated, on compact subsets and for the C^∞ topology on the coefficients of the vector fields, by *aberrant* systems. We call aberrant any system $\boldsymbol{L}=(L_1,\cdots,L_n)$ of n smooth vector fields in an open subset Ω' of Ω (defining a CR structure on Ω') that has the following property:

(0.11) *Whatever $p \in \Omega'$ and $\delta > 0$, every germ at p of a $C^{1+\delta}$ solution of the homogeneous differential equations*

(0.12) $L_j h = 0\,, \quad j = 1, \cdots, n\,,$

is the germ of a constant function.

Note that if such a system $\boldsymbol{L}$ is sufficiently close to a basis of $\mathscr{C}$ over some compact set, it will automatically possess Property (0.6) there. In practice we may limit our attention to systems $\boldsymbol{L}$ that have that property.

REMARK. When $n \geq 2$ every system $\boldsymbol{L}$ that has Property (0.6) is hypoelliptic and even 1/2-subelliptic (see [1]). In particular, any distribution solution of (0.12) in an arbitrary open subset of Ω' is a C^∞ function in that subset. Thus Condition (0.11) is equivalent to the following one:

(0.13) *Whatever $p \in \Omega'$ every germ at p of a distribution solution of* (0.12) *is the germ of a constant function.*

THEOREM II. *Suppose that the* CR *structure $\mathscr{C}$ has Property* (0.6). *Any linear basis of $\mathscr{C}$ over a neighborhood of a compact subset K of Ω is the limit, for the C^∞ topology on a possibly smaller open neighborhood of K, of a sequence of systems of vector fields which have Property* (0.11).

Needless to say the only bases of $\mathscr{C}$ we consider here are made up of C^∞ sections of $\mathscr{C}$.

The proof of Th. II is based on Th. I and on a Baire's category argument inspired by the classical work of Hans Lewy [3]. Thus the proof is not constructive, in contrast with that of Th. I and with Nirenberg's construction in [4]. The reader will notice that the solutions (of class $C^{1+\delta}$ for some $\delta > 0$ when $n = 1$) of the aberrant homogeneous equations, *in any open subset of Ω*, are locally constant. In Nirenberg's example the open sets had to contain a central point. And the aberrant systems, far from being rare, are in fact dense. It is highly likely that, if one is willing to define to appropriate C^∞ topology on the set of CR structures satisfying (0.6), the latter assertion could be precisely restated as *the density of the aberrant CR structures*. We have not attempted to do so here.

The present article is essentially self-contained. Some portions of the reasoning of [2] have therefore been repeated.

1. Perturbations of locally integrable structures

We begin by considering a locally integrable CR structure on Ω. An arbitrary point p_0 of Ω has an open neighborhood U in which there are (real) local coordinates $x_1, \cdots, x_n$, $y_1, \cdots, y_n$, u and $n+1$ C^∞ CR functions $z_1, \cdots, z_n$, w, such that

$$(1.1)\qquad z_j = x_j + iy_j \qquad (i=\sqrt{-1},\ j=1, \cdots, n);$$
$$w = u + i\phi(z, \bar{z}, u).$$

We shall always assume that the coordinates and the CR functions (1.1) all vanish at the point p_0. Henceforth we refer to it as "the origin" (and identify U to an open neighborhood of the origin in $\boldsymbol{R}^{2n+1}$). It is standard to effect some simplifications of the Taylor expansion of ϕ about the origin, by means of holomorphic substitutions of $(z_1, \cdots, z_n, w)$. One may suppose that

$$(1.2)\qquad \phi(z, \bar{z}, u) = \sum_{j,k=1}^{n} c_{jk} z_j \bar{z}_k + O\left(|z|\ |u| + |u|^2 + |z|^3\right).$$

It is well known that the hypothesis that the Levi form of the structure (at p_0) is nondegenerate is equivalent with the property

$$(1.3)\qquad \det(c_{jk}) \neq 0.$$

The hypothesis (0.6) about the signature of the Levi form means that, possibly after a nonsingular $\boldsymbol{C}$-linear transformation of z we may assume

$$(1.4)\qquad \phi(z, \bar{z}, u) = |z_1|^2 - |z'|^2 + O\left(|z|\ |u| + |u|^2 + |z|^3\right),$$

where we have used the notation $z' = (z_2, \cdots, z_n)$.

