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On the indices of minimal orbits of Hermann actions

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Abstract. We give a formula to determine the indices of special (non-totally geodesic) minimal orbits of Hermann actions. Also, we give examples of such minimal orbits of Hermann actions and calculate their indices by using the formula.

Key words: Symmetric space, Hermann action, index, Jacobi operator, Casimir operator.

1. Introduction

In 1987, Y. Ohnita [O] gave a formula to calculate the indices (and nullities) of totally geodesic submanifolds in a symmetric space N of compact type and showed that the indices of all Helgason spheres in every simply connected irreducible compact symmetric space are equal to zero, that is, they are stable. In 1993, O. Ikawa [I1] investigated the Jacobi operator of equivariant minimal homogeneous submanifold in a Riemannian homogeneous space. In 1995, by using Ohnita's index formula, M. S. Tanaka [Tan] determined the stability of all polars and meridians in every simply connected irreducible compact symmetric space. Note that polars and meridians are totally geodesic. In 2008, by using the index formula, T. Kimura [Ki] determined the stability of all totally geodesic singular orbits of all cohomogeneity one actions on every simply connected irreducible compact symmetric space. In 2009, by using this index formula, T. Kimura and M. S. Tanaka [KT] determined the stability of all maximal totally geodesic submanifolds in every simply connected irreducible compact symmetric space of rank two. Let $N = G/K$ be a symmetrc space of compact type equipped with the G-invariant metric induced from the Killing form of the Lie algebra of G. In this paper, we treat only a symmetric space of compact type equipped with such a G-invariant metric. Let H be a symmetric subgroup of G (i.e., $(\mathrm{Fix}\,\tau)_0 \subset H \subset \mathrm{Fix}\,\tau$ for some involution τ of G), where $\mathrm{Fix}\,\tau$ is the fixed point group of τ and $(\mathrm{Fix}\,\tau)_0$ is the identity component of $\mathrm{Fix}\,\tau$. The natural action of H on N is called a *Hermann action* (see [HPTT], [Kol]). Let θ

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be an involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$. According to [Co], in the case where G is simple, we may assume that $\theta\circ\tau = \tau\circ\theta$ by replacing H to a suitable conjugate group of H if necessary except for the following three Hermann actions:

(i) $Sp(p+q) \curvearrowright SU(2p+2q)/S(U(2p-1)\times U(2q+1)) \quad (p \geq q+2)$,
(ii) $U(p+q+1) \curvearrowright Spin(2p+2q+2)/Spin(2p+1)\times_{\mathbf{Z}_2} Spin(2q+1) \quad (p \geq q+1)$,
(iii) $Spin(3)\times_{\mathbf{Z}_2} Spin(5) \curvearrowright Spin(8)/\omega(Spin(3)\times_{\mathbf{Z}_2} Spin(5))$,

where ω is the triality automorphism of $Spin(8)$. Here we note that we remove transitive Hermann actions.

Assumption In the sequel, we assume that $\theta\circ\tau = \tau\circ\theta$.

Let $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of G, K and H, respectively. Denote the involutions of $\mathfrak{g}$ induced from θ and τ by the same symbols θ and τ, respectively. Set $\mathfrak{p} := \mathrm{Ker}(\theta + \mathrm{id})$ and $\mathfrak{q} := \mathrm{Ker}(\tau + \mathrm{id})$. The vector space $\mathfrak{p}$ is identified with $T_{eK}(G/K)$, where e is the identity element of G. Take a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{p}\cap\mathfrak{q}$. For each $\beta \in \mathfrak{b}^*$, we set $\mathfrak{p}_\beta := \{X \in \mathfrak{p} \mid \mathrm{ad}(b)^2(X) = -\beta(b)^2 X \ (\forall\, b \in \mathfrak{b})\}$ and $\triangle' := \{\beta \in \mathfrak{b}^* \setminus \{0\} \mid \mathfrak{p}_\beta \neq \{0\}\}$. This set $\triangle'$ is a root system. Note that we call $\triangle'$ a root system because β's ($\beta \in \triangle'$) give a root system in the vector subspace spanned by them (in the sense of [He]) even if they do not span $\mathfrak{b}^*$. Let $\Pi' = \{\beta_1, \ldots, \beta_r\}$ be the simple root system of the positive root system $\triangle'_+$ of $\triangle'$ under a lexicographic ordering of $\mathfrak{b}^*$. Set ${\triangle'}^V_+ := \{\beta \in \triangle'_+ \mid \mathfrak{p}_\beta \cap \mathfrak{q} \neq \{0\}\}$ and ${\triangle'}^H_+ := \{\beta \in \triangle'_+ \mid \mathfrak{p}_\beta \cap \mathfrak{h} \neq \{0\}\}$. Define a subset $\widetilde{C}$ of $\mathfrak{b}$ by

$$\widetilde{C} := \left\{ b \in \mathfrak{b} \mid 0 < \beta(b) < \pi\big(\forall\, \beta \in {\triangle'}^V_+\big),\ \ -\frac{\pi}{2} < \beta(b) < \frac{\pi}{2}\big(\forall\, \beta \in {\triangle'}^H_+\big) \right\}.$$

The closure $\overline{\widetilde{C}}$ of $\widetilde{C}$ is a simplicial complex. Set $C := \mathrm{Exp}(\widetilde{C})$, where Exp is the exponential map of G/K at eK. Each principal H-orbit passes through only one point of C and each singular H-orbit passes through only one point of $\mathrm{Exp}(\partial\widetilde{C})$. For each simplex σ of $\overline{\widetilde{C}}$, only one minimal H-orbit through $\mathrm{Exp}(\sigma)$ exists. See proofs of Theorems A and B in [Koi1] (also [I2]) about this fact. Also, it is known that only one minimal H-orbit through $\mathrm{Exp}(\sigma)$ is unstable if σ is not a vertex (see the proof of Theorem 2.24 in [I2]). Denote by $D(H)$ the set of all equivalence classes of (finite dimensional) irreducible complex representations of H and $\rho_{G/H} : H \to GL(\mathfrak{q})$ the isotropy representation of G/H, that is, $\rho_{G/H}(h) := \mathrm{Ad}_G(h)|_{\mathfrak{q}}$ $(h \in H)$, where Ad_G is the adjoint representation of G. Denote by μ the equivalence class of the complexification of $\rho_{G/H}$. Denote by $B_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$. For $\beta \in \triangle'_+$, we set $m_\beta := \dim \mathfrak{p}_\beta$, $m_\beta^V := \dim(\mathfrak{p}_\beta \cap \mathfrak{q})$ and $m_\beta^H := \dim(\mathfrak{p}_\beta \cap \mathfrak{h})$. Also, let $\beta = \sum_{i=1}^r n_i^\beta \beta_i$, $(\beta \in \triangle'_+)$. Let Z_0 be a point of $\mathfrak{b}$. We consider the following two conditions for Z_0:

$$
(\mathrm{I}) \quad \begin{cases}
\beta(Z_0) \equiv 0, \dfrac{\pi}{6}\, \dfrac{\pi}{3}\, \dfrac{\pi}{2}, \dfrac{2\pi}{3}\, \dfrac{5\pi}{6} \, (\mathrm{mod}\, \pi) \ (\forall \beta \in \triangle'_+) \ \& \\
\displaystyle\sum_{\beta \in \triangle'^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{6} \, (\mathrm{mod}\, \pi)} 3 n_i^\beta m_\beta^V + \sum_{\beta \in \triangle'^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{3} \, (\mathrm{mod}\, \pi)} n_i^\beta m_\beta^V \\
\displaystyle\quad + \sum_{\beta \in \triangle'^H_+ \text{ s.t. } \beta(Z_0) \equiv \frac{2\pi}{3} \, (\mathrm{mod}\, \pi)} 3 n_i^\beta m_\beta^H + \sum_{\beta \in \triangle'^H_+ \text{ s.t. } \beta(Z_0) \equiv \frac{5\pi}{6} \, (\mathrm{mod}\, \pi)} n_i^\beta m_\beta^H \\
\displaystyle\quad = \sum_{\beta \in \triangle'^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{2\pi}{3} \, (\mathrm{mod}\, \pi)} n_i^\beta m_\beta^V + \sum_{\beta \in \triangle'^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{5\pi}{6} \, (\mathrm{mod}\, \pi)} 3 n_i^\beta m_\beta^V \\
\displaystyle\quad + \sum_{\beta \in \triangle'^H_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{6} \, (\mathrm{mod}\, \pi)} n_i^\beta m_\beta^H + \sum_{\beta \in \triangle'^H_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{3} \, (\mathrm{mod}\, \pi)} 3 n_i^\beta m_\beta^H \\
\hfill (i = 1, \ldots, r).
\end{cases}
$$

and

$$
(\mathrm{II}) \quad \begin{cases}
\beta(Z_0) \equiv 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4} \, (\mathrm{mod}\, \pi) \ (\forall \beta \in \triangle'_+) \ \& \\
\displaystyle\sum_{\beta \in \triangle'^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{4} \, (\mathrm{mod}\, \pi)} n_i^\beta m_\beta^V + \sum_{\beta \in \triangle'^H_+ \text{ s.t. } \beta(Z_0) \equiv \frac{3\pi}{4} \, (\mathrm{mod}\, \pi)} n_i^\beta m_\beta^H \\
\displaystyle\quad = \sum_{\beta \in \triangle'^V_+ \text{ s.t. } \beta(Z_0) \equiv \frac{3\pi}{4} \, (\mathrm{mod}\, \pi)} n_i^\beta m_\beta^V + \sum_{\beta \in \triangle'^H_+ \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{4} \, (\mathrm{mod}\, \pi)} n_i^\beta m_\beta^H \\
\hfill (i = 1, \ldots, r).
\end{cases}
$$

Denote by H_{Z_0} the isotropy group of the H-action at $\mathrm{Exp}\, Z_0$. For simplicity, we set $L := H_{Z_0}$ and denote the identity component of L by L_0. Set $M := H(\mathrm{Exp}\, Z_0)(= H/L)$ and $\widehat{M} := H/L_0$, and define a covering map $\psi : \widehat{M} \to M$ by $\psi(hL_0) = hL$ $(h \in H)$. Denote by ι the inclusion map of M into G/K and set $\widehat{\iota} := \iota \circ \psi$. In the sequel, we regard $\widehat{M}$ as a submanifold in G/K immersed by $\widehat{\iota}$. Also, denote by $\mathfrak{h}_{Z_0}$ (or $\mathfrak{l}$) the Lie algebra of L. We showed that M is minimal and that $\mathfrak{h}$ admits a natural reductive decomposition $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}_{\mathfrak{h}}$ (see Theorem A in [Koi2] or the proof of Theorem A of this paper). Furthermore, we ([Koi2]) showed that the induced metric on the submanifold M in G/K coincides with the H-invariant metric arising from the restriction $cB_{\mathfrak{g}}|_{\mathfrak{m}_{\mathfrak{h}}\times\mathfrak{m}_{\mathfrak{h}}}$ of some constant-multiple $cB_{\mathfrak{g}}$ of $B_{\mathfrak{g}}$ to $\mathfrak{m}_{\mathfrak{h}} \times \mathfrak{m}_{\mathfrak{h}}$ if one of the following conditions holds:

