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Period-additivity and multistability in piecewise smooth systems

Hunseok Kang and Ah Reum Lee

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Abstract. Piecewise smooth systems have been consistently considered and investigated in nonlinear dynamics due to their practical applications. In this paper, we study a generic type of piecewise smooth dynamical system to deal with period-additivity and multistability in the system; an arithmetic sequence of periodic attractors appearing in the period-adding bifurcation and the coexistence of multiple attractors in the system. We state a physical observation of the phenomena and then provide rigorous mathematical arguments and numerical simulations.

Key words: Piecewise smooth system, Period-adding bifurcation, Multistability.

1. Introduction

Bifurcation and stability have attracted constant attention and interest in dynamical systems. In this paper, we perform bifurcation analysis and examine stability to understand the dynamics of a generic type of piecewise smooth dynamical system. It is well-known that maps that are piecewise smooth and depend smoothly on a parameter possess rich and interesting dynamics, and in particular, such maps exhibit various kinds of bifurcation phenomena such as border-collision in the phase space. See [5], [6]. The systems are applicable to represent adequate mathematical models for many processes in science and engineering fields. They arise naturally in physical systems including grazing impacting systems in mechanical oscillators [4], piecewise linear electronic circuits [5], [8], and cardiac dynamics [7].

The purpose of the research is to explore period additivity and multistability in piecewise smooth systems. We consider a two-dimensional piecewise smooth system in which a linear map is continuously combined with a nonlinear map. For such a system, the phase space can be divided into two regions where the dynamics in each region is different from each other but are nonetheless smooth, and a border that separates the two regions. We shall investigate its dynamics properties which are involved in bifurcation

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and stability. As a parameter in the system increases, the periods of the stable periodic attractors follow an arithmetic increasing sequence. This phenomenon is usually called an arithmetic period-adding bifurcation. On the other hand, multistability is characterized by the coexistence of multiple attractors, and it is common in nonlinear dynamical systems. In such a case, starting the system from a different initial condition can result in a completely different asymptotic state.

In Section 2, we describe the system model to be considered in this paper and give a linear analysis of the system. To find the underlying arithmetic rule among the periods, in Section 3, we investigate the dynamical mechanism of arithmetic period-adding bifurcation for specific parameter settings. In Section 4, we look into the existence and appearance of multistability rigorously. Conclusions are given in Section 5.

2. System Description

We define a piecewise smooth dynamical system F_μ with one border and two smooth regions which is induced by two-dimensional maps H and L_μ as follows:

$$(x_{k+1}, y_{k+1}) = F_\mu(x_k, y_k) = \begin{cases} H(x_k, y_k) & \text{if } x_k \leq \mu, \\ L_\mu(x_k, y_k) & \text{if } x_k > \mu, \end{cases} \tag{1}$$

where

$$\begin{aligned} H(x,y) &= (a - x^2 + by,\ x), \\ L_\mu(x,y) &= (a + cx + by - (\mu + c)\mu,\ dx + (1-d)\mu), \end{aligned} \tag{2}$$

and a, b, c, d and μ are system constants. The system (1) was firstly introduced in [6] for the research of border-collision bifurcation [6]. It is characterized by nondifferentiability on the border, i.e., two distinct one-sided derivatives on the border, which result in various kinds of bifurcation phenomena in the dynamics. See [1], [3], [6].

The system (1) is composed of two distinct two-dimensional maps; one is nonlinear and the other is linear. Obviously, $H(\mu, y) = L_\mu(\mu, y)$ for all y, and thus, F_μ is continuous everywhere. The system F_μ may possess at most three fixed points. One of them is obtained from $L_\mu(x,y) = (x,y)$

$$(x_{L_\mu}, y_{L_\mu}) = \left(\frac{\mu^2 - (b - bd - c)\mu - a}{bd + c - 1}, \frac{d\mu^2 - (1 - c - d)\mu - ad}{bd + c - 1} \right),$$

and the other two are from $H(x, y) = (x, y)$

$$(x_H^{\pm}, y_H^{\pm}) = \left(\frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2}, \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2} \right),$$

We propose the criteria for determining stabilities of the system at the three fixed points above. The following results are straight-forwardly calculated. See [3] for details.