Observing that dz_j, $d\bar{z}_k$ (j, $k=1, \cdots, n$), together with dw, make up a linear basis of $\boldsymbol{C}T_p^*\Omega$ at every point p of U we introduce the dual basis of $\boldsymbol{C}T_p\Omega$. This defines $2n+1$ smooth vector fields in U, $M_0, M_1, \cdots, M_n$, $L_1, \cdots, L_n$, by the conditions:

$$(1.5)\qquad \begin{aligned} &L_j z_k = L_j w = 0, \qquad L_j \bar{z}_k = \delta_{jk} \text{ (Kronecker index)} \\ &M_j \bar{z}_k = M_j w = 0, \qquad M_j z_k = \delta_{jk}, \quad \textit{if } j,\ k = 1, \cdots, n, \end{aligned}$$

$$(1.6)\qquad M_0 z_k = M_0 \bar{z}_k = 0, \qquad k = 1, \cdots, n, \qquad M_0 w = 1.$$

It follows immediately from (1.5)-(1.6) that, everywhere in U,

$$(1.7)\qquad \begin{aligned} &[L_j, L_k] = [L_j, M_l] = [M_l, M_m] = 0, \\ &\quad j,\ k = 1, \cdots, n, \qquad l,\ m = 0, 1, \cdots, n. \end{aligned}$$

More explicit descriptions of those vector fields are easy to obtain. First of all,

$$(1.8)\qquad M_0 = w_u^{-1} \frac{\partial}{\partial u}.$$

Then

(1.9) $$L_j=\frac{\partial}{\partial\bar{z}_j}-w_{\bar{z}_j}M_0,\qquad M_j=\frac{\partial}{\partial z_j}-w_{z_j}M_0,\qquad j=1,\cdots,n.$$

In slightly different notation, for $j=1,\cdots,n$,

(1.10) $$L_j=\frac{\partial}{\partial\bar{z}_j}-i\lambda_j\frac{\partial}{\partial u},\qquad M_j=\frac{\partial}{\partial z_j}-i\mu_j\frac{\partial}{\partial u},$$

where

(1.11) $$\lambda_j=\phi_{\bar{z}_j}/(1+i\phi_u),\qquad \mu_j=\phi_{z_j}/(1+i\phi_u).$$

The commutation relations (1.7) are then equivalent to the equations

(1.12) $$L_j\lambda_k=L_k\lambda_j,\qquad L_j\mu_k=M_k\lambda_j,\qquad M_j\mu_k=M_k\mu_j,$$
$$\text{if}\quad j,\ k=1,\cdots,n,$$

(1.13) $$L_jw_u^{-1}=-iM_0\lambda_j,\qquad M_jw_u^{-1}=-iM_0\mu_j,\qquad j=1,\cdots,n.$$

We shall also make use of the following differential operator

(1.14) $$M=\sum_{k=1}^{n}z_kM_k.$$

Note that

(1.15) $$M=\sum_{k=1}^{n}z_k\frac{\partial}{\partial z_k}-i\mu\frac{\partial}{\partial u},$$

where

(1.16) $$\mu=\sum_{k=1}^{n}z_k\mu_k.$$

Evidently M commutes with each L_j, which is equivalent to saying that

(1.17) $$L_j\mu=M\lambda_j,\qquad j=1,\cdots,n.$$

For use below we note that $L_j\overline{w}=L_j(w+\overline{w})=2L_ju$, i.e.,

(1.18) $$L_j\overline{w}=-2i\lambda_j.$$

Likewise,

(1.19) $$M\overline{w}=-2i\mu.$$

Let us denote by u, v the coordinates in $\boldsymbol{R}^2$. Consider any function $f\in C^\infty(\boldsymbol{R}^2)$ whose support is contained in the sector

(1.20) $$\{(u,v)\in\boldsymbol{R}^2;\ |u|\le v\}.$$

Possibly after contracting U about the origin we may state:

LEMMA 1.1. *Whatever the integer m, the function $f(w)/z_1^m$ is smooth in U and vanishes to infinite order on the subspace $z_1=0$.*

PROOF. Let $v=\phi(z,\bar{z},u)$ (then $w=u+iv$). From (1.4) & (1.20) we derive

$$|u|+|z'|^2 \leq |z_1|^2+\text{const}\left(|z|\;|u|+|u|^2+|z|^3\right)$$

on supp $(f \circ w)$. As a consequence, and provided U is small enough, we have

$$|u|+|z|^2 \leq \text{const.}\;|z_1|^2, \qquad \forall (z,u)\in \text{supp}\,(f\circ w).$$

When $z_1 \to 0$ in $\boldsymbol{C}^1$, the point (z,u) converges to the origin in U and w converges to 0 in the sector (1.20). It suffices then to note that f vanishes to infinite order at the origin.

Let g be another C^∞ function in $\boldsymbol{R}^2$ with support contained in the sector (1.20). Let us set

$$F=\frac{f(w)/z_1}{1+f(w)/w_u z_1}, \qquad G=\frac{g(w)/z_1^2}{1-\mu g(w)/z_1^2}. \tag{1.21}$$

We are assuming, henceforth, that U is small enough that both $|f(w)/w_u z_1|$ and $|g(w)\mu/z_1^2|$ are very small compared to *one*. This is possible thanks to Lemma 1.1.