(I_1) (I) holds, $\triangle'^V_+ \cap \triangle'^H_+ = \emptyset$, $\beta(Z_0) \equiv 0,\ \pi/3,\ 2\pi/3 \pmod{\pi}$ for all $\beta \in \triangle'^V_+$ and $\beta(Z_0) \equiv \pi/6,\ \pi/2,\ 5\pi/6 \pmod{\pi}$ for all $\beta \in \triangle'^H_+$,

(I_2) (I) holds, $\triangle'^V_+ \cap \triangle'^H_+ = \emptyset$, $\beta(Z_0) \equiv 0,\ \pi/6,\ 5\pi/6 \pmod{\pi}$ for all $\beta \in \triangle'^V_+$ and $\beta(Z_0) \equiv \pi/3,\ \pi/2,\ 2\pi/3 \pmod{\pi}$ for all $\beta \in \triangle'^H_+$,

(II_1) (II) holds, $\triangle'^V_+ \cap \triangle'^H_+ = \emptyset$, $\beta(Z_0) \equiv 0,\ \pi/4,\ 3\pi/4 \pmod{\pi}$ for all $\beta \in \triangle'^V_+$ and $\beta(Z_0) \equiv \pi/4,\ \pi/2,\ 3\pi/4 \pmod{\pi}$ for all $\beta \in \triangle'^H_+$

(see Theorems $C \sim F$ in [Koi2]). Here we note that, when G is simple, there exists an inner automorphism ρ of G with $\rho(K) = H$ by Proposition 4.39 of [I2]. Denote by H^s the semi-simple part of H and $\mathfrak{h}^s$ the Lie algebra of H^s. Let k be the positive integer defined by

$$k := \begin{cases} 1 & (G/H : \text{Hermite type}) \\ 3 & (G/H : \text{quarternionic Kähler type}) \\ 0 & (G/H : \text{other}). \end{cases} \tag{1.1}$$

Easily we can show $H = S^k \cdot H^s$, where k is as above. Denote by $H^s_{Z_0}$ the isotropy group of H^s at $\operatorname{Exp} Z_0$. For simplicity, we set $L^s := H^s_{Z_0}$ and denote the identity component of L^s by L^s_0. Denote by $\mathfrak{l}^s$ the Lie algebra of L^s and $\mathfrak{z}$ the center of $\mathfrak{h}$ and $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{b})$ the centralizer of $\mathfrak{b}$ in $\mathfrak{h}$. In the case where G/H is of Hermite type or quarternionic Kähler type, we assume that $\operatorname{cohom} H = \operatorname{rank} G/K$ holds, where $\operatorname{cohom} H$ is the cohomogeneity of the H-action. From this assumption, $\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{p}$ and hence $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{b}) = \mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b})$. Also, we have $\mathfrak{z} \subset \mathfrak{z}_{\mathfrak{h}}(\mathfrak{b})$ (see Page 92 of [Tak]). Hence we obtain $\mathfrak{z} \subset \mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b})$. On the other hand, according to (3.1), we have $\mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) \subset \mathfrak{l}$. Therefore, we obtain $\mathfrak{z} \subset \mathfrak{l}$ and hence $L = S^k \cdot L^s$. From this relation, it follows that $M = H/L = H^s/L^s$ and that $\widehat{M} = H/L_0 = H^s/L^s_0$. Clearly we have $\mathfrak{h}^s = \mathfrak{l}^s + \mathfrak{m}_{\mathfrak{h}}$. Let $(\rho^S_{H^s})_{Z_0} : L^s \to GL(T^\perp_{\operatorname{Exp} Z_0} M)$ the slice representation of the H^s-action at $\operatorname{Exp} Z_0$, where $T^\perp_{\operatorname{Exp} Z_0} M$ is the normal space of M at $\operatorname{Exp} Z_0$. Set $\mathfrak{m} := (\exp Z_0)^{-1}_*(T_{\operatorname{Exp} Z_0} M)$ and $\mathfrak{m}^\perp :=$

$(\exp Z_0)_*^{-1}(T^\perp_{\mathrm{Exp}\,Z_0}M)$. Let $I(\exp Z_0) : G \to G$ be the inner automorphism by $\exp Z_0$. Easily we can show $I(\exp(-Z_0))(L^s) \subset K$ and hence

$$\mathrm{Ad}_G(\exp(-Z_0))(\mathfrak{l}^s) \subset \mathfrak{k}. \tag{1.2}$$

Also we can show

$$\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp) \subset \mathfrak{q}. \tag{1.3}$$

See (3.5) about the proof of (1.3). Set $\mathfrak{q}^s := \mathfrak{z} + \mathfrak{q}$. Also, let $\rho_{G/H} : H \to GL(\mathfrak{q})$ be the isotropy representation of G/H and $\rho_{G/H^s} : H^s \to GL(\mathfrak{q}^s)$ the isotropy representation of G/H^s. Define the representation $\sigma_{Z_0} : L_0^s \to GL(\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))$ by

$$\sigma_{Z_0}(l)(w) := (\rho_{G/H^s}(l))(w) \quad (l \in L_0^s,\ w \in \mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp)).$$

Under the identification of $T^\perp_{\mathrm{Exp}\,Z_0}M$ and $\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp)$, the restriction $(\rho^S_{H^s})_{Z_0}|_{L_0^s}$ of $(\rho^S_{H^s})_{Z_0}$ to L_0^s is identified with σ_{Z_0}. We regard $(\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}$ as a L_0^s-module associated with the complexification $\sigma^{\mathbf{c}}_{Z_0} : L_0^s \to GL((\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}})$ of σ_{Z_0}. Denote by μ the equivalence class of the complexification $\rho^{\mathbf{c}}_{G/H} : H \to GL(\mathfrak{q}^{\mathbf{c}})$ of $\rho_{G/H}$ and $\mu|_{H^s}$ the equivalence class of of the restriction $\rho^{\mathbf{c}}_{G/H}|_{H^s}$ of $\rho^{\mathbf{c}}_{G/H}$ to H^s.

In this paper, we prove the following result.

Theorem A *Let G/K be an irreducible simply connected symmetric space of compact type, $H \curvearrowright G/K$ a Hermann action and Z_0 an element of $\mathfrak{b}$ such that (H, Z_0) satisfies one of the above conditions* (I_1), (I_2) *or* (II_1)*. Furthermore, assume that* $\mathrm{cohom}\,H = \mathrm{rank}\,G/K$ *holds. Let M, $\widehat{M}$, H^s and L_0 be as above. Then the orbit M* (*hence $\widehat{M}$*) *is minimal* (*but not totally geodesic*) *and the index $i(\widehat{M})$ of $\widehat{M}$ is given by*

$$i(\widehat{M}) = \sum_{\lambda \in D_{G/H}} m_\lambda \cdot \dim \mathrm{Hom}_{L_0^s}(V_{\rho_\lambda}, (\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}).$$

Here $D_{G/H} := \{\lambda \in D(H^s) \mid a_\lambda > a_{\mu|_{H^s}}\}$, where a_λ (*resp. $a_{\mu|_{H^s}}$*) *is the eigenvalue of the Casimir operator of an irreducible complex representation belonging to λ* (*resp. $\mu|_{H^s}$*) *with respect to $B_{\mathfrak{g}}|_{\mathfrak{h}^s \times \mathfrak{h}^s}$, V_{ρ_λ} is the representation space of an irreducible representation ρ_λ belonging to λ, m_λ is the dimension of V_{ρ_λ} and $\mathrm{Hom}_{L_0^s}(V_{\rho_\lambda}, (\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}})$ is the L_0^s-module* *of all L_0^s-homomorphisms from V_{ρ_λ} to* $(\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}$.

Remark 1.1 (i) In general, we have $i(M) \leq i(\widehat{M})$. In particular, if L is connected, then we have $M = \widehat{M}$.

(ii) If G/H is of Hermite-type, then the isotrpy representation $\rho_{G/H}$ of G/H is an irreducible complex representation of H and, when its equivalence class is denoted by ν, we have $\mu = \nu \oplus \nu$ and $a_\mu = a_\nu$.

In the final section, we give examples of a Hermann action $H \curvearrowright G/K$ and $Z_0 \in \mathfrak{b}$ as in Theorem A and calculate the indices of $\widehat{M}$ for some of the examples by using Theorem A.

2. Basic notions and facts

In this section, we recall some basic notions and facts.

Jacobi operators

Let $f : (M, g) \hookrightarrow (\widetilde{M}, \widetilde{g})$ be a minimal isometric immersion of a compact Riemannian manifold (M, g) into another Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Denote by $T^\perp M$ the normal bundle of f and $\Gamma(T^\perp M)$ the space of all normal vector fields of f. Also, denote by ∇(resp. $\nabla^\perp$) the Levi-Civita connection of g (resp. the normal connection of f) and A the shape tensor of f. Let f_t $(-\varepsilon < t < \varepsilon)$ be a C^∞-family of immersions of M into $\widetilde{M}$ with $f_0 = f$, where ε is a positive number. Define a map $F : M \times (-\varepsilon, \varepsilon) \to \widetilde{M}$ by $F(x, t) := f_t(x)$ $((x, t) \in M \times (-\varepsilon, \varepsilon))$. Denote by $\mathrm{Vol}(M, f_t^*\widetilde{g})$ the volume of $(M, f_t^*\widetilde{g})$ and dv the volume element of g, where $f_t^*\widetilde{g}$ is the metric induced form $\widetilde{g}$ by f_t. Then we have the following second variational formula:

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Vol}(M, f_t^*\widetilde{g}) = \int_M \widetilde{g}\left(\mathcal{J}\left(F_*\left(\left.\frac{\partial}{\partial t}\right|_{t=0}\right)_\perp\right), F_*\left(\left.\frac{\partial}{\partial t}\right|_{t=0}\right)_\perp\right) dv$$

(see Theorem 3.2.2 in [S]). Here F_* is the differential of F, $(\cdot)_\perp$ is the normal component of $(\cdot)$, $\mathcal{J}$ is the Jacobi operator of f (or M), which is defined by $\mathcal{J} := -\triangle^\perp + \mathcal{R} - \mathcal{A}\,(: \Gamma(T^\perp M) \to \Gamma(T^\perp M))$ (where $\triangle^\perp$ is the rough Laplacian operator defined by ∇ and $\nabla^\perp$, $\mathcal{A}$ is defined by $g(\mathcal{A}(v), w) = \mathrm{Tr}(A_v \circ A_w)$ $(v, w \in \Gamma(T^\perp M))$ and $\mathcal{R}$ is defined by $g(\mathcal{R}(v), w) = -\mathrm{Tr}(R(\cdot, v)w)$ $(v, w \in \Gamma(T^\perp M))$). Set $E_\lambda^\perp := \{v \in \Gamma(T^\perp M) \mid \mathcal{J}(v) = \lambda v\}$ for each $\lambda \in \mathbb{R}$. The dimension of $\sum_{\lambda<0} E_\lambda^\perp$ (resp. $E_0^\perp$) is called the *index* (resp. *nullity*) of f (or M).

