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# On coretractable modules

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**Abstract.** Let $R$ be any ring. We prove that every right $R$-module is coretractable if and only if $R$ is right perfect and every right $R$-module is small coretractable if and only if all torsion theories on $R$ are cohereditary. We also study mono-coretractable modules. We show that coretractable modules are a proper generalization of mono-coretractable modules.

*Key words*: Coretractable module, Kasch module.

## 1. Introduction

Let $M$ be an $R$-module. $M$ is called *coretractable* if $Hom(M/N, M)$ is nonzero for all proper submodules $N$ of $M$ (see [2]). Amini, Ershad and Sharif study these modules and proved in [2, Theorem 2.14] that $R$ is a right Kasch ring if and only if $R_R$ is a coretractable module. Recall that a module $M$ is a *Kasch* module if every simple module in $\sigma[M]$ can be embedded in $M$ (see [3]). Therefore $R_R$ is a Kasch module if and only if $R_R$ is a coretractable module by [2, Theorem 2.14]. In Section 2, firstly we generalize this result (see Theorem 2.1). Mainly the purpose of Section 2 is to investigate rings whose all right modules are coretractable (see Theorem 2.7). We also prove that being coretractable is a Morita invariant property.

Let $R$ be any ring and let $M$ be any module. We will call $M$ *mono-coretractable* if for every submodule $N$ of $M$ there is a monomorphism from $M/N$ to $M$. Mono-coretractable modules are defined as *co-epi-retractable* modules in [7]. We should also note that saying "$R_R$ is mono-coretractable" is the same with saying "$R$ is a co-pri ring" in [7]. In Section 3, we study mono-coretractable modules. We are giving an example of a coretractable module which is not mono-coretractable (see Example 3.6).

Throughout this paper rings will have a nonzero identity element and modules will be unitary right modules. We follow [1], [4] and [5] for the terms not defined here.

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## 2. Coretractable Modules

Let $M$ be an $R$-module. $M$ is called *coretractable* if $Hom(M/N, M)$ is nonzero for all proper submodules $N$ of $M$. Let $R$ be a commutative domain. Then $R_R$ cannot be coretractable. For, let $A$ be a nonzero proper right ideal of $R$. Let $f : R/A \longrightarrow R$ be any $R$-homomorphism. Since $f(R/A)A = 0$, $f = 0$. This is also clear by [10, Proposition 1.44]. Let $M_R$ be a module such that any simple module in $\sigma[M]$ is $M$-cyclic, i.e., isomorphic to a factor module of $M$. In this case if $M$ is coretractable, then $M$ is a Kasch module. Because, let $S$ be a simple module in $\sigma[M]$. Then $S \cong M/N$ for some submodule $N$ of $M$. Since $M$ is coretractable, there is a nonzero homomorphism from $M/N$ to $M$. Thus there exists a nonzero homomorphism, which is a monomorphism, from $S$ to $M$. Therefore $M$ is Kasch. On the other hand, if $M$ is a finitely generated Kasch module, then it is easy to see that $M$ is a coretractable module. So we can give the following result which generalizes [2, Theorem 2.14]:

**Theorem 2.1** *Let $M_R$ be a finitely generated self-generator module. Then $M$ is coretractable if and only if it is Kasch.*

Theorem 2.1 gives us several examples as we see in the following:

**Example 2.2** (1) Let $F$ be a field. Then the ring $R = F\text{x}F\text{x}F\text{x}\cdots$ is not a Kasch ring and so $R$ is not coretractable as an $R$-module.

(2) Suppose that $R$ is a semiperfect ring in which $\text{Soc}(R_R)$ is essential in ${}_RR$. Then $R$ is right Kasch by [10, Lemma 1.48], and so $R_R$ is coretractable.

(3) Assume that $R$ is a right self-injective, semiperfect ring with $\text{Soc}(R_R)$ essential in $R_R$. Then $R$ is right and left Kasch by [10, Lemma 1.49] and so $R_R$ and ${}_RR$ are coretractable.