LEMMA 1.2. *The (smooth) vector fields in U,*

$$\tilde{L}_j=L_j+i\lambda_j F M_0, \qquad j=1,\cdots,n, \tag{1.22}$$

commute pairwise. So do the vector fields

$$L_j^\sharp=L_j+\lambda_j G M, \qquad j=1,\cdots,n. \tag{1.23}$$

Note that there is no $\sqrt{-1}$ in front of λ_j in Eq. (1.23).

PROOF. Straightforward differentiation shows that

$$L_j F+F^2 L_j(w_u^{-1})=\left[1+f(w)/w_u z_1\right]^{-2} L_j\left[f(w)/z_1\right], \tag{1.24}$$

$$L_j G-G^2 L_j \mu=\left[1-\mu g(w)/z_1^2\right]^{-2} L_j\left[g(w)/z_1^2\right]. \tag{1.25}$$

By virtue of (1.18),

$$L_j\left[f(w)/z_1\right]=-2i\lambda_j f_{\bar{w}}(w)/z_1, \qquad L_j\left[g(w)/z_1^2\right]=-2i\lambda_j g_{\bar{w}}(w)/z_1^2.$$

If we combine this with (1.24) & (1.25) respectively we see that there are C^∞ functions F_1, G_1 in U such that

(1.26) $$L_j F + F^2 L_j(w_u^{-1}) = F_1 \lambda_j ,$$

(1.27) $$L_j G - G^2 L_j \mu = G_1 \lambda_j ,$$

for every $j=1, \cdots, n$. We have (cf. (1.7))

$$[\tilde{L}_j, \tilde{L}_k] = \left[L_j(\lambda_k F) - L_k(\lambda_j F)\right] iM_0$$
$$- F^2(\lambda_j M_0 \lambda_k - \lambda_k M_0 \lambda_j)\, M_0 = i\phi_0 M_0 .$$

If we take (1.12) and (1.13) into account, we get

$$\phi_0 = \lambda_k\left[L_j F + F^2 L_j(w_u^{-1})\right] - \lambda_j\left[L_k F + F^2 L_k(w_u^{-1})\right] \equiv 0 \text{ by (1.26)} .$$

Likewise,

$$[L_j^\sharp, L_k^\sharp] = \phi M ,$$

with

$$\phi = \lambda_k L_j G - \lambda_j L_k G + G^2(\lambda_j M \lambda_k - \lambda_k M \lambda_j)$$
$$= \lambda_k(L_j G - G^2 L_j \mu) - \lambda_j(L_k G - G^2 L_k \mu) \text{ by (1.17)} .$$

It suffices to apply (1.27) to conclude that ϕ vansihes identically.

COROLLARY 1.1. *Suppose that fg vanishes identically. Then the n smooth vector fields in U,*

(1.28) $$\Lambda_j = L_j + \lambda_j(iFM_0 + GM) , \qquad j = 1, \cdots, n ,$$

commute pairwise.

PROOF. Indeed, at each point of U, Λ_j is equal to infinite order either to $\tilde{L}_j$ for every j, or to $L_j^\sharp$ for every j.

REMARK 1.1. For every $j=1, \cdots, n$, $L_j - \Lambda_j$ vanishes to infinite order at the origin.

2. Reduction to the case $n=1$

Let $t=(t_1, \cdots, t_n) \in \boldsymbol{R}^n$ be a point such that

(2.1) $$t_1^2 - (t_2^2 + \cdots + t_n^2) = 1 .$$

We shall always assume that $|t| \leq R < +\infty$ $(R>1)$. In the sequel ζ will denote a complex variable (in $\boldsymbol{C}^1$). Fixing t as said, we call U^t the image of a suitably small open neighborhood of the origin in $\boldsymbol{C}^1 \times \boldsymbol{R}^1$ under the mapping $(\zeta, u) \mapsto (z_1, \cdots, z_n, u)$, defined by the equations

(2.2) $$z_j = \zeta t_j , \qquad j = 1, \cdots, n .$$

Thus U^t is a smooth 3-dimensional submanifold of U. We shall use the notation

$$\phi^t(\zeta, \bar{\zeta}, u) = \phi(\zeta t, \bar{\zeta} t, u), \qquad w^t = u + i\phi^t(\zeta, \bar{\zeta}, u). \tag{2.3}$$

According to (1.4) we have

$$\phi^t = |\zeta|^2 + O\left(|\zeta|\,|u| + |u|^2 + |\zeta|^3\right). \tag{2.4}$$

The functions ζ, w^t define a CR structure on U^t. The CR bundle of this structure is spanned by the vector field

$$L^t = \frac{\partial}{\partial \bar{\zeta}} - i\lambda^t \frac{\partial}{\partial u}, \qquad \lambda^t = \phi^t_{\bar{\zeta}} / w^t_u. \tag{2.5}$$