The eigenvalues of the Casimir operators

For a compact Lie group H, denote by $D(H)$ the set of all equivalence classes of (finite dimensional) irreducible complex representations of H. Fix an $\mathrm{Ad}(H)$-invariant inner product $\langle\ ,\ \rangle$ of the Lie algebra $\mathfrak{h}$ of H. Let ρ be an irreducible complex representation of H. The Casimir operator C_ρ of ρ with respect to $\langle\ ,\ \rangle$ is defined by $C_\rho := \sum_{i=1}^m \rho_{*e}(e_i)^2$, where $(e_1, \ldots, e_m)$ is an orthonormal base of $\mathfrak{h}$ with respect to $\langle\ ,\ \rangle$ and e is the identity element of H. Assume that H is semi-simple and connected. Fix a Cartan subalgebra $\widetilde{\mathfrak{a}}$ of the Lie algebra $\mathfrak{h}$ of H. Let $\triangle$ be the root system of $\mathfrak{h}$ with respect to $\widetilde{\mathfrak{a}}$, $\triangle_+$ the positive root system of $\triangle$ under some lexicographic ordering of the dual space $\widetilde{\mathfrak{a}}^*$ of $\widetilde{\mathfrak{a}}$ and $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a simple root system of $\triangle_+$. Define $\Lambda_i \in \widetilde{\mathfrak{a}}^*$ $(i = 1, \ldots, r)$ by $2\langle\alpha_j, \Lambda_i\rangle/\langle\alpha_j, \alpha_j\rangle = \delta_{ij}$ $(1 \le i,\ j \le r)$. It is known that an injection of $D(H)$ into $\mathbb{Z}_+\{\Lambda_1, \ldots, \Lambda_r\}$ $(:= \{\sum_{i=1}^r z_i\Lambda_i \mid z_i \in \mathbb{Z}_+\})$ is given by assigning the highest weight of ρ to each $\lambda = [\rho] \in D(H)$, where $[\rho]$ is the equivalence class of an irreducible complex representation ρ of H. Denote by $\widehat{D(H)}$ the image of this injection. Then the quotient group $\mathbb{Z}_+\{\Lambda_1, \ldots, \Lambda_r\}/\widehat{D(H)}$ is isomorphic to the fundamental group $\pi_1(H)$ of H. Denote by $(z_1, \ldots, z_r)$ the equivalence class of the irreducible complex representation of H corresponding to $\sum_{i=1}^r z_i\Lambda_i$. If H is simple, then we have $C_\rho = a_\rho\,\mathrm{id}_{\mathfrak{h}}$ for some $a_\rho \in \mathbb{R}$ (by Schur's lemma), where $\mathrm{id}_{\mathfrak{h}}$ is the identity transformation of $\mathfrak{h}$. According to the Freudenthal's formula, we have

$$a_\rho = -\left\langle \Lambda, \Lambda + \sum_{\alpha\in\triangle_+} \alpha \right\rangle, \tag{2.1}$$

where Λ is the highest weight of ρ.

Irreducible complex representations of $\boldsymbol{T^r}$, $\boldsymbol{Spin(2r)}$ and $\boldsymbol{Spin}$ $\boldsymbol{(2r+1)}$

For each $(m_1, \ldots, m_r) \in \mathbb{Z}^r$, an ireducible complex representation ρ of r-dimensional torus group $T^r (= SO(2)^r = U(1)^r)$ is defined by

$$\rho(z_1, \ldots, z_r)(w) := z_1^{m_1} \cdots z_r^{m_r} w \ \ ((z_1, \ldots, z_r) \in T^r = U(1)^r,\ w \in \mathbb{C}).$$

Denote by $(m_1 - \cdots - m_r)$ the equivalence class of this representation. Let $D(T^r)$ be the set of all the equivalence classes of irreducible complex representations of T^r. Then it is known that $D(T^r) = \{(m_1 - \cdots - m_r) \mid$

$(m_1, \ldots, m_r) \in \mathbb{Z}^r\}$ holds (see [KO] for example).

Let $H = Spin(2r)$ or $Spin(2r+1)$, and $\widetilde{\mathfrak{a}}^*$ and $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be as above. Also, let $\{\beta_1, \ldots, \beta_r\}$ be the base of $\widetilde{\mathfrak{a}}^*$ defined by $\alpha_i = \beta_i - \beta_{i+1}$ $(i = 1, \ldots, r-1)$ and

$$\begin{cases} \alpha_r = \beta_{r-1} + \beta_r & (H = Spin(2r)) \\ \alpha_r = \beta_r & (H = Spin(2r+1)). \end{cases}$$

For an irrecducible complex representation ρ of H, the highest weight Λ of ρ is expressed as $\Lambda = \sum_{i=1}^r m_i \beta_i$ for some $(m_1, \ldots, m_r) \in \mathbb{Z}^r + \{(0, \ldots, 0), (1/2, \ldots, 1/2)\}$. Then we denote the equivalence class of ρ by $(m_1 \cdots m_r)^\bullet$. It is known that

$$D(H) = \left\{ (m_1 \cdots m_r)^\bullet \mid (m_1, \ldots, m_r) \in \mathbb{Z}^r + \left\{ (0, \ldots, 0), \left(\frac{1}{2}, \ldots, \frac{1}{2} \right) \right\} \right\}$$

and that

$$D(H/\{\pm 1\}) = \{ (m_1 \cdots m_r)^\bullet \mid (m_1, \ldots, m_r) \in \mathbb{Z}^r \},$$

where $H/\{\pm 1\} = SO(2r)$ or $SO(2r+1)$ (see Chapter 9 of [KO] for example).

The canonical connection

Let H/L be a reductive homogeneous space and $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$ be a reductive decomposition (i.e., $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$), where $\mathfrak{h}$ (resp. $\mathfrak{l}$) is the Lie algebra of H (resp. L). Also, let $\pi : P \to H/L$ be a principal G-bundle, where G is a Lie group. Assume that H acts on P as $\pi(h \cdot u) = h \cdot \pi(u)$ for any $u \in P$ and any $h \in H$. Then there uniquely exists a connection ω of P such that, for any $X \in \mathfrak{m}$ and any $u \in P$, $t \mapsto (\exp tX)(u)$ is a horizontal curve with respect to ω, where exp is the exponential map of H. This connection ω is called the *canonical connection* of P associated with the reductive decomposition $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$.

The rough Laplacian operator with respect to the canonical connection

Let H be a Lie group and H/L be a reductive homogeneous space with a reductive decomposition $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H. The subspace $\mathfrak{m}$ is identified with $T_{eL}(H/L)$. Let B be an $\mathrm{Ad}(H)$-invariant inner product of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$ is an orthogonal decomposition with respect to B. Denote by $\langle\ ,\ \rangle$ the H-invariant metric on H/L induced from $B|_{\mathfrak{m}\times\mathfrak{m}}$ and ∇ the Levi-Civita connection of $\langle\ ,\ \rangle$. Let $\pi : H \to H/L$ be the natural projection, $\sigma : L \to GL(W)$ a unitary representation of L and $E_\sigma := H \times_{\sigma(L)} W$ the associated complex vector bundle of the L-bundle $\pi : H \to H/L$ with respect to σ. The Lie group H acts on H and H/L naturally. Also, each $h(\in H)$ gives a linear isomorphism of W onto the fibre $(E_\sigma)_{\pi(h)}$. Denote by $\Gamma(E_\sigma)$ the space of all sections of E_σ and set $C^\infty(H,W)_\sigma := \{f \in C^\infty(H,W) \mid f(hl) = \sigma(l^{-1})f(h)\,(\forall\, h \in H, \forall\, l \in L)\}$, where $C^\infty(H,W)$ is the space of all W-valued C^∞-functions on H. Define a map $\Psi : \Gamma(E_\sigma) \to C^\infty(H,W)_\sigma$ by $\Psi(\xi)(h) = h^{-1}\cdot\xi_{\pi(h)}$ ($\xi \in \Gamma(E_\sigma)$, $h \in H$). This map Ψ is a linear isomorphism preserving the H-action. Take an orthonormal base $(e_1,\dots,e_m)$ of $\mathfrak{h}$ with respect to B with $e_i \in \mathfrak{l}$ ($i = 1,\dots,n$) and $e_b \in \mathfrak{m}$ ($b = n+1,\dots,m$), where $n := \dim\mathfrak{l}$. Let $\mathcal{C}_H$ ($: C^\infty(H,W) \to C^\infty(H,W)$) be the Casimir differential operator of H with respect to B, that is, $\mathcal{C}_H(f) = \sum_{i=1}^m \widetilde{e}_i(\widetilde{e}_i f)$, where $\widetilde{e}_i$ is the left-invariant vector field induced from e_i. Also, let $\mathcal{C}_\sigma$ be the Casimir operator of σ with respect to $B|_{\mathfrak{l}\times\mathfrak{l}}$. For $f \in C^\infty(H,W)_\sigma$, we can show $\mathcal{C}_H(f) = \mathcal{C}_\sigma\circ f + \sum_{b=n+1}^m \widetilde{e}_b(\widetilde{e}_b f)$. Let ∇^ω be the connection of E_σ induced from the canonical connection ω of $\pi : H \to H/L$ with respect to the reductive decomposition $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$ and $\triangle^{E_\sigma}$ the rough Laplacian operator of E_σ with respect to ∇^ω and ∇. Set $\widetilde{\triangle^{E_\sigma}} := \Psi\circ\triangle^{E_\sigma}\circ\Psi^{-1}$. Then we have $\widetilde{\triangle^{E_\sigma}} f = \sum_{b=n+1}^m \widetilde{e}_b(\widetilde{e}_b f)$ ($f \in C^\infty(H,W)_\sigma$) by Proposition 2.3 of [O]. Furthermore, by Corollary 2.5 of [O], we have the following relation.