Proposition 1 (a) *If the inequality* $-1 < bd < 1 - |c|$ *holds, then the fixed point* (x_{L_μ}, y_{L_μ}) *is a sink for* L_μ *if exists.*

(b) *If the inequalities*

$$|b| < 1 \quad \textit{and} \quad \left| \frac{4a}{(b-1)^2} - 1 \right| < 2 \tag{3}$$

hold, then the fixed point (x_H^+, y_H^+) *is a sink for* H *whereas the other fixed point* (x_H^-, y_H^-) *is a saddle for* H *if exist.*

Finally, we consider a particular value μ_0 for the border $x = \mu$ of the system, that is,

$$\mu_0 = \frac{b - 1 + \sqrt{(b-1)^2 + 4a}}{2}. \tag{4}$$

Then it makes the following relation:

$$(x_{L_\mu}, y_{L_\mu}) = \left(x_H^+, y_H^+\right) \quad \text{if and only if} \quad \mu = \mu_0.$$

From now on, we fix $\mu = \mu_0$ so that both of the maps H and L_μ have the same fixed point on the border, which is

$$(x_{L_\mu}, y_{L_\mu}) = \left(x_H^+, y_H^+\right) = (\mu_0, \mu_0).$$

3. Additivity of periodicity

In this section, we probe mathematical properties of the system F_μ in (1) focusing on its periodic attractors when $\mu = \mu_0$. We shall ascertain that, as the parameter c increases while other parameters are fixed, periodic attractors are continuously created, one of them is terminated before another is created, and their periodicity follows an arithmetic sequence. To verify these, we examine the following:

1. the existence and stabilities of periodic orbits;
2. the way of the periodic attractors being arranged.

In order to provide a detailed mathematical arguments illustrating such phenomena, we assign the following values to the parameters:

$$a = 1.5,\ b = -0.9,\ d = 1.0, \tag{5}$$

and c is varied as a bifurcation parameter. The value μ_0 given in (4) is $\mu_0 = 0.6$. As assumed in Section 2, the border $\mu = \mu_0 = 0.6$ is fixed so the system to be considered is as follows:

$$(x_{n+1}, y_{n+1}) = \begin{cases} (1.5 - x_n^2 - 0.9y_n,\, x_n) & \text{if } x_n \le 0.6, \\ (1.5 + cx_n - 0.9y_n - 0.6(c+0.6),\, x_n) & \text{if } x_n > 0.6. \end{cases} \tag{6}$$

Here, the parameters' values are selected so as to avoid some undesirable cases where round-off errors of irrationals for μ_0 in numerical computations result in some critical fallacies.

In addition, we restrict our attention to the period-n orbits which are obtained from the equations

$$L_\mu^{(i-1)} \circ H \circ L_\mu^{(n-i)}(x, y) = (x, y), \tag{7}$$

for $i = 1, 2, \ldots, n$, which are quadratic. Each equation in (7) makes two distinct solutions. Clearly, one of them is the fixed point. We denote the fixed point by $\mathbf{P}^*$, i.e., $\mathbf{P}^* := (0.6, 0.6)$, and the other solution by $(p_n^{(i)}, q_n^{(i)})$. The inequalities

$$p_n^{(i)} \ge \mu_0 \quad \text{and} \quad p_n^{(n)} \le \mu_0, \tag{8}$$

for $i = 1, 2, \dots, n-1$, guarantee that $\{(p_n^{(i)}, q_n^{(i)}) : i = 1, 2, \dots, n\}$ exists as a period-n orbit by the definition given in (1). A straight-forward calculation indicates that neither $H^n(x, y) = (x, y)$ nor $L_\mu^n(x, y) = (x, y)$ provide any period-n orbit in the system for $n \geq 2$.

3.1. Fixed point

Using Proposition 1(a), one can find an interval $\mathbf{K}$ such that for each $c \in \mathbf{K}$ the fixed point $\mathbf{P}^*$ is a sink for both H and L_μ if exists;

$$\mathbf{K} := (-1 + b, 1 - b) = (-1.9, 1.9). \tag{9}$$

In addition, the inequality in Proposition 1(b) holds under the values of parameters in (5). Thus, for each $c \in \mathbf{K}$, the fixed point $\mathbf{P}^*$ is a sink for both L_μ and H. From now on, we only deal with the case where $c \in \mathbf{K}$.