It is readily checked that, along U^t,

$$L^t = \sum_{j=1}^n t_j L_j, \qquad \lambda^t = \sum_{j=1}^n t_j \lambda_j. \tag{2.6}$$

We may also introduce the vector fields

$$M^t = \sum_{j=1}^n t_j M_j = \frac{\partial}{\partial \zeta} - i\mu^t \frac{\partial}{\partial u}, \qquad \mu^t = \phi^t_\zeta / w^t_u, \tag{2.7}$$

$$M^t_0 = (w^t_u)^{-1} \frac{\partial}{\partial u}. \tag{2.8}$$

Since the vector field M (see (1.15)) annihilates all the functions $t_j z_k - t_k z_j$ (j, $k = 1, \cdots, n$) at every point of U^t it is tangent to this submanifold, and we have, along U^t,

$$M = \zeta M^t. \tag{2.9}$$

(There is an awkwardness in the notation: μ^t is *not* the restriction of the function μ of (1.16) to U^t, but $\zeta\mu^t$ is.) If then we call F^t and G^t the restrictions to U^t of the functions F and G defined in (1.21) we may consider the analogue in U^t of the vector fields (1.28):

$$\Lambda^t = \sum_{j=1}^n t_j \Lambda_j \Big|_{U^t} = L^t + \lambda^t (iF^t M^t_0 + G^t \zeta M^t). \tag{2.10}$$

We consider now an arbitrary C^1 solution h of the system of equations, in an open neighborhood $U_* \subset U$ of the origin,

$$\Lambda_j h = 0, \qquad j = 1, \cdots, n. \tag{2.11}$$

If h^t denotes the restriction of h to $U^t \cap U_*$ we have, there,

$$\Lambda^t h^t = 0. \tag{2.12}$$

In the next section we are going to show that there exist functions f and g as in (1. 21) such that, whatever the vector t verifying (2. 1) (and $|t|\leq R$), the equation (2. 12) implies that the differential of h^t vanishes at the origin. Let us show here that the latter, in turn, implies

$$dh|_0=0\,. \tag{2.13}$$

Indeed we have

$$\frac{\partial h^t}{\partial u}(0,0)=\frac{\partial h}{\partial u}\bigg|_0, \tag{2.14}$$

$$\frac{\partial h^t}{\partial \zeta}(0,0)=\sum_{j=1}^n t_j\frac{\partial h}{\partial z_j}\bigg|_0, \qquad \frac{\partial h^t}{\partial \bar\zeta}(0,0)=\sum_{j=1}^n t_j\frac{\partial h}{\partial \bar z_j}\bigg|_0. \tag{2.15}$$

Thus, if $dh^t=0$ at the origin all the right-hand sides in (2. 14) & (2. 15) vanish at the origin. But the set of vectors t such that (2. 1) and $|t|\leq R$ hold generate the whole space $\boldsymbol{R}^n$, and thus the vanishing of those right-hand sides allows us to conclude that (2. 13) is valid.

3. The case $n=1$

We return to the CR-structure on the 3-dimensional manifold U^t defined by the functions ζ, w^t (Sect. 2).

We notice that, when $u=0$, there is an open disk in ζ-plane, centered at the origin, Δ, such that

$$\phi^t>0 \ \ in \ \ \Delta\backslash\{0\}\,.$$

Recall that $\phi^t=0$ when $\zeta=\mathrm{u}=0$. Consequently, and possibly after contracting Δ, there is a C^∞ function of (u,t) in an open subset of $\boldsymbol{R}^{n+1}$, $\mathcal{O}$, which we describe below, $\zeta_0^t(u)$, valued in Δ, such that

$$\zeta_0^t(0)=0\,, \tag{3.1}$$

$$C^{-1}\left|\zeta-\zeta_0^t(u)\right|^2\leq \tag{3.2}$$
$$\left|\phi^t(\zeta,\zeta,u)-\phi_0^t(u)\right|\leq C\left|\zeta-\zeta_0^t(u)\right|^2, \qquad \zeta\in\Delta\,,$$

where

$$\phi_0^t(u)=\phi^t\left(\zeta_0^t(u),\ \overline{\zeta_0^t(u)},\ u\right), \qquad (u,t)\in\mathcal{O}\,, \tag{3.3}$$

and C is a constant >0. The subset $\mathcal{O}$ is a product $U_0\times\Theta$, with U_0 a suitably small interval in $\boldsymbol{R}^1$ centered at zero, and Θ a suitable open neighborhood of the subset of $\boldsymbol{R}^n$ defined by (2. 1) and by $|t|\leq R$. We may then find a number $\varepsilon>0$ such that the sector in the (u,v)-plane

$$|u| < \varepsilon v$$

lies *above* the curve $v=\phi_0^t(u)$ - whatever $t\in\Theta$.