Lemma 2.1 ([O]) *For each $f \in C^\infty(H,W)_\sigma$, we have*

$$\widetilde{\triangle^{E_\sigma}} f = \mathcal{C}_H(f) - \mathcal{C}_\sigma\circ f.$$

3. Proof of Theorem A

In this section, we shall prove Theorem A. We use the notations in Introduction. Let (H, Z_0) be as in the statement of Theorem A. Denote by $\langle\ ,\ \rangle$ the G-invariant metric of G/K induced from $B_{\mathfrak{g}}|_{\mathfrak{p}\times\mathfrak{p}}$. We shall describe some subspaces stated in Introduction explicitly. Set

$$\triangle'^V_{Z_0} := \left\{\beta \in \triangle'^V_+ \mid \beta(Z_0) \equiv 0 \ (\mathrm{mod}\,\pi)\right\}$$

and

$$\triangle'^{H}_{Z_0} := \left\{\beta \in \triangle'^{H}_{+} \mid \beta(Z_0) \equiv \frac{\pi}{2} \ (\mathrm{mod}\, \pi)\right\}.$$

Clearly the Lie algebra $\mathfrak{l}$ is given by

$$\mathfrak{l} = \mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) + \sum_{\beta\in\triangle'^{V}_{Z_0}} (\mathfrak{k}_\beta \cap \mathfrak{h}) + \sum_{\beta\in\triangle'^{H}_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{h}). \tag{3.1}$$

Easily we can show that $\mathfrak{m}_{\mathfrak{h}}$ is given by

$$\mathfrak{m}_{\mathfrak{h}} = \mathfrak{z}_{\mathfrak{p}\cap\mathfrak{h}}(\mathfrak{b}) + \sum_{\beta\in\triangle'^{V}_{+}\setminus\triangle'^{V}_{Z_0}} (\mathfrak{k}_\beta \cap \mathfrak{h}) + \sum_{\beta\in\triangle'^{H}_{+}\setminus\triangle'^{H}_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{h}). \tag{3.2}$$

From these relations, it follows that the decompositions $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}_{\mathfrak{h}}$ and $\mathfrak{h}^s = \mathfrak{l}^s + \mathfrak{m}_{\mathfrak{h}}$ are reductive, respectively. Easily we can show that $\mathfrak{m}$ is given by

$$\mathfrak{m} = \mathfrak{z}_{\mathfrak{p}\cap\mathfrak{h}}(\mathfrak{b}) + \sum_{\beta\in\triangle'^{V}_{+}\setminus\triangle'^{V}_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{q}) + \sum_{\beta\in\triangle'^{H}_{+}\setminus\triangle'^{H}_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{h}) \tag{3.3}$$

and hence

$$\mathfrak{m}^\perp = \mathfrak{b} + \sum_{\beta\in\triangle'^{V}_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{q}) + \sum_{\beta\in\triangle'^{H}_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{h}). \tag{3.4}$$

Furthermore, we can show

$$\begin{aligned} &\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp) \\ &\quad = \mathfrak{b} + \sum_{\beta\in\triangle'^{V}_{Z_0}} \big(\cos(\mathrm{ad}(Z_0))(\mathfrak{p}_\beta \cap \mathfrak{q}) + \sin(\mathrm{ad}(Z_0))(\mathfrak{p}_\beta \cap \mathfrak{q})\big) \\ &\qquad + \sum_{\beta\in\triangle'^{H}_{Z_0}} \big(\cos(\mathrm{ad}(Z_0))(\mathfrak{p}_\beta \cap \mathfrak{h}) + \sin(\mathrm{ad}(Z_0))(\mathfrak{p}_\beta \cap \mathfrak{h})\big) \\ &\quad = \mathfrak{b} + \sum_{\beta\in\triangle'^{V}_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{q}) + \sum_{\beta\in\triangle'^{H}_{Z_0}} (\mathfrak{k}_\beta \cap \mathfrak{q}) \qquad (\subset \mathfrak{q}), \end{aligned} \tag{3.5}$$

where $\cos(\mathrm{ad}(Z_0))$ and $\sin(\mathrm{ad}(Z_0))$ are defined by

$$\cos(\mathrm{ad}(Z_0)) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \mathrm{ad}(Z_0)^{2k}$$

$$\sin(\mathrm{ad}(Z_0)) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \mathrm{ad}(Z_0)^{2k+1},$$

respectively. The relations (3.1) $\sim$ (3.3) will be used in the proof of Theorem A and (3.1) $\sim$ (3.5) will be used in the proof of Propositions 4.1$\sim$4.8.

Proof of Theorem A. From (3.1) and (3.2), we have $B_{\mathfrak{g}}(\mathfrak{l}, \mathfrak{m}_{\mathfrak{h}}) = 0$. Denote by g_I the induced metric on the submanifold $\widehat{M}$ in G/K. By imitating the discussion in the proof of Theorem A in [Koi2], we can show that $(\psi^* g_I)_{eL} = cB_{\mathfrak{g}}|_{\mathfrak{m}_{\mathfrak{h}} \times \mathfrak{m}_{\mathfrak{h}}}$ ($c = 3/4$ in case of (I_1), $c = 1/4$ in case of (I_2) and $c = 1/2$ in case of (II_1)), where we use (3.2) and (3.3). Let ω be the canonical connection of the principal L_0-bundle $\pi : H^s \to H^s/L_0^s (= \widehat{M})$ with respect to the reductive decomposition $\mathfrak{h}^s = \mathfrak{l}^s + \mathfrak{m}_{\mathfrak{h}}$ and $F^{\perp}(\widehat{M})$ the normal frame bundle of $\widehat{M}$. Note that $F^{\perp}(\widehat{M})$ is identified with the induced bundle $\psi^*(F^{\perp}(M))$ ($\subset \widehat{M} \times F^{\perp}(M)$) of $F^{\perp}(M)$ by ψ. Define a map $\eta : H^s \to F^{\perp}(\widehat{M})$ by $\eta(h) = (hL_0^s, h_* u_0)$ ($h \in H^s$), where u_0 is a fixed normal frame of M at $\mathrm{Exp}\, Z_0$. This map η is an embedding. By identifying H^s with $\eta(H^s)$, we regard $\pi : H^s \to H^s/L_0^s (= \widehat{M})$ as a subbundle of $F^{\perp}(\widehat{M})$. Denote by the same symbol ω the connection of $F^{\perp}(\widehat{M})$ induced from ω and ∇^{ω} the linear connection on $T^{\perp}\widehat{M}$ associated with ω. Denote by $\nabla^{\perp}$ the normal connection of the submanifold $\widehat{M}$. By imitating the discussion in the proof of Theorem A in [Koi2], we can show that $\nabla^{\omega} = \nabla^{\perp}$. Denote by $E_{\sigma_{Z_0}}$ the associated vector bundle $H^s \times_{\sigma_{Z_0}} \mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^{\perp})$ of the principal L_0-bundle $\pi : H^s \to H^s/L_0$ with respect to σ_{Z_0}, where σ_{Z_0} is as stated in Introduction. Since σ_{Z_0} is identified with $(\rho_H^S)_{Z_0}|_{L_0^s}$ as stated in Introduction, $E_{\sigma_{Z_0}}$ is identified with the normal bundle $T^{\perp}\widehat{M}$ of $\widehat{M}$ under the correspondence $h \cdot v \leftrightarrow (hL_0, h_*((\exp Z_0)_*(\mathrm{Ad}_G(\exp(-Z_0))(v))))$ ($h \in H^s$, $v \in \mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^{\perp})$). Also we note that $T^{\perp}\widehat{M}$ is identified with the induced bundle $\psi^*(T^{\perp}M)$ ($\subset \widehat{M} \times T^{\perp}M$) of $T^{\perp}M$ by ψ. Let $\Psi : \Gamma(E_{\sigma_{Z_0}}) \to C^{\infty}(H^s, \mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^{\perp}))_{\sigma_{Z_0}}$ be a diffeomorphism defined in the previous section. Denote by ∇ the Levi-Civita connection of $\psi^* g_I$.

Since $\psi^* g_I$ coincides with the H^s-invariant metric induced from $cB_{\mathfrak{g}}|_{\mathfrak{m}_{\mathfrak{h}}\times\mathfrak{m}_{\mathfrak{h}}}$ and $\nabla^\omega = \nabla^\perp$, it follows from Lemma 2.1 that the rough Laplacian operator $\triangle^\perp$ of $E_{\sigma_{Z_0}}$ with respect to $\nabla^\perp$ and ∇ satisfies

$$(\Psi\circ\triangle^\perp\circ\Psi^{-1})(f) = \mathcal{C}_{H^s}(f) - \mathcal{C}_{\sigma_{Z_0}}\circ f \qquad (f\in C^\infty(H^s, \mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))_{\sigma_{Z_0}}, \quad (3.6)$$

where $\mathcal{C}_{H^s}$ is the Casimir differential operator of H^s with respect to $cB_{\mathfrak{g}}|_{\mathfrak{h}^s\times\mathfrak{h}^s}$ and $\mathcal{C}_{\sigma_{Z_0}}$ is the Casimir operator of σ_{Z_0} with respect to $cB_{\mathfrak{g}}|_{\mathfrak{l}\times\mathfrak{l}}$. Let $\mathcal{R}$ and $\mathcal{A}$ be the operators defined for $\widehat{M}$ in similar to $\mathcal{R}$ and $\mathcal{A}$ stated in the previous section, respectively. Then, by using Lemma 4.1 of [I1], we can show

$$(\Psi\circ\mathcal{R}\circ\Psi^{-1})(f) = \sum_{i=1}^{n}[(e_i)_{\mathfrak{p}}, [(e_i)_{\mathfrak{p}}, f]]_{\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp)} \qquad (f\in C^\infty(H^s, \mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))_{\sigma_{Z_0}})) \quad (3.7)$$

and

$$(\Psi\circ\mathcal{A}\circ\Psi^{-1})(f) = -\sum_{i=1}^{n}[(e_i)_{\mathfrak{k}}, [(e_i)_{\mathfrak{k}}, f]]_{\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp)} \qquad (f\in C^\infty(H^s, \mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))_{\sigma_{Z_0}})) \quad (3.8)$$

where $(e_1,\dots,e_n)$ is an orthonormal base of $\mathfrak{m}_{\mathfrak{h}}$ with respect to $cB_{\mathfrak{g}}|_{\mathfrak{m}_{\mathfrak{h}}\times\mathfrak{m}_{\mathfrak{h}}}$, and $(\cdot)_{\mathfrak{k}}$, $(\cdot)_{\mathfrak{p}}$ and $(\cdot)_{\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp)}$ is the $\mathfrak{k}$-component, $\mathfrak{p}$-component and $\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp)$-component of $(\cdot)$, respectively. From (3.6), (3.7) and (3.8), the Jacobi operator $\mathcal{J}$ of $\widehat{M}$ is given by

$$(\Psi\circ\mathcal{J}\circ\Psi^{-1})(f) = -\mathcal{C}_{H^s}(f) + \mathcal{C}_{\rho_{G/H^s}}\circ f \qquad (f\in C^\infty(H^s, \mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))_{\sigma_{Z_0}}). \quad (3.9)$$

Easily we can show

$$\rho_{G/H^s}(h) = \mathrm{id}_{\mathfrak{z}}\oplus\rho_{G/H}(h)$$

for any $h \in H^s$, and hence

$$\mathcal{C}_{\rho_{G/H^s}} = 0_{\mathfrak{z}} \oplus \frac{a_{\mu|_{H^s}}}{c}\,\mathrm{id}_{\mathfrak{q}},$$

where $0_{\mathfrak{z}}$ is the zero map from $\mathfrak{z}$ to oneself and $a_{\mu|_{H^s}}$ is as in the statement of Theorem A. Hence we have

$$(\Psi \circ \mathcal{J} \circ \Psi^{-1})(f) = -\mathcal{C}_{H^s}(f) + \frac{a_{\mu|_{H^s}}}{c} f \qquad (f \in C^\infty(H^s, \mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))_{\sigma_{Z_0}}). \tag{3.10}$$