3.2. Period-2 attractor

From Eq. (7) at $n = 2$, we obtain a set $\mathbf{P}_2$ of two points

$$\mathbf{P}_2 = \left\{ \left(p_2^{(i)}, q_2^{(i)}\right) \middle| i = 1, 2 \right\},$$

where

$$p_2^{(1)} = q_2^{(2)} = \mu_0 - 19\delta_2(c),$$
$$p_2^{(2)} = q_2^{(1)} = \mu_0 - 10c\delta_2(c),$$

and

$$\delta_2(c) = \frac{120c + 361}{1000c^2}.$$

However, for any $c \in \mathbf{K}$, at least one of the values $p_2^{(1)}$ and $p_2^{(2)}$ does not satisfy the inequalities in (8), where $\mathbf{K}$ is given in (9). This means that the set $\mathbf{P}_2$ cannot be selected as a period-2 orbit of the system F_μ, and hence, there is no period-2 orbit when $c \in \mathbf{K}$.

3.3. Period-3 attractor

From Eq. (7) at $n = 3$, we obtain a set $\mathbf{P}_3$ of three points

$$\mathbf{P}_3 = \left\{ \left(p_3^{(i)}, q_3^{(i)}\right) \middle| i = 1, 2, 3 \right\},$$

where

$$p_3^{(1)} = q_3^{(2)} = \mu_0 - (90c + 100)\delta_3(c),$$
$$p_3^{(2)} = q_3^{(3)} = \mu_0 - (100c + 81)\delta_3(c),$$
$$p_3^{(3)} = q_3^{(1)} = \mu_0 - (100c^2 - 90)\delta_3(c),$$

and

$$\delta_3(c) = \frac{1200c^2 + 1800c + 649}{1000(10c^2 - 9)^2}. \tag{10}$$

Plugging $p_3^{(1)}$, $p_3^{(2)}$ and $p_3^{(3)}$ into the inequalities in (8), we find an interval $\mathbf{I}_3 \subset \mathbf{K}$ so that for $c \in \mathbf{I}_3$ the set $\mathbf{P}_3$ is accepted as a period-3 orbit. It is a closed interval $\mathbf{I}_3 = [c_3^{(1)}, c_3^{(2)}]$, where

$$c_3^{(1)} = -\frac{81}{100} \quad \text{and} \quad c_3^{(2)} = -\frac{3}{4} + \frac{\sqrt{78}}{60} \approx -0.6028.$$

This means that for each $c \in \mathbf{I}_3$ there exists a period-3 orbit of F_μ, which is $\mathbf{P}_3$.

We show that the period-3 orbit $\mathbf{P}_3$ is attracting. Let $\mathrm{D}(H \circ L_\mu^2)$ be the Jacobian matrix of the map $H \circ L_\mu^2$. The characteristic polynomial χ_3 of $\mathrm{D}(H \circ L_\mu^2)$ at $(p_3^{(1)}, q_3^{(1)})$ is

$$\chi_3(\lambda) = \lambda^2 - \left(\frac{6}{5}c^2 + \frac{9}{5}c + \frac{1189}{500}\right)\lambda + \frac{729}{1000}.$$

For each $c \in (c_3^{(1)}, c_3^{(2)})$, one can confirm that

$$\chi_3(-1) > 0,\ \chi_3'(-1) < 0,\ \chi_3(1) > 0,\ \chi_3'(1) > 0 \text{ and } 0 < \chi_3(0) < 1.$$

This implies that $\mathrm{D}(H \circ L_\mu^2)$ at $(p_3^{(1)}, q_3^{(1)})$ has two eigenvalues whose magnitudes are less than 1. Hence, $\mathbf{P}_3$ is a period-3 attractor for the system F_μ with $H \circ L_\mu^2$.

On the other hand, the characteristic polynomial $\bar{\chi}_3$ of $\mathrm{D}(H \circ L_\mu^2)$ at $\mathbf{P}^*$ is

$$\bar{\chi}_3(\lambda) = \lambda^2 + \left(\frac{6}{5}c^2 + \frac{9}{5}c - \frac{27}{25}\right)\lambda + \frac{729}{1000}.$$

It is easy to show that for each $c \in (c_3^{(1)}, c_3^{(2)})$ the inequality

$$\bar{\chi}_3(-1) \cdot \bar{\chi}_3(1) < 0$$

holds. Thus, the fixed point $\mathbf{P}^*$ is a saddle for the maps $L_\mu^{i-1} \circ H \circ L_\mu^{3-i}$ $(i = 1, 2, 3)$. For this reason, $\mathbf{P}^*$ loses the stability and becomes unstable. This phenomenon is called a *dangerous border-collision bifurcation*. See [1], [2] for details.