By virtue of (3. 2) there is $\delta>0$ such that, given any point $(u, v)\in \boldsymbol{R}^2$ such that

(3. 4) $$v>\phi_0^t(u)\,, \qquad u^2+v^2<\delta^2\,, \qquad |u|<\varepsilon v\,,$$

the equation

(3. 5) $$\phi^t(\zeta, \bar\zeta, u) = v$$

defines a smooth closed curve $\gamma^t(u, v)$ in ζ-plane, contained in the disk Δ and winding around $\zeta_0^t(u)$.

Following [2] we select two sequences $\{A_j\}$, $\{B_j\}$ $(j=1, 2, \cdots)$ of compact subsets of the plane converging to $\{0\}$, and further submitted to the requirement that every one of them be convex and contained in the region (3. 4) and that they be pairwise disjoint, more precisely that

(3. 6) *the projections on the u-axis of the A_j and of the B_k be pairwise disjoint* (for all j, $k=1, \cdots$).

We also require that the interior of each A_j and of each B_k be nonempty. We choose the functions f, g in (1. 21) as follows:

(3. 7) $f\equiv 0$ *in the complement of* $\bigcup_{j=1}^{+\infty} A_j$, $g\equiv 0$ *in the complement of* $\bigcup_{j=1}^{+\infty} B_j$;

(3. 8) *for every* $j=1, 2, \cdots, f>0$ (resp., $g>0$) *in the interior of* A_j (resp. B_j).

LEMMA 3. 1. *Let P, Q be two continuous functions in an open neighborhood of the origin, U_*^t, in U^t such that there is a C^1 function χ in U_*^t satisfying there*

(3. 9) $$L^t\chi = (f\mathrm{o}w^t)\, P + (g\mathrm{o}w^t)\, Q\,.$$

Then necessarily $P=Q=0$ at the origin.

PROOF. For the sake of simplicity we shall omit the superscripts t and reason as if U_*^t were equal to U^t. It will be evident that the reasoning applies when U_*^t is smaller. We call $\mathscr{G}$ the complement in the set (3. 4) of the union of all the sets A_j and B_k; note that $\mathscr{G}$ is open. When $w\in\mathscr{G}$ we have

(3. 10) $$L\chi\equiv 0\,.$$

Consider then the function of $w=u+iv$ in $\mathscr{G}$,

$$I(w)=\oint_{\gamma(u,v)}\chi(\zeta,\bar{\zeta},u)\,d\zeta\,.$$

We contend that, in $\mathscr{G}$, $I(w)$ vanishes identically. It suffices to show that

$$\frac{\partial I}{\partial\bar{w}}\equiv 0\,. \tag{3.11}$$

For then $I(w)$ is holomorphic in $\mathscr{G}$. But $I(w)$ tends to zero as w converges to any point of the curve $v=\phi_0(u)$ - simply because the cycle $\gamma(u,v)$ contracts to a point. By taking advantage of (3.6) and of the fact that the compact sets A_j and B_j are convex, the propagation of zeros of a holomorphic function at once implies that $I(w)\equiv 0$ in $\mathscr{G}$, as the latter set is connected.

Eq. (3.5) defines a smooth map

$$\mathscr{G}\times S^1\ni(u+iv,\theta)\mapsto\zeta(u,v,\theta)\in\Delta\,.$$

Because of (3.10) we have, in $\mathscr{G}$ (cf. (2.7), (2.8)),

$$d\chi=M\chi d\zeta+M_0\chi dw\,,$$

and thus $\chi_{\bar{w}}=(M\chi)\,\zeta_{\bar{w}}\,,\qquad \chi_\theta=(M\chi)\,\zeta_\theta\,.$

For each fixed $u+iv\in\mathscr{G}$, as θ winds around the unit circle S^1, $\zeta(u,v,\theta)$ winds around $\zeta_0(u)$ on the curve $\gamma(u,v)$. Thus we have

$$I(w)=\int_0^{2\pi}\chi(\zeta,\bar{\zeta},u)\,\zeta_\theta d\theta\,,\qquad \zeta=\zeta(u,v,\theta)\,.$$

Consequently,

$$\begin{aligned}I_{\bar{w}}&=\int_0^{2\pi}\Big[(M\chi)\,\zeta_{\bar{w}}\zeta_\theta+\chi\zeta_{\bar{w}\theta}\Big]\,d\theta\\&=\int_0^{2\pi}\frac{\partial}{\partial\theta}(\chi\zeta_{\bar{w}})\,d\theta\end{aligned}$$

whence (3.11).