Let $\lambda (= [\rho_\lambda])$ be an element of $D(H^s)$. Define a map $\eta_{\rho_\lambda} : V_{\rho_\lambda} \otimes \mathrm{Hom}_{L_0^s}(V_{\rho_\lambda}, (\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}) \to C^\infty(H^s, (\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}})_{(\sigma_{Z_0})^{\mathbf{c}}}$ by

$$(\eta_{\rho_\lambda}(v \otimes \phi))(h) := \phi(\rho_\lambda(h^{-1})(v))$$
$$(v \in V_{\rho_\lambda},\ \phi \in \mathrm{Hom}_{L_0^s}(V_{\rho_\lambda}, (\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}),\ h \in H^s).$$

This map η_{ρ_λ} is injective. Denote by $E_{(\sigma_{Z_0})^{\mathbf{c}}}$ the associated complex vector bundle $H^s \times_{(\sigma_{Z_0})^{\mathbf{c}}} (\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}$ of $\pi : H^s \to H^s/L_0^s$ with respect to $(\sigma_{Z_0})^{\mathbf{c}}$, which is identified with the complexification $(T^\perp\widehat{M})^{\mathbf{c}}$ of $T^\perp\widehat{M}$. Define a diffeomorphism $(\Psi^s)^{\mathbf{c}} : \Gamma(E_{(\sigma_{Z_0})^{\mathbf{c}}}) \to C^\infty(H^s, (\mathrm{Ad}_G(\exp Z_0) \cdot (\mathfrak{m}^\perp))^{\mathbf{c}})_{(\sigma_{Z_0})^{\mathbf{c}}}$ by $\Psi^{\mathbf{c}}(\xi)(h) := h^{-1} \cdot \xi_{\pi(h)}$ ($\xi \in \Gamma(E_{(\sigma_{Z_0})^{\mathbf{c}}})$, $h \in H^s$). Set $\Gamma_\lambda((T^\perp\widehat{M})^{\mathbf{c}}) := (\Psi^{\mathbf{c}})^{-1}(\eta_{\rho_\lambda}(V_{\rho_\lambda} \otimes \mathrm{Hom}_{L_0^s}(V_{\rho_\lambda}, (\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}))$. Then, according to Peter-Weyl theorem for vector bundles (see Page P173 of [B]), $\sum_{\lambda \in D(H^s)} \Gamma_\lambda((T^\perp\widehat{M})^{\mathbf{c}})$ (direct sum) is uniformly dense in $\Gamma((T^\perp\widehat{M})^{\mathbf{c}})$ with respect to the uniformly topology. Also, it follows from (3.10) that

$$(\mathcal{J})^{\mathbf{c}}(f) = \frac{a_{\mu|_{H^s}} - a_\lambda}{c} f \quad (f \in \Gamma_\lambda((T^\perp\widehat{M})^{\mathbf{c}})). \tag{3.11}$$

From this relation, we have

$$i(\widehat{M}) = \sum_{\lambda \in D_{G/H}} m_\lambda \cdot \dim \mathrm{Hom}_{L_0^s}(V_{\rho_\lambda}, (\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}),$$

where $D_{G/H}$ is as in the statement of Theorem A. This completes the proof. $\square$

4. Examples

In this section, we give examples of a Hermann action $H \curvearrowright G/K$ and $Z_0 \in \mathfrak{b}$ as in Theorem A and calculate the index of the minimal orbit $M := H(\operatorname{Exp} Z_0)$ for some of the examples by using Theorem A. We use the notations in Introduction. Denote by $\triangle$ the root system of the symmetric space G/K with respect to a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ including $\mathfrak{b}$. First we give examples of (H, Z_0) satisfying the condition (I_1).

Example 1 We consider the isotropy action of $SU(3n+3)/SO(3n+3)$. Then we have $\triangle = \triangle'$, which is of $(\mathfrak{a}_{3n+2})$-type. Also, we have $\triangle'_+ = {\triangle'}^V_+$ and hence ${\triangle'}^H_+ = \emptyset$. Let $\Pi = \{\beta_1, \ldots, \beta_{3n+2}\}$ be a simple root system of $\triangle'_+$, where we order $\beta_1, \ldots, \beta_{3n+2}$ as the Dynkin diagram of $\triangle'_+$ is as in Figure 1. We have $\triangle'_+ = \{\beta_i + \cdots + \beta_j \mid 1 \leq i, j \leq 3n+2\}$. For any $\beta \in \triangle'_+$, we have $m_\beta = 1$. Let Z_0 be the point of $\mathfrak{b}$ defined by $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \pi/3$ and $\beta_i(Z_0) = 0$ $(i \in \{1, \ldots, 3n+2\} \setminus \{n+1, 2n+2\})$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

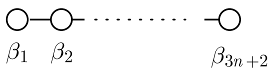


Figure 1.

Example 2 We consider the isotropy action of $SU(6n+6)/Sp(3n+3)$. Then we have $\triangle = \triangle'$, which is of $(\mathfrak{a}_{3n+2})$-type. Also, we have $\triangle'_+ = {\triangle'}^V_+$ and hence ${\triangle'}^H_+ = \emptyset$. Let $\Pi = \{\beta_1, \ldots, \beta_{3n+2}\}$ be a simple root system of $\triangle'_+$, where we order $\beta_1, \ldots, \beta_{3n+2}$ as above. We have $m_\beta = 4$ for any $\beta \in \triangle'_+$. Let Z_0 be the point of $\mathfrak{b}$ defined by $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \pi/3$ and $\beta_i(Z_0) = 0$ $(i \in \{1, \ldots, 3n+2\} \setminus \{n+1, 2n+2\})$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

Example 3 We consider the isotropy action of $SU(3)/S(U(1) \times U(2))$ (2-dimensional complex projective space). Then we have $\triangle = \triangle'$, which is of $(\mathfrak{b}c_1)$-type. Also, we have $\triangle'_+ = {\triangle'}^V_+$ and hence ${\triangle'}^H_+ = \emptyset$. Let $\Pi = \{\beta\}$ be a simple root system of $\triangle'_+$. We have $\triangle'_+ = \{\beta, 2\beta\}$ and $m_\beta = 2$ and $m_{2\beta} = 1$. Let Z_0 be the point of $\mathfrak{b}$ defined by $\beta(Z_0) = \pi/3$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

Example 4 We consider the isotropy action of $Sp(3n+2)/U(3n+2)$. Then we have $\triangle = \triangle'$, which is of $(\mathfrak{c}_{3n+2})$-type. Also, we have $\triangle'_+ = {\triangle'}^V_+$ and hence ${\triangle'}^H_+ = \emptyset$. Let $\Pi = \{\beta_1, \dots, \beta_{3n+2}\}$ be a simple root system of $\triangle'_+$, where we order $\beta_1, \dots, \beta_{3n+2}$ as the Dynkin diagram of $\triangle'_+$ is as in Figure 2. We have $m_\beta = 1$ for any $\beta \in \triangle'_+$. Let Z_0 be the point of $\mathfrak{b}$ defined by $\beta_{n+1}(Z_0) = \beta_{3n+2}(Z_0) = \pi/3$ and $\beta_i(Z_0) = 0$ ($i \in \{1, \dots, 3n+2\} \setminus \{n+1, 3n+2\}$). This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

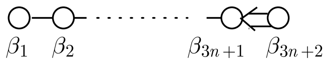


Figure 2.

Let G^*/K be the dual of G/K, that is, $G^* = \exp_{G^{\mathbb{C}}}(\mathfrak{k} + \sqrt{-1}\mathfrak{p})$ and set $H^* := \exp_{G^{\mathbb{C}}}(\mathfrak{h} \cap \mathfrak{k} + \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p}))$, where $\exp_{G^{\mathbb{C}}}$ is the exponential map of $G^{\mathbb{C}}$. Then the natural action $H^* \curvearrowright G^*/K$ also is called a *Hermann action* (or *Hermann type action*) and $H^* \curvearrowright G^*/K$ (resp. $H \curvearrowright G/K$) is called the *dual action* of $H \curvearrowright G/K$ (resp. $H^* \curvearrowright G^*/K$).

Example 5 We consider the dual action $\rho_1(SO(3)) \curvearrowright SU(3)/SO(3)$ of the Hermann action $SO_0(1,2) \curvearrowright SL(3,\mathbb{R})/SO(3)$, where ρ_1 is an inner automorphism of $SU(3)$. Then $\triangle = \triangle'$ is of ($\mathfrak{a}_2$)-type. Let $\Pi = \{\beta_1, \beta_2\}$ be a simple root system of $\triangle'_+$. Then we have ${\triangle'}^V_+ = \{\beta_1\}$, ${\triangle'}^H_+ = \{\beta_2, \beta_1+\beta_2\}$ and hence ${\triangle'}^V_+ \cap {\triangle'}^H_+ = \emptyset$. Also we have $m_{\beta_1} = m_{\beta_2} = m_{\beta_1+\beta_2} = 1$. Let Z_0 be the point of $\mathfrak{b}$ satisfying $(\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/6)$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

Example 6 We consider the dual action $\rho_2(Sp(3)) \curvearrowright SU(6)/Sp(3)$ of the Hermann action $Sp(1,2) \curvearrowright SU^*(6)/Sp(3)$, where ρ_2 is an inner automorphism of $SU(6)$. Then $\triangle = \triangle'$ is of ($\mathfrak{a}_2$)-type. Let $\Pi = \{\beta_1, \beta_2\}$ be a simple root system of $\triangle'_+$. Then we have ${\triangle'}^V_+ = \{\beta_1\}$, ${\triangle'}^H_+ = \{\beta_2, \beta_1 + \beta_2\}$ and hence ${\triangle'}^V_+ \cap {\triangle'}^H_+ = \emptyset$. Also we have $m_{\beta_1} = m_{\beta_2} = m_{\beta_1+\beta_2} = 4$. Let Z_0 be the point of $\mathfrak{b}$ satisfying $(\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/6)$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

Example 7 We consider the dual action $\rho_3(SU(2)) \curvearrowright Sp(2)/U(2)$ of the Hermann action $U(1,1) \curvearrowright Sp(2,\mathbb{R})/U(2)$, where ρ_3 is an inner automorphism of $Sp(2)$. Then $\triangle = \triangle'$ is of ($\mathfrak{c}_2$)-type. Let $\Pi = \{\beta_1, \beta_2\}$ be a simple root system of $\triangle'_+$, where we order β_1 and β_2 as the Dynkin diagram of $\triangle'_+$ is as in Figure 3. Then we have ${\triangle'}^V_+ = \{\beta_2, 2\beta_1+\beta_2\}$, ${\triangle'}^H_+ = \{\beta_1, \beta_1+\beta_2\}$ and hence ${\triangle'}^V_+ \cap {\triangle'}^H_+ = \emptyset$. Also we have $m_{\beta_1} = m_{\beta_2} = m_{\beta_1+\beta_2} = m_{2\beta_1+\beta_2} = 1$.