3.4. Period-4 attractor

From Eq. (7) at $n = 4$, we obtain a set $\mathbf{P}_4$ of four points

$$\mathbf{P}_4 = \left\{ \left(p_4^{(i)}, q_4^{(i)}\right) \;\middle|\; i = 1, 2, 3, 4 \right\},$$

where

$$\begin{aligned}
p_4^{(1)} &= q_4^{(2)} = \mu_0 - (900c^2 + 190)\delta_4(c), \\
p_4^{(2)} &= q_4^{(3)} = \mu_0 - 1810c\delta_4(c), \\
p_4^{(3)} &= q_4^{(4)} = \mu_0 - (1000c^2 - 171)\delta_4(c), \\
p_4^{(4)} &= q_4^{(1)} = \mu_0 - (1000c^3 - 1800c)\delta_4(c),
\end{aligned}$$

where

$$\delta_4(c) = \frac{12000c^3 + 18000c^2 - 21600c + 361}{400000c^2(5c^2 - 9)^2}.$$

Plugging $p_4^{(1)}$, $p_4^{(2)}$, $p_4^{(3)}$ and $p_4^{(4)}$ into the inequalities in (8), we find an interval $\mathbf{I}_4 \subset \mathbf{K}$ so that for $c \in \mathbf{I}_4$ the set $\mathbf{P}_4$ is valid as a period-4 orbit. It is a closed interval $\mathbf{I}_4 = [c_4^{(1)}, c_4^{(2)}]$, where

$$c_4^{(1)} = \sqrt{\frac{171}{1000}} \approx 0.4135 \quad \text{and} \quad c_4^{(2)} \approx 0.7744,$$

and $c_4^{(2)}$ is the largest root of the equation $\delta_4(c) = 0$. Thus, for each $c \in \mathbf{I}_4$, there exists a period-4 orbit of F_μ, which is $\mathbf{P}_4$.

We show that the period-4 orbit $\mathbf{P}_4$ is attracting. Let $\mathrm{D}(H \circ L_\mu^3)$ be the Jacobian matrix of the map $H \circ L_\mu^3$. The characteristic polynomial χ_4 of $\mathrm{D}(H \circ L_\mu^3)$ at $(p_4^{(1)}, q_4^{(1)})$ is

$$\chi_4(\lambda) = \lambda^2 - \left(\frac{6}{5}c^3 + \frac{9}{5}c^2 - \frac{54}{25}c + \frac{8461}{5000}\right)\lambda + \frac{6561}{10000}.$$

For each $c \in (c_4^{(1)}, c_4^{(2)})$, one can confirm that

$$\chi_4(-1) > 0,\ \chi_4'(-1) < 0,\ \chi_4(1) > 0,\ \chi_4'(1) > 0,\ \text{and } 0 < \chi_4(0) < 1.$$

This implies that $\mathrm{D}(H \circ L_\mu^3)$ at $(p_4^{(1)}, q_4^{(1)})$ has two eigenvalues whose magnitudes are less than 1. Hence, $\mathbf{P}_4$ is a period-4 attractor for the system F_μ with $H \circ L_\mu^3$.

On the other hand, the characteristic polynomial $\bar{\chi}_4$ of $\mathrm{D}(H \circ L_\mu^3)$ at $\mathbf{P}^*$ is

$$\bar{\chi}_4(\lambda) = \lambda^2 + \left(\frac{6}{5}c^3 + \frac{9}{5}c^2 - \frac{54}{25}c - \frac{81}{50}\right)\lambda + \frac{6561}{10000}.$$

It is easy to show that for each $c \in (c_4^{(1)}, c_4^{(2)})$, the inequality

$$\bar{\chi}_4(-1) \cdot \bar{\chi}_4(1) < 0$$

holds. Thus, the fixed point $\mathbf{P}^*$ is a saddle for $L_\mu^{i-1} \circ H \circ L_\mu^{4-i}$ $(i = 1, 2, 3, 4)$. Thus, similarly, $\mathbf{P}^*$ loses the stability and becomes unstable.