Availing ourselves once again of (3.6) we select smooth closed curves in $\mathscr{G}$, c_j, c'_j such that, for each $j=1,2,\cdots$, c_j (resp., c'_j) winds around (once) A_j (resp., B_j) and whose interior does not intersect any other set A_k nor any B_l (resp., any other set B_k nor any A_l) for k, $l=1,2,\cdots$, $k\neq j$. Since $I(w)\equiv 0$ in $\mathscr{G}$, we have trivially

$$\oint_{c_j}\oint_{\gamma(u,v)}\chi(\zeta,\bar{\zeta},u)\,d\zeta dw=0 \tag{3.12}$$

and likewise for c'_j. For each j, the mapping

$$(u+iv, \theta)\mapsto\big(\zeta(u, v, \theta), u\big)$$

is a diffeomorphism of $c_j\times S^1$ onto a 2-dimensional torus $T_j\subset\Delta\times U_0$. Call $\hat{T}_j$ its interior. When c'_j is substituted for c_j we use the notation T'_j and $\hat{T}'_j$. By (3.12) the integral of the two-form $\chi d\zeta\wedge dw$ on T_j (resp., T'_j) is equal to zero. By Stokes' theorem the integral on $\hat{T}_j$ (resp., $\hat{T}'_j$) of

$$d(\chi dz\wedge dw)=L\chi d\bar{z}\wedge dz\wedge dw$$

must also be zero. According to (3.9) we have, for every $j=1, 2, \cdots$,

$$\int_{\hat{T}_j} f(w)\, P(z, \bar{z}, u)\, w_u dxdydu=0\,, \tag{3.13}$$

$$\int_{\hat{T}'_j} g(w)\, Q(z, \bar{z}, u)\, w_u dxdydu=0\,. \tag{3.14}$$

Note that the intersection of supp(f∘w) with $\hat{T}_j$ is defined by the fact that $w\in A_j$. The intersection of supp(g∘w) with $\hat{T}'_j$ is likewise defined by the fact that $w\in B_j$. And f (resp., g), which is nonnegative everywhere, is strictly positive at some point of A_j (resp. B_j). As $j\to+0$ the solid tori $\hat{T}_j$ and $\hat{T}'_j$ converge to the set $\{0\}$, and w_u converges to *one*. If P were $\neq 0$ at the origin, as $j\to+\infty$ the argument of the integrand in (3.13) would not vary enough for that equation to hold, and the same is true for Q and (3.14).

We shall apply Lemma 3.1 to the function $\chi=h^t$, the trace on U^t of a C^1 solution of (2.11). By (1.21) we have

$$F^t=\mathrm{f}\circ w^t/\zeta(t_1+\mathrm{f}\circ w^t/w^t_u\zeta)\,,$$

$$G^t=\mathrm{g}\circ w^t/\zeta^2\big(t_1^2-\mu^t(\mathrm{g}\circ w^t)/\zeta\big)$$

(see remark following (2.9)). In view of (2.10) we therefore take

$$P=-i\zeta^{-1}\lambda^t(t_1 w^t_u+\mathrm{f}\circ w^t/\zeta)^{-1}\chi_u\,,$$

$$Q=-\zeta^{-1}\lambda^t\big(t_1^2-\mu^t(\mathrm{g}\circ w^t)/\zeta\big)^{-1}M^t\chi\,.$$

Note that P and Q are indeed continuous (including at the origin), as Lemma 3.1 requires. By (2.4) we see that, on support of f∘w^t and g∘w^t, $\phi^t_\zeta-\zeta$ vanishes to second order at the origin. Since $w^t_u|_0=1$ the same is true of $\lambda^t-\zeta$. Likewise, on those supports, $\mu^t-\bar{\zeta}$ vanishes to second order at the origin, and therefore $M^t|_0=\dfrac{\partial}{\partial\zeta}$. If $P=Q=0$ at the origin it follows that we have

$$\chi_u = \chi_\zeta = 0$$

at the same point. But the equation (2. 12) implies then that we must also have $\chi_{\bar{\zeta}}=0$ at the origin, whence $dh^t|_0=0$, which is what we sought.

4. End of the proof of Theorem I. Proof of Theorem II

We now consider an abstract CR structure $\mathscr{C}$ on Ω whose Levi from is nowhere degenerate and has a signature that conforms to the hypothesis (0. 6) (see Introduction). Given any point p_0 of Ω we can find a local chart $(U, x_1, \cdots, x_n, y_1, \cdots, y_n, u)$ centered at p_0 such that the CR bundle is spanned over U by the vector fields

$$L_j = \frac{\partial}{\partial \bar{z}_j} - i\lambda_j \frac{\partial}{\partial u} - \sum_{k=1}^{n} \lambda_{jk} \frac{\partial}{\partial z_k}, \qquad j=1, \cdots, n, \tag{4. 1}$$

with

$$\lambda_j = \lambda_{jk} = 0 \ \textit{at}\ p_0 \ \textit{for all}\ j,\ k=1, \cdots, n. \tag{4. 2}$$

(That the above local chart is centered at p_0 means that all the local coordinates x_j, y_k, u vanish at p_0.) Condition (4. 2) allows us to solve the following "initial value problems"

$$\begin{aligned} &L_j\zeta_{j'} = 0, \qquad \zeta_{j'} - z_{j'} = 0(|z|^2+u^2), \qquad j'=1, \cdots, n, \\ &L_j\omega = 0, \qquad \omega - u = 0(|z|^2+u^2), \qquad j=1, \cdots, n, \end{aligned} \tag{4. 3}$$

in the ring of formal power series (in x_j, y_k, u). Having done this we select at random $n+1$ C^∞ functions $Z_1, \cdots, Z_n$, w whose Taylor expansions at the origin are equal to the formal power series $\zeta_1, \cdots, \zeta_n$, ω respectively. We have then

(4. 4) $\quad L_jZ_k$, L_jw *vanish to infinite order at* p_0 $(j, k=1, \cdots, n)$.