Let Z_0 be the point of $\mathfrak{b}$ satisfying $(\beta_1(Z_0), \beta_2(Z_0)) = (-\pi/6, \pi/3)$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

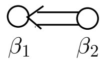


Figure 3.

Example 8 We consider the dual action $\rho_4(Sp(2)) \curvearrowright (Sp(2) \times Sp(2))/Sp(2)$ of the Hermann action $Sp(1,1) \curvearrowright Sp(2,\mathbb{C})/Sp(2)$, where ρ_4 is an automorphism of $Sp(2) \times Sp(2)$. Then $\triangle = \triangle'$ is of $(\mathfrak{c}_2)$-type. Let $\Pi = \{\beta_1, \beta_2\}$ be a simple root system of $\triangle'_+$, where we order β_1 and β_2 as the Dynkin diagram of $\triangle'_+$ is as in Figure 3. Then we have ${\triangle'}^V_+ = \{\beta_2, 2\beta_1 + \beta_2\}$, ${\triangle'}^H_+ = \{\beta_1, \beta_1 + \beta_2\}$ and hence ${\triangle'}^V_+ \cap {\triangle'}^H_+ = \emptyset$. Also we have $m_{\beta_1} = m_{\beta_2} = m_{\beta_1+\beta_2} = m_{2\beta_1+\beta_2} = 2$. Let Z_0 be the point of $\mathfrak{b}$ satisfying $(\beta_1(Z_0), \beta_2(Z_0)) = (-\pi/6, \pi/3)$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

Example 9 We consider the dual action $\rho_5(F_4) \curvearrowright E_6/F_4$ of the Hermann action $F_4^{-20} \curvearrowright E_6^{-26}/F_4$, where ρ_6 is an inner automorphism of E_6. Then $\triangle = \triangle'$ is of $(\mathfrak{a}_2)$-type. Let $\Pi = \{\beta_1, \beta_2\}$ be a simple root system of $\triangle'_+$. Then we have ${\triangle'}^V_+ = \{\beta_1\}$, ${\triangle'}^H_+ = \{\beta_2, \beta_1 + \beta_2\}$ and hence ${\triangle'}^V_+ \cap {\triangle'}^H_+ = \emptyset$. Also we have $m_{\beta_1} = m_{\beta_2} = m_{\beta_1+\beta_2} = 8$. Let Z_0 be the point of $\mathfrak{b}$ satisfying $(\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/6)$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

Example 10 We consider the dual action $\rho_6(SO(4)) \curvearrowright G_2/SO(4)$ of the Hermann action $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \curvearrowright G_2^2/SO(4)$, where ρ_6 is an inner automorphism of G_2. Then $\triangle = \triangle'$ is of $(\mathfrak{g}_2)$-type. Let $\Pi = \{\beta_1, \beta_2\}$ be a simple root system of $\triangle'_+$, where we order β_1 and β_2 as the Dynkin diagram of $\triangle'_+$ is as in Figure 4. Then we have ${\triangle'}^V_+ = \{\beta_1, 3\beta_1 + 2\beta_2\}$, ${\triangle'}^H_+ = \{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2\}$ and hence ${\triangle'}^V_+ \cap {\triangle'}^H_+ = \emptyset$. Also we have $m_{\beta_1} = m_{\beta_2} = m_{\beta_1+\beta_2} = m_{2\beta_1+\beta_2} = m_{3\beta_1+\beta_2} = m_{3\beta_1+2\beta_2} = 1$. Let Z_0 be the point of $\mathfrak{b}$ satisfying $(\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/2)$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

Example 11 We consider the dual action $\rho_7(G_2) \curvearrowright (G_2 \times G_2)/G_2$ of the Hermann action $G_2^2 \curvearrowright G_2^{\mathbf{c}}/G_2$, where ρ_7 is an automorphism of $G_2 \times G_2$. Then $\triangle = \triangle'$ is of $(\mathfrak{g}_2)$-type. Let $\Pi = \{\beta_1, \beta_2\}$ be a simple root system of $\triangle'_+$, where we order β_1 and β_2 as the Dynkin diagram of $\triangle'_+$ is as in Figure

4. Then we have ${\triangle'}_+^V = \{\beta_1, 3\beta_1+2\beta_2\}$, ${\triangle'}_+^H = \{\beta_2, \beta_1+\beta_2, 2\beta_1+\beta_2, 3\beta_1+\beta_2\}$ and hence ${\triangle'}_+^V \cap {\triangle'}_+^H = \emptyset$. Also we have $m_{\beta_1} = m_{\beta_2} = m_{\beta_1+\beta_2} = m_{2\beta_1+\beta_2} = m_{3\beta_1+\beta_2} = m_{3\beta_1+2\beta_2} = 2$. Let Z_0 be the point of $\mathfrak{b}$ satisfying $(\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/2)$. This point Z_0 satisfies the condition (I_1) (see Section 4 of [Koi2]).

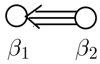


Figure 4.

First we prepare the following lemma.

Lemma 4.1 *Let $G/K, H, L, \theta, \tau$ and M be as in Introduction. If both the symmetric space $H/H\cap K$ and the principal orbit of the isotropy action of the symmetric space* $\mathrm{Fix}(\theta\circ\tau)_0/H\cap K$ *are simply connected, then so is also M.*

Proof. Easily we can show $H(eK) = H/H\cap K$ and $\exp^\perp(T^\perp_{eK}H(eK)) = \mathrm{Fix}(\theta\circ\tau)_0/H\cap K$, where $\exp^\perp$ is the normal exponential map of $H(eK)$. Let F be a principal orbit of the isotropy action of $\mathrm{Fix}(\theta\circ\tau)_0/H\cap K$ and M' the principal orbit of the H-action including F. Then we can show that the focal map of M' onto $H(eK)$ is a fibration having F as the standard fibre. Hence it follows from the assumption that M' is simply connected. Let pr be the natural projection of M' onto M. In the case where M is a singular orbit, pr is the focal map of M' onto M and it is a fibration with connected fibre, where we note that the fibre is the image of a principal orbit of the direct sum representation of some s-representations by the normal exponential map (of M) and hence it is connected. In the case where M is a principal orbit, pr is the end-point map (which is a diffeomorphism) of M' onto M. In both cases, pr is a fibration with connected fibre. Hence, since M' is simply connected, so is also M. $\square$

For the representations ρ_{λ_i} of H_i $(i=1,\dots,k)$, we define the representation $\rho_{\lambda_1}\text{-}\cdots\text{-}\rho_{\lambda_k}$ of $H_1\times\cdots\times H_k$ by $(\rho_{\lambda_1}\text{-}\cdots\text{-}\rho_{\lambda_k})(h_1,\dots,h_k)(v_1\otimes\cdots\otimes v_k) := \rho_{\lambda_1}(h_1)(v_1)\otimes\cdots\otimes\rho_{\lambda_k}(h_k)(v_k)$ $(h_i\in H_i,\ v_i\in V_{\rho_{\lambda_i}})$ (the representation space of $\rho_{\lambda_1}\text{-}\cdots\text{-}\rho_{\lambda_k}$ is $V_{\rho_{\lambda_1}}\otimes\cdots\otimes V_{\rho_{\lambda_k}}$). Denote by $(\lambda_1\text{-}\cdots\text{-}\lambda_k)$ the equivalence class of $\rho_{\lambda_1}\text{-}\cdots\text{-}\rho_{\lambda_k}$. By using Theorem A, we shall calculate the indices of some of the minimal orbits $M = H(\mathrm{Exp}\,Z_0)$ in Examples $1\sim 11$.

First we consider the case of $n=2$ in Example 1 (i.e., the case where $G/K = SU(9)/SO(9)$, $H = SO(9)$ and $M = SO(9)(\mathrm{Exp}\, Z_0)$ ($\beta_3(Z_0) = \beta_6(Z_0) = \pi/3$, $\beta_i(Z_0) = 0\,(i \neq 3,6)$)). Since $SO(9)$ is simple, we have $H^s = H$. The equivalence class μ of the complexification of the isotropy representation of G/H is equal to $(2\,0\,0\,0)$. Hence, according to Table 1 in [MP], all of the equivalence classes λ's of irreducible complex representations of $Spin(9)$ with $a_\lambda > a_{\mu|_{H^s}}$ consist of $(0\,0\,0\,0)$, $(1\,0\,0\,0)$, $(0\,0\,0\,1)$ and $(0\,1\,0\,0)$. These equivalence classes $(0\,0\,0\,0)$, $(1\,0\,0\,0)$, $(0\,0\,0\,1)$ and $(0\,1\,0\,0)$ are equal to $(0\,0\,0\,0)^\bullet$, $(1\,0\,0\,0)^\bullet$, $(\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2})^\bullet$ and $(1\,1\,0\,0)^\bullet$, respectively. Hence, $(0\,0\,0\,1)$ is not the equivalence classes of the irreducible complex representations of $SO(9)$. From this fact and $H = H^s$, we have

$$D_{G/H} = \{(0\,0\,0\,0),\,(1\,0\,0\,0),\,(0\,1\,0\,0)\}.$$

On the other hand, since $\triangle' = \triangle$ is $(\mathfrak{a}_8)$-type, $\beta_3(Z_0) = \beta_6(Z_0) = \pi/3$ and since $\beta_i(Z_0) = 0\,(i \neq 3,6)$, we have ${\triangle'}^V_{Z_0} = \{\beta_1, \beta_2, \beta_1+\beta_2, \beta_4, \beta_5, \beta_4+\beta_5, \beta_7, \beta_8, \beta_7+\beta_8\}$ and ${\triangle'}^H_{Z_0} = \emptyset$. Also we have $\mathfrak{z} = \mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) = \{0\}$ and $\mathfrak{l}^s = \mathfrak{l} = \sum_{i\in\{1,\,4,\,7\}}(\mathfrak{h}_{\beta_i} + \mathfrak{h}_{\beta_{i+1}} + \mathfrak{h}_{\beta_i+\beta_{i+1}})$. Also we have $\dim \mathfrak{h}_\beta = 1$ for all $\beta \in {\triangle'}^V_+$. Hence we have $\mathfrak{l}^s = 3\mathfrak{so}(3)$. Hence we have $L_0^s = SO(3)^3$. Hence, by using Table 2 (the branching rules) in [MP], we have the following table:

Table 2.