3.5. Arithmetic sequence of periodicity

Using the same argument described in Section 3.3 and 3.4, one can continue to find periodic attractors. A period-n orbit $\mathbf{P}_n$ is composed of the solutions to the equations $L_\mu^i \circ H \circ L_\mu^{n-i}(x, y) = (x, y)$ $(i = 1, 2, \ldots, n)$ as follows:

$$\mathbf{P}_n = \left\{ \left(p_n^{(i)}, q_n^{(i)}\right) \;\middle|\; i = 1, 2, \ldots, n \right\}.$$

Applying the inequalities (8), one can find the closed interval $\mathbf{I}_n$ satisfying the following properties:

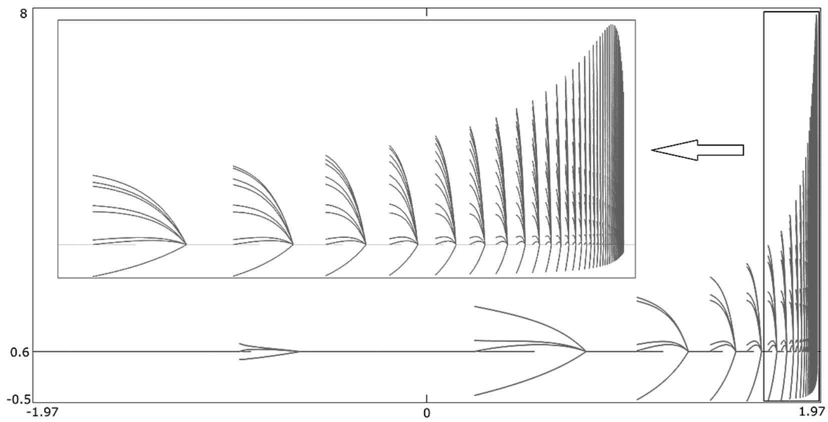

Figure 1. Period-adding bifurcation at $a = 1.542$, $b = -0.97$, and $d = 1$. c is the leading bifurcation parameter, and it varies from -1.97 to 1.97.

1. for each $c \in \mathbf{I}_n$, the set $\mathbf{P}_n$ is valid as a period-n attractor of F_μ;
2. as $c \in \mathbf{K}$ increases, the periodic attractors $\mathbf{P}_n$ appear in succession;
3. the periodicity of the newly created periodic attractor increases by 1 as c increases.

One can also figure out how the stability at the fixed point $\mathbf{P}^*$ changes as c varies:

1. for each $c \in \mathbf{K} \setminus \bigcup \mathbf{I}_n$, the fixed point $\mathbf{P}^*$ is a sink for F_μ (See Section 3.1.);
2. for each $c \in \mathbf{I}_n$, the stability at the fixed point is dominated by the saddles generated by $L_\mu^i \circ H \circ L_\mu^{n-i}$ $(i = 1, 2, \ldots, n)$, but not determined by L_μ nor by H;
3. as $c \in \mathbf{K}$ increases, the appearance and disappearance of dangerous border-collision bifurcation repeat, and thus, the stability at the fixed point changes accordingly.

Figure 1 exhibits a bifurcation diagram related to the period-additivity.

4. Multiple attractors

Multistability, as characterized by the coexistence of multiple attractors, is common in nonlinear dynamical systems. In this case, starting the system from a different initial condition can result in a completely different asymptotic state. We work on the multistability of the system, which means, for each $n \geq 3$, there exists a subinterval $\mathbf{J}_n \subset \mathbf{I}_n$ such that for each $c \in \mathbf{J}_n$ the system possesses a fixed point attractor as well as a period-n attractor.

Considering the case $n = 3$, we study the mechanism of how the multiple attractors are formulated so as to confirm the coexistence of the period-3 attractor $\mathbf{P}_3$ with the fixed point attractor $\mathbf{P}^*$. In Section 3.3, we have seen that $L_\mu^{i-1} \circ H \circ L_\mu^{3-i}$ $(i = 1, 2, 3)$ generates $\mathbf{P}_3$ but neither H^3 nor L_μ^3 can provide any period-3 orbit. Thus, we examine whether $H^{i-1} \circ L_\mu \circ H^{3-i}$ $(i = 1, 2, 3)$ create any period-3 orbit. The solutions to the equations $H^{i-1} \circ L_\mu \circ H^{3-i}(x, y) = (x, y)$ constitute a set $\tilde{\mathbf{P}}_3$:

$$\tilde{\mathbf{P}}_3 = \left\{ \left(\tilde{p}_3^{(i)}, \tilde{q}_3^{(i)}\right) \mid i = 1, 2, 3 \right\},$$

where

$$\begin{aligned}
\tilde{p}_3^{(1)} &= \tilde{q}_3^{(2)} = \frac{\sqrt[3]{\alpha}}{100c} + \frac{13(4c^2 + 7c + 3)}{c\sqrt[3]{\alpha}} - \frac{3 + c}{5c} \\
\tilde{p}_3^{(2)} &= \tilde{q}_3^{(3)} = \frac{1}{9c + 10}\left(\frac{237}{50} + \frac{27c}{5} + \frac{81}{10}\tilde{p}_3^{(1)} - 10\left(\tilde{p}_3^{(1)}\right)^2\right) \\
\tilde{p}_3^{(3)} &= \tilde{q}_3^{(1)} = \frac{1}{9c + 10}\left(\frac{57}{5} + 9c - 9\tilde{p}_3^{(1)} - 10c\left(\tilde{p}_3^{(1)}\right)^2\right),
\end{aligned}$$

and

$$\begin{aligned}
\alpha = -500\big(&250c^3 + 1040.1c^2 + 1279c + 486 \\
&- \sqrt{-(6172c^4 + 16318c^3 + 14284.79c^2 + 4158c + 10.8)(9c + 10)^2}\big).
\end{aligned}$$

The set $\tilde{\mathbf{P}}_3$ is valid as a period-3 orbit of F_μ if the following inequalities hold:

$$\tilde{p}_3^{(1)} \geq \mu_0, \quad \tilde{p}_3^{(2)} \leq \mu_0, \quad \text{and} \quad \tilde{p}_3^{(3)} \leq \mu_0. \tag{11}$$

A closed interval $\mathbf{J}_3 = [-81/100, -431/540]$ is obtained from the inequalities (11). This means, for each $c \in \mathbf{J}_3$, the system F_μ has a period-3 orbit $\tilde{\mathbf{P}}_3$, which is different from $\mathbf{P}_3$.

The stabilities of F_μ at $\tilde{\mathbf{P}}_3$ and $\mathbf{P}^*$ are given as follows. Let $D(H^2 \circ L_\mu)$

be the Jacobian matrix of $H^2 \circ L_\mu$. The characteristic polynomial η_3 of $D(H^2 \circ L_\mu)$ at $(\tilde{p}_3^{(1)}, \tilde{q}_3^{(1)})$ is

$$\eta_3(\lambda) = \lambda^2 + \left(4c\big(\tilde{p}_3^{(3)}\big)^2 + \frac{18}{5}\tilde{p}_3^{(3)} - \frac{9c}{10}\right)\lambda + \frac{729}{1000}.$$

For each $c \in \mathbf{J}_3$, the inequality $\eta_3(-1)\eta_3(1) < 0$ holds. Thus, the period-3 orbit $\tilde{\mathbf{P}}_3$ is a saddle orbit for the map $H^2 \circ L_\mu$, and hence, $\tilde{\mathbf{P}}_3$ is unstable.

On the other hands, the characteristic polynomial $\bar{\eta}_3$ of $D(H^2 \circ L_\mu)$ at $\mathbf{P}^*$ is

$$\bar{\eta}_3(\lambda) = \lambda^2 + \left(\frac{27}{50}c + \frac{54}{25}\right)\lambda + \frac{729}{1000}.$$

For each $c \in \mathbf{J}_3$, one can confirm that

$$\bar{\eta}_3(-1) > 0,\ \bar{\eta}_3'(-1) < 0,\ \bar{\eta}_3(1) > 0,\ \bar{\eta}_3'(1) > 0 \text{ and } 0 < \bar{\eta}_3(0) < 1.$$

This implies that $\mathrm{D}(H^2 \circ L_\mu)$ at $\mathbf{P}^*$ has two eigenvalues whose magnitudes are less than 1. Hence, the fixed point $\mathbf{P}^*$ is a sink for the system F_μ with $H^2 \circ L_\mu$.