(4) (see [10, Page, 214]) Let ${}_DV_D$ and ${}_DP_D$ be nonzero bimodules over a division ring $D$, and suppose a bimap $V\text{x}V \longrightarrow P$ is given. Write $R = [D, V, P] = D \oplus V \oplus P$ and define a multiplication on $R$ by

$$(d + v + p)(d_1 + v_1 + p_1) = dd_1 + (dv_1 + vd_1) + (dp_1 + vv_1 + pd_1).$$

Then $R$ is a (an associative) ring. The ring $R$ has a matrix representation as

$$R = \left\{ \begin{bmatrix} d & v & p \\ 0 & d & v \\ 0 & 0 & d \end{bmatrix} : d \in D, v \in V, p \in P \right\}.$$

By [10, Proposition 9.14], $R$ is right and left Kasch, and so $R_R$ and ${}_RR$ are coretractable.

Following [12], if for any module $M$, $\overline{Z}(M) = \cap\{N \mid M/N \text{ is small}\} = M$, then $M$ is called *noncosingular*.

**Proposition 2.3** *Let $R$ be a right perfect ring. Let $M$ be a noncosingular projective right $R$-module. Then $M$ is coretractable if and only if $M$ is semisimple.*

*Proof.* The sufficiency is clear. Conversely, suppose that $M$ is coretractable. Let $N$ be a proper submodule of $M$. Then there exists a nonzero homomorphism $f : M/N \longrightarrow M$. Let $\operatorname{Ker} f = T/N$. Since $M/T$ is noncosingular by [12, Proposition 2.4], $\operatorname{Im} f$ is noncosingular. Then by [12, Lemma 2.3(2)], $\operatorname{Im} f$ is coclosed in $M$. Since $M$ is lifting, $\operatorname{Im} f$ is a direct summand of $M$. So, $T$ is a proper direct summand of $M$, which contains $N$. This means that $N$ cannot be essential in $M$. Thus $M$ is semisimple. □

**Lemma 2.4** *Let $M$ be a quasi-injective module and $N \leq M$. Then $N$ is coretractable if and only if for all submodules $L$ of $M$ contained properly in $N$, the set*

$$\mathcal{A}_L = \{f : M \longrightarrow M \mid f(N) \subseteq N, L \subseteq \operatorname{Ker} f, N \nsubseteq \operatorname{Ker} f\}$$

*is nonempty.*

*Proof.* ($\Rightarrow$) Let $N$ be coretractable and $L$ a proper submodule of $N$. Then there exists a nonzero homomorphism $f : N/L \longrightarrow N$. Let $i : N \longrightarrow M$ and $i_L : N/L \longrightarrow M/L$ be inclusion maps. Since $M$ is $M/L$-injective, there exists a nonzero homomorphism $g : M/L \longrightarrow M$ such that $gi_L = if$. Consider the nonzero homomorphism $g\pi : M \longrightarrow M$, where $\pi : M \longrightarrow M/L$ is the natural epimorphism. It is easy to see that $L \subseteq \operatorname{Ker} g\pi, N \nsubseteq \operatorname{Ker} g\pi$ and $g\pi(N) \subseteq N$. Therefore $\mathcal{A}_L$ is nonempty.

($\Leftarrow$) Let $L$ be a proper submodule of $N$. By hypothesis, the set $\mathcal{A}_L$ is nonempty. Therefore there exists a nonzero homomorphism $f : M \longrightarrow M$ such that $f(N) \subseteq N$, $L \subseteq \operatorname{Ker} f$ and $N \nsubseteq \operatorname{Ker} f$. Define the homomorphism

$g : N/L \longrightarrow N$, $g(n + L) = f(n)$. Clearly, $g$ is nonzero. Thus $N$ is coretractable. □

**Proposition 2.5** *Let $R_R$ be injective and $I$ a nonzero proper right ideal of $R$. Then $I_R$ is coretractable