λ	$\lambda\vert_{L_0^s}$	m_λ
$(0\,0\,0\,0)$	(0-0-0)	1
$(1\,0\,0\,0)$	(2-0-0) $\oplus$ (0-2-0) $\oplus$ (0-0-2)	9
$(0\,1\,0\,0)$	(2-0-0) $\oplus$ (0-2-0) $\oplus$ (0-0-2) (2-2-0) $\oplus$ (2-0-2) $\oplus$ (0-2-2)	36
$\mu = (2\,0\,0\,0)$	2(0-0-0) $\oplus$ (4-0-0) $\oplus$ (0-4-0) $\oplus$ (0-0-4) (2-2-0) $\oplus$ (2-0-2) $\oplus$ (0-2-2)	44

Also we have $\dim \mathfrak{m}^\perp = 17$. Hence we have

$$[(\sigma_{Z_0})^{\mathbf{c}}] = 2(0\text{-}0\text{-}0) \oplus (4\text{-}0\text{-}0) \oplus (0\text{-}4\text{-}0) \oplus (0\text{-}0\text{-}4).$$

Thus the isomorphicity of the L_0-module $(\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}$ associated with the representation $(\sigma_{Z_0})^{\mathbf{c}}$ is analyzed completely. Therefore, according to Theorem A, it follows from Table 2 and this fact that the index of $\widehat{M}$ is equal to 2. Also, since Z_0 belongs to an (open) 1-simplex (which we denote by σ) of the simplicial complex $\overline{\overline{\widetilde{C}}}$, M is not stable. In fact, when M moves along σ as $SO(9)$-orbits, its volume decreases. Thus we obtain the following result.

Proposition 4.2 *Let $\Pi = \{\beta_1, \ldots, \beta_8\}$ be the simple root system of the positive root system $\triangle_+$ of $SU(9)/SO(9)$ ($\underset{\beta_1}{\circ}\!-\!\underset{\beta_2}{\circ}\!-\cdots-\!\underset{\beta_8}{\circ}$) and Z_0 the element of $\mathfrak{b}$ with $\beta_3(Z_0) = \beta_6(Z_0) = \pi/3$ and $\beta_i(Z_0) = 0$ $(i \neq 3, 6)$. Then the orbit $M := SO(9)(\mathrm{Exp}(Z_0))$ of the isotropy action of $SU(9)/SO(9)$ is minimal (but not totally geodesic) and we have $1 \leq i(M) \leq i(\widehat{M}) = 2$, where $\widehat{M}$ is the above covering of M.*

Next we consider the case of $n = 1$ in Example 2 (i.e., the case where $G/K = SU(12)/Sp(6)$, $H = Sp(6)$, and $M = Sp(6)(\mathrm{Exp}\, Z_0)$ $(\beta_2(Z_0) = \beta_4(Z_0) = \pi/3,\ \beta_i(Z_0) = 0\ (i \neq 2, 4))$. Since $Sp(6)$ is simple, we have $H^s = H$. The equivalence class μ of the complexification of the isotropy representation of G/H is $(0\,1\,0\,0\,0\,0)$. Hence, according to Table 1 in [MP], we have $D_{G/H} = \{(0\,0\,0\,0\,0\,0), (1\,0\,0\,0\,0\,0)\}$. On the other hand, since $\triangle' = \triangle$ is $(\mathfrak{a}_5)$-type, $\beta_2(Z_0) = \beta_4(Z_0) = \pi/3$ and since $\beta_i(Z_0) = 0\,(i \neq 2, 4)$, ${\triangle'}^V_{Z_0} = \{\beta_1, \beta_3, \beta_5\}$ and ${\triangle'}^H_{Z_0} = \emptyset$. Hence we have $\mathfrak{l} = \mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) + \mathfrak{h}_{\beta_1} + \mathfrak{h}_{\beta_3} + \mathfrak{h}_{\beta_5}$ and $\dim \mathfrak{h}_{\beta_i} = 4$ $(i = 1, 3, 5)$. Also we have $\mathfrak{z} = \{0\}$ and $\mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) = 6\mathfrak{s}p(1)$ From these facts, we have $\mathfrak{l}^s = 3\mathfrak{sp}(2)$. Therefore, we have $L_0^s = L_0 = Sp(2)^3$. Hence, by using Table 2 (the branching rules) in [MP], we have the following table:

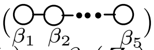
Table 3.

λ	$\lambda\vert_{L_0^s}$	m_λ
$(0\,0\,0\,0\,0\,0)$	(00-00-00)	1
$(1\,0\,0\,0\,0\,0)$	(10-00-00) $\oplus$ (00-10-00) $\oplus$ (00-00-10)	12
$\mu = (0\,1\,0\,0\,0\,0)$	2(00-00-00) $\oplus$ (01-00-00) $\oplus$ (00-01-00) $\oplus$ (00-00-01) $\oplus$(10-10-00) $\oplus$ (10-00-10) $\oplus$ (00-10-10)	65

Also, we have $\dim \mathfrak{m}^\perp = 17$. Hence we have

$$[(\sigma_{Z_0})^{\mathbf{c}}] = 2(00\text{-}00\text{-}00) \oplus (01\text{-}00\text{-}00) \oplus (00\text{-}01\text{-}00) \oplus (00\text{-}00\text{-}01).$$

Thus the isomorphicity of the L_0-module $(\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}$ associated with the representation $(\sigma_{Z_0})^{\mathbf{c}}$ is analyzed completely. Therefore, according to Theorem A, it follows from Table 3 and this fact that the index of $\widehat{M}$ is equal to 2. On the other hand, principal orbits of this isotropy action are diffeomorphic to $Sp(6)/Sp(1)^6$, which is simply connected. Also we have $H/H \cap K$ is the one-point set because $H = K$. Hence, it follows from Lemma 4.1 that M is simply connected, that is, $M = \widehat{M}$. Therefore we obtain the following result.

Proposition 4.3 *Let $\Pi = \{\beta_1, \dots, \beta_5\}$ be the simple root system of the positive root system $\triangle_+$ of $SU(12)/Sp(6)$ (*  *) and Z_0 the element of $\mathfrak{b}$ with $\beta_2(Z_0) = \beta_4(Z_0) = \pi/3$ and $\beta_1(Z_0) = \beta_3(Z_0) = \beta_5(Z_0) = 0$. Then the orbit $M := Sp(6)(\mathrm{Exp}(Z_0))$ of the isotropy action of $SU(12)/Sp(6)$ is minimal (but not totally geodesic) and we have $i(M) = 2$.*

Next we consider the case of Example 3 (i.e., $G/K = SU(3)/S(U(1) \times U(2))$, $H = S(U(1) \times U(2))$, and $M = S(U(1) \times U(2))(\mathrm{Exp}\, Z_0)$ $(\beta(Z_0) = \pi/3)$. Clearly we have $H^s = SU(2)$. Since M is a geodesic sphere in G/K, it is simply connected and of dimension three. Hence we have $L = L_0 = U(1)$ and $L_0^s = \{e\}$, where e is the identity element of G. Since G/H is Hermite-type, the isotropy representation of G/H is regarded as an irreducible complex representation of $H \cong U(2)$ and it is equal to $(1\,0)^\bullet$. The equivalence class μ of its complexification is equal to $(1\,0)^\bullet \oplus (1\,0)^\bullet$. Hence we have $\mu|_{H^s} = (1\,0)^\bullet|_{H^s} \oplus (1\,0)^\bullet|_{H^s} = (1) \oplus (1)$ and hence $a_{\mu|_{H^s}} = a_{(1)}$. Hence, according to (2.3) and (2.18) in [MP] and and the Freudenthal's formula, we have $D_{G/H} = \{(0)\}$. On the other hand, since $\triangle' = \triangle$ is $(\mathfrak{bc}_1)$-type and since $\beta(Z_0) = \pi/3$, we have ${\triangle'}^V_{Z_0} = {\triangle'}^H_{Z_0} = \emptyset$. Also, we have $\dim(\mathfrak{m}^\perp)^{\mathbf{c}} = 1$. According to Theorem A, it follows from these facts and $L^s = L_0^s = \{e\}$ that the index of M is equal to 1. Thus we obtain the following result.

Proposition 4.4 *Let $\triangle_+ = \{\beta, 2\beta\}$ be the positive root system of $SU(3)/S(U(1)\times U(2))$ and Z_0 the element of $\mathfrak{b}$ with $\beta(Z_0) = \frac{\pi}{3}$. Then the orbit $M := S(U(1)\times U(2))(\mathrm{Exp}(Z_0))$ of the isotropy action of $SU(3)/S(U(1)\times U(2))$ is minimal (but not totally geodesic) and we have $i(M) = 1$.*

Remark 4.1 This result has already been proved in [G] in different method.

Next we consider the case of Example 6 (i.e., $G/K = SU(6)/Sp(3)$, $H = \rho_2(Sp(3))$ and $M = \rho_2(Sp(3))(\mathrm{Exp}\, Z_0)$ $((\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/6))$. Since $Sp(3)$ is simple, we have $H^s = H = Sp(3)$. Since the equivalence class $\mu|_{H^s}$ of the complexification of the restriction of the isotropy representation of G/H to H^s is $(0\,1\,0)$. Hence, according to Table 1 in [MP], we have

$$D_{G/H} = \{(0\,0\,0),\ (1\,0\,0)\}.$$

On the other hand, since $\triangle' = \triangle$ is $(\mathfrak{a}_2)$-type and since $(\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/6)$, we have ${\triangle'}^V_+ = \{\beta_1\}$, ${\triangle'}^H_+ = \{\beta_2, \beta_1 + \beta_2\}$ and ${\triangle'}^V_{Z_0} = {\triangle'}^H_{Z_0} = \emptyset$. Also we have $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{b}) = \mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) = \mathfrak{sp}(1)^3$. From these facts, we have $\mathfrak{l} = \mathfrak{sp}(1)^3$ and hence $L_0 = L_0^s = Sp(1)^3$. Hence, by using Table 2 (the branching rules) in [MP], we have the following table:

Table 6.

λ	$\lambda\vert_{L_0^s}$	m_λ
$(0\,0\,0)$	(0-0-0)	1
$(1\,0\,0)$	(1-0-0) $\oplus$ (0-1-0) $\oplus$ (0-0-1)	6
$\mu\vert_{H^s} = (0\,1\,0)$	2(0-0-0) $\oplus$ (1-1-0) $\oplus$ (1-0-1) $\oplus$ (0-1-1)	14

Also, we have $\dim \mathfrak{m}^\perp = 2$. Hence we have $[(\sigma_{Z_0})^{\mathbf{c}}] = 2(0\text{-}0))$. Thus the isomorphicity of the L_0^s-module $(\mathrm{Ad}_G(\exp Z_0)(\mathfrak{m}^\perp))^{\mathbf{c}}$ associated with the representation $(\sigma_{Z_0})^{\mathbf{c}}$ is analyzed completely. Therefore, according to Theorem A, it follows from Table 6 and this fact that the index of $\widehat{M}$ is equal to 2. On the other hand, we have $H/H \cap K = Sp(3)/Sp(1) \times Sp(2)$ (whcih is simply connected) and $\mathrm{Fix}(\theta \circ \tau)_0/H \cap K = (SU(4)/Sp(2)) \times U(1)$. The principal orbit of the isotropy action of $(SU(4)/Sp(2)) \times U(1)$ is diffeomorphic to $S^3 \times S^3$, which is simply connected. Hence, it follows from Lemma 4.1 that M is simply connected, that is, $M = \widehat{M}$. Therefore we obtain the following result.