Possibly after contracting U about p_0 we have the right to use $\mathrm{Re}\, Z_j$, $\mathrm{Im}\, Z_k$, $\mathrm{Re}\, w$ as local coordinates (this follows from (4. 3)). These functions we presently call x_j, y_k, u respectively. In the new coordinates the vector fields L_j still have the expressions (4. 1) but now with the additional properties that

(4. 5) $\quad i\lambda_j - w_{\bar{z}_j}/w_u$ *and* λ_{jk} *vanish to infinite order at* p_0 *for all* j, $k=1, \cdots, n$.

Define then the vector fields in U

$$L_j^0 = \frac{\partial}{\partial \bar{z}_j} - w_{\bar{z}_j} w_u^{-1} \frac{\partial}{\partial u}, \qquad j=1, \cdots, n. \tag{4. 6}$$

For each j, $L_j - L_j^0$ vanishes to infinite order at p_0. The L_j^0 commute pairwise and they define a CR structure $\mathscr{C}^0$ on U, obviously integrable. Possibly after contracting U about p_0 we may assume that the Levi form of $\mathscr{C}^0$ satisfies Condition (0. 6) of the Introduction.

The argument in Sections 1, 2, 3 shows how to construct vector fields $\Lambda_1, \cdots, \Lambda_n$, having the following properties:

(4. 7) *$\Lambda_j - L_j^0$ vanishes to infinite order at p_0 $(j=1, \cdots, n)$;*

(4. 8) *any C^1 function h in an open neighborhood $U_* \subset U$ of p_0 which satisfies, in U_*,*

(4. 9) $\Lambda_j h = 0$, $\quad j=1, \cdots, n$,
is such that $dh|_{p_0} = 0$.

Property (4. 7) implies that, for each j, $L_j - \Lambda_j$ vanishes to infinite order at p_0.

Let $g \in C^\infty(\boldsymbol{R}^{2n+1})$ vanish identically outside the ball of radius one and be identically equal to one inside the ball of radius 1/2. It is elementary that, given any sequence of numbers $r_\nu \searrow 0$, if we define

$$g_\nu(x, y, u) = g(x/r_\nu, y/r_\nu, u/r_\nu),$$

then the coefficients of $g_\nu(L_j - \Lambda_j)$ converge to zero in $C^\infty(U)$. Define then

$$L_j^{(\nu)} = g_\nu \Lambda_j + (1-g_\nu) L_j, \qquad j=1, \cdots, n. \tag{4. 10}$$

Note that, for each j, $L_j^{(\nu)} = \Lambda_j$ when $g_\nu = 1$, in particular in a full neighborhood of the origin, and $L_j^{(\nu)} = L_j$ in the complement of supp g_ν. Moreover, the coefficients of $L_j^{(\nu)}$ converge to the corresponding ones of L_j, in $C^\infty(U)$.

Call $\mathscr{C}^{(\nu)}$ the CR structure on Ω which is equal to the original CR structure $\mathscr{C}$ in $\Omega \backslash$ supp g_ν, and to the one defined by the vector fields $L_j^{(\nu)}$ in U. This makes sense in view of what has just been said. Whatever ν, every germ of C^1 CR function at p_0 in the sense of $\mathscr{C}^{(\nu)}$ has a differential that vanishes at p_0.

We now proceed with the proof of Th. II.

Let Ω' be an open subset of Ω with compact closure. Suppose that the boundary of Ω' is a C^∞ hypersurface, and that Ω' lies on one side only of it. Then every C^∞ function in the closure $\bar{\Omega}'$ of Ω' extends as a C^∞ function to the whole of Ω. Let $(CT\Omega)^n$ denote the Whitney sum over Ω of n copies of the vector bundle $CT\Omega$, and $C^\infty(\bar{\Omega}'; (CT\Omega)^n)$ the space of C^∞ sections of $(CT\Omega)^n$ over $\bar{\Omega}'$, equipped with its natural C^∞ topology. It is a Fréchet space; its topology can be defined by a metric for which it is a complete metric space. Let $\gamma^n(\bar{\Omega}')$ denote the *closed* subspace consisting of those systems of vector fields, $\boldsymbol{L} = (L_1, \cdots, L_n)$, satisfying the formal integrability condition