In Section 3.3, it is shown that for $c \in \mathbf{I}_3$ the fixed point $\mathbf{P}^*$ loses the stability due to the maps $L_\mu^{i-1} \circ H \circ L_\mu^{3-i}$ $(i = 1, 2, 3)$. In this section, it is seen that for $c \in \mathbf{J}_3 \subset \mathbf{I}_3$, the stability at $\mathbf{P}^*$ is recovered by the maps $H^{i-1} \circ L_\mu \circ H^{3-i}$ $(i = 1, 2, 3)$. This can be confirmed by observing the stable and unstable manifolds of $\mathbf{P}^*$ by $H^{i-1} \circ L_\mu \circ H^{3-i}$ and $L_\mu^{i-1} \circ H \circ L_\mu^{3-i}$, respectively. Therefore, for each $c \in \mathbf{J}_3$, there exist two attractors: the fixed point attractor $\mathbf{P}^*$ and the period-3 attractor $\mathbf{P}_3$. Figure 2 illustrates this situation.

In this way, one can continue to find the multistability for the period-n orbit for each $n \geq 3$, and eventually, reach to the following conclusions.

1. Every period-n attractor begins with a multistability. For each $n \geq 3$ there exists an interval $\mathbf{J}_n \subset \mathbf{I}_n$ such that for each $c \in \mathbf{J}_n$ the system F_μ possesses two attractors simultaneously: the fixed point attractor $\mathbf{P}^*$ and the period-n attractor $\mathbf{P}_n$.
2. Each multistability starts at the moment of the occurrence of saddle-node bifurcation and ends up with the disappearance of the attracting fixed point due to the occurrence of dangerous border-collision bifurcation.

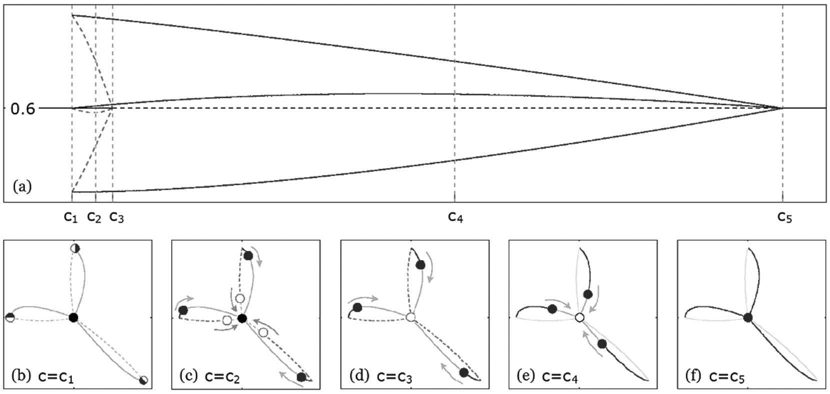

Figure 2. Stabilities of periodic attractors and fixed point for $c \in \mathbf{I}_3 = [c_1, c_5]$. Multistability appears when $c \in \mathbf{J}_3 = (c_1, c_3)$. As c reaches from c_1 to c_3, the period-3 saddle orbit $\tilde{\mathbf{P}}_3$ is merged with the sink $\mathbf{P}^*$, and so, F_μ loses the stability at $\mathbf{P}^*$ and multistability ends when $c = c_3$. In contrast, as c reaches to c_5 from c_3, the period-3 attractor $\mathbf{P}_3$ is merged with the saddle $\mathbf{P}^*$, and so, F_μ recovers the stability at $\mathbf{P}^*$.

5. Conclusion

Nonsmooth dynamical systems arise commonly in physical and engineering applications and they permit behaviors that usually find no counterparts in smooth systems. One of the common examples of multistability is KAM islands in a Hamiltonian system.

Piecewise smooth dynamical systems are of particular interest in physical and engineering fields. We have discussed the problem of period-adding bifurcation and multistability in the system which have not been observed in smooth systems. We have provided physical analysis and mathematical arguments to establish our finding. For the period-adding bifurcation, we have shown that as a leading parameter in the system increases, the periods of the newly created periodic attractors follows an arithmetic sequence. In addition, we have identified the saddle-node bifurcation as the mechanism to create another periodic attractors in the presence of an existing periodic one which produces multistability.

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Hunseok Kang
Department of Mathematics
Soongsil university
Seoul 156-743, Korea
E-mail: hkang@ssu.ac.kr