Proposition 4.5 *Let $\Pi = \{\beta_1, \beta_2\}$ be the simple root system of the positive root system $\triangle_+$ of $SU(6)/Sp(3)$ ($\underset{\beta_1}{\circ}\!\!-\!\!\!-\!\!\underset{\beta_2}{\circ}$) and Z_0 the element of $\mathfrak{b}$ with*

$(\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/6)$. *Then the orbit* $M := \rho_2(Sp(3))(\mathrm{Exp}(Z_0))$ *of the dual action* $\rho_2(Sp(3)) \curvearrowright SU(6)/Sp(3)$ *of* $Sp(1,2) \curvearrowright SU^*(6)/Sp(3)$ *is minimal* (*but not totally geodesic*) *and we have* $i(M) = 2$.

Next we consider the case of Example 7 (i.e., $G/K = Sp(2)/U(2)$, $H = \rho_3(U(2))$ and $M = \rho_3(U(2))(\mathrm{Exp}\, Z_0)$ $((\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/6))$. Clearly we have $H^s = SU(2)$. Since the equivalence class $\mu|_{H^s}$ of the complexification of the restriction of the isotropy representation of G/H to H^s is $(2) \oplus (2)$. Hence we have $D_{G/H} = \{(0), (1)\}$, On the other hand, since $\triangle' = \triangle$ is $(\mathfrak{c}_2)$-type and since $(\beta_1(Z_0), \beta_2(Z_0)) = (-\pi/6, \pi/3)$, we have ${\triangle'}^V_{Z_0} = \{2\beta_1 + \beta_2\}$ and ${\triangle'}^H_{Z_0} = \emptyset$. Also we have $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{b}) = \mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) = \{0\}$. From these facts, we have $\mathfrak{l}^s = \mathfrak{so}(2)$ and hence $L_0^s = SO(2)$. Denote by $\widetilde{\lambda}$ the canonical extension of $\lambda \in D(SU(2))$ to $U(2)$ and T^2 a maximal torus of $U(2)$. By noticing these facts and using Weyl's character formula (see Page 409 of [KO] for example), we have the following table:

Table 7.

λ	$\widetilde{\lambda}$	$\widetilde{\lambda}\|_{T^2}$	$\lambda\|_{L_0^s}$	m_λ
(0)	$(0\,0)^\bullet$	$(0\text{-}0)$	(0)	1
(1)	$(\frac{1}{2}\ (-\frac{1}{2}))^\bullet$	$(\frac{1}{2}\text{-}(-\frac{1}{2})) \oplus ((-\frac{1}{2})\text{-}\frac{1}{2})$	$(\frac{1}{2}) \oplus (-\frac{1}{2})$	2
(2)	$(1\,(-1))^\bullet$	$(1\text{-}(-1)) \oplus (0\text{-}0) \oplus ((-1)\text{-}1)$	$(1) \oplus (0) \oplus (-1)$	3

Easily we can show that $\dim \mathfrak{m}^\perp = 3$ and furthermore $[(\sigma_{Z_0})^{\mathbf{c}}] = (1) \oplus (0) \oplus (-1)$. Therefore, according to Theorem A, it follows from Table 7 and this fact that the index of $\widehat{M}$ is equal to 1. Also, since Z_0 belongs to an (open) 1-simplex (whcih we denote by σ) of the simplicial complex $\widetilde{\widetilde{C}}$, M is not stable. In fact, when M moves along σ as $\rho_3(U(2))$-orbits, its volume decreases. Thus we obtain the following result.

Proposition 4.6 *Let* $\Pi = \{\beta_1, \beta_2\}$ *be the simple root system of the positive root system* $\triangle_+$ *of* $Sp(2)/U(2)$ ($\underset{\beta_1}{\circ}\!\Leftarrow\!\underset{\beta_2}{\circ}$) *and* Z_0 *the element of* $\mathfrak{b}$ *with* $(\beta_1(Z_0), \beta_2(Z_0)) = (-\pi/6, \pi/3)$. *Then the orbit* $M := \rho_3(U(2))(\mathrm{Exp}(Z_0))$ *of the dual action* $\rho_3(U(2)) \curvearrowright Sp(2)/U(2)$ *of* $U(1,1) \curvearrowright Sp(2,\mathbb{R})/U(2)$ *is minimal (but not totally geodesic) and we have* $i(M) = 1$.

Next we consider the case of Example 8 (i.e., $G/K = (Sp(2) \times Sp(2))/Sp(2)$, $H = \rho_4(Sp(2))$ and $M = \rho_4(Sp(2))(\mathrm{Exp}\, Z_0)$ $((\beta_1(Z_0), \beta_2(Z_0)) =$

$(-\pi/6, \pi/3))$. Clearly we have $H^s = H = Sp(2)$. Since the equivalence class $\mu|_{H^s}$ of the complexification of the restriction of the isotropy representation of G/H to H^s is $(2\,0)$. Hence we have $D_{G/H} = \{(0\,0), (1\,0), (0\,1)\}$. On the other hand, since $\triangle' = \triangle$ is $(\mathfrak{b}_2)$-type and since $(\beta_1(Z_0), \beta_2(Z_0)) = (-\pi/6, \pi/3)$, we have ${\triangle'}^V_{Z_0} = \{\beta_2\}$ and ${\triangle'}^H_{Z_0} = \emptyset$. Also we have $\dim \mathfrak{z}_{\mathfrak{k}}(\mathfrak{b}) = \dim \mathfrak{z}_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) = 2$. From these facts, we have $\mathfrak{l}^s = \mathfrak{u}(2) = \mathfrak{so}(2) + \mathfrak{su}(2)$ and hence $L_0^s = U(2)$. Hence, by using Table 2 (the branching rules) in [MP], we have the following table:

Table 8.

λ	$\lambda\vert_{L_0^s}$	m_λ
$(0\,0)$	(0-0)	1
$(1\,0)$	(0-3)	4
$(0\,1)$	(0-4)	5
$\mu = \mu\vert_{H^s} = (2\,0)$	(0-6) $\oplus$ (0-2)	10

Also, we have $\dim \mathfrak{m}^\perp = 4$. Hence we have $[(\sigma_{Z_0})^{\mathbf{c}}] = $ (0-0) $\oplus$ (0-2). Therefore, according to Theorem A, it follows from Table 9 and this fact that the index of $\widehat{M}$ is equal to 1. On the other hand, we have $H/H\cap K = Sp(2)/Sp(1)\times Sp(1)$ (whcih is simply connected) and $\mathrm{Fix}(\theta\circ\tau)_0/H\cap K = S^3\times S^3$. The principal orbit of the isotropy action of $S^3\times S^3$ is diffeomorphic to $S^2\times S^2$, which is simply connected. Hence, it follows from Lemma 4.1 that M is simply connected, that is, $M = \widehat{M}$. Therefore we obtain the following result.

Proposition 4.7 *Let* $\Pi = \{\beta_1, \beta_2\}$ *be the simple root system of the positive root system* $\triangle_+$ *of* $(Sp(2)\times Sp(2))/Sp(2)$ () *and* Z_0 *the element of* $\mathfrak{b}$ *with* $(\beta_1(Z_0), \beta_2(Z_0)) = (-\pi/6, \pi/3)$. *Then the orbit* $M := \rho_4(Sp(2))\cdot(\mathrm{Exp}(Z_0))$ *of the dual action* $\rho_4(Sp(2)) \curvearrowright Sp(2)/U(2)$ *of* $Sp(1,1) \curvearrowright Sp(2,\mathbb{C})/Sp(2)$ *is minimal* (*but not totally geodesic*) *and we have* $i(M) = 1$.

Next we consider the case of Example 9 (i.e., $G/K = E_6/F_4$, $H = \rho_5(F_4)$ and $M = \rho_5(F_4)(\mathrm{Exp}\,Z_0)$ $((\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/6))$. Clearly we have $H^s = H = F_4$. Since the equivalence class μ of the complexification of the isotropy representation of G/H is $(0\,0\,0\,1)$, we have $D_{G/H} = \{(0\,0\,0\,0)\}$. On the other hand, since $\triangle' = \triangle$ is $(\mathfrak{a}_2)$-type and since $(\beta_1(Z_0), \beta_2(Z_0)) =$

$(\pi/3, -\pi/6)$, we have $\triangle'^V_{Z_0} = \triangle'^H_{Z_0} = \emptyset$. Also we have $z_{\mathfrak{k}}(\mathfrak{b}) = z_{\mathfrak{k}\cap\mathfrak{h}}(\mathfrak{b}) = \mathfrak{so}(8)$. From these facts, we have $\mathfrak{l}^s = \mathfrak{so}(8)$ and hence $L_0^s = SO(8)$. Hence, by using Table 2 (the branching rules) in [MP], we have the following table:

Table 9.

λ	$\lambda\vert_{L_0^s}$	m_λ
$(0\,0\,0\,0)$	$(0\,0\,0\,0)$	1
$(0\,0\,0\,1)$	$2(0\,0\,0\,0) \oplus (1\,0\,0\,0) \oplus (0\,0\,1\,0) \oplus (0\,0\,0\,1)$	26

Also, we have $\dim \mathfrak{m}^\perp = 2$. Hence we have $[(\sigma_{Z_0})^{\mathbf{c}}] = 2(0\,0\,0\,0)$. Therefore, according to Theorem A, it follows from Table 9 and this fact that the index of $\widehat{M}$ is equal to 2. On the other hand, we have $H/H\cap K = F_4/\mathrm{Spin}(9)$ (whcih is simply connected) and $\mathrm{Fix}(\theta\circ\tau)_0/H\cap K = S^9\times S^1$. The principal orbit of the isotropy action of $S^9\times S^1$ is diffeomorphic to S^8, which is simply connected. Hence, it follows from Lemma 4.1 that M is simply connected, that is, $M = \widehat{M}$. Therefore we obtain the following result.

Proposition 4.8 *Let $\Pi = \{\beta_1, \beta_2\}$ be the simple root system of the positive root system $\triangle_+$ of E_6/F_4 ($\underset{\beta_1}{\circ}\!\!-\!\!\underset{\beta_2}{\circ}$) and Z_0 the element of $\mathfrak{b}$ with $(\beta_1(Z_0), \beta_2(Z_0)) = (\pi/3, -\pi/6)$. Then the orbit $M := \rho_5(F_4)(\mathrm{Exp}(Z_0))$ of the dual action $\rho_5(F_4) \curvearrowright E_6/F_4$ of $F_4^{-20} \curvearrowright E_6^{-26}/F_4$ is minimal (but not totally geodesic) and we have $i(M) = 2$.*

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