(4.11) *at every point p of $\bar{\Omega}'$, for every pair j, $k=1, \cdots, n$, the bracket $[L_j, L_k]$ is a linear combination of $L_1, \cdots, L_n$.*

The additional condition that $L_1, \cdots, L_n$ be linearly independent at every point of $\bar{\Omega}'$ defines an open subset of $\gamma^n(\bar{\Omega}')$, and the further condition that the system $\boldsymbol{L}$ obeys (0.6) defines an open subset of the latter open subset, which we assume nonempty and denote by

$$\gamma^{(n-1,1)}(\bar{\Omega}').$$

The Baire's category theorem applies to this set (equipped with the induced topology).

Let us use a Riemannian metric in Ω and the associated norm on the cotangent spaces. Let $\{U_m\}$ $(m=1, 2, \cdots)$ be a sequence of open balls making up a basis of the topology of Ω'. For any m call $\mathscr{A}_m$ the subset of $\gamma^{(n-1,1)}(\bar{\Omega}')$ consisting of those systems $\boldsymbol{L}$ that have the following property:

(4.12) *There is a solution of class $C^{1+1/m}$, h, of the equations*

(4.13) $$L_j h=0\,, \qquad j=1, \cdots, n\,,$$

in $\bar{U}_m$ whose norm in $C^{1+1/m}(\bar{U}_m)$ does not exceed m and is such, moreover, that, everywhere in U_m,

(4.14) $$m^{-1} \leq |dh|\,.$$

Let $\boldsymbol{L}^{(\nu)}$ $(\nu=1, 2, \cdots)$ be a sequence in $\mathscr{A}_m$ converging to a system $\boldsymbol{L}\in\gamma^{(n-1,1)}(\bar{\Omega}')$. For each ν we can select a solution $h^{(\nu)}\in C^{1+1/m}(\bar{U}_m)$ of the equations $L_j^{(\nu)}h=0$, $j=1, \cdots, m$, such that $|dh^{(\nu)}|\geq m^{-1}$. By the compactness of the embedding $C^{1+1/m}(\bar{U}_m)\to C^1(\bar{U}_m)$ and possibly after replacing the sequence $\{h^{(\nu)}\}$ by one of its subsequences, we may assume that it converges in $C^1(\bar{U}_m)$ to a solution h of (4.13) - which must also satisfy (4.14). In other words the closure $\bar{\mathscr{A}}_m$ of $\mathscr{A}_m$ in $\gamma^{(n-1,1)}(\bar{\Omega}')$ *is* contained in the subset $\mathscr{B}_m$ of $\gamma^{(n-1,1)}(\bar{\Omega}')$ consisting of the systems $\boldsymbol{L}$ that have the following property:

(4.15) *There is a C^1 solution h of the homogeneous equations* (4.13) *in U_m such that* (4.14) *holds.*

But the interior of $\mathscr{B}_m$ must be empty. For the reasoning in Sections 1, 2, 3 and in the first part of the present section has shown that, given any point of U_m, p, and any system $\boldsymbol{L}\in\mathscr{B}_m$, there is another system $\tilde{\boldsymbol{L}}\in\gamma^{(n-1,1)}(\bar{\Omega}')$ which is as close as we wish to $\boldsymbol{L}$ in the C^∞ sense, and is such that every C^1 solution in U_m of the equations $L_j h=0$ $(j=1, \cdots, n)$ must satisfy $dh|_p=0$. Just apply that reasoning in an open neighborhood Ω'' of $\bar{\Omega}'$ to which $\boldsymbol{L}$ has been extended — in the place of Ω.

The above implies that *the complement of the union of the sets* $\bar{\mathscr{A}}_m$ *is dense in* $\gamma^{(n-1,1)}(\bar{\Omega}')$, and so is therefore the complement of the union of the sets $\mathscr{A}_m$. Let $\boldsymbol{L}$ be an element of the latter complement, and h a $C^{1+\delta}$ solution of Eq. (4.13) in some open subset U of Ω' (for some $\delta>0$). If dh were different from zero at some point p of U there would be an infinite sequence of integers $m\geq 1$ and a constant $c>0$ such that the $C^{1+\delta}$ norm of h in $\bar{U}_m$ is $\leq c$ and that, in the same set,

$$c^{-1}\leq |dh| .$$

By taking m large enough that $m^{-1}\leq \mathrm{Min}\,(\delta, c)$, we would be able to conclude that h is of class $C^{1+1/m}$ and satisfies (4.14) in U_m and therefore that $\boldsymbol{L}\in\mathscr{A}_m$, contrary to our hypothesis. We reach thus the conclusion that every system $\boldsymbol{L}$ in the complement of the union of the sets $\mathscr{A}_m$ has Property (0.11) (Introduction). This obviously completes the proof of Th. II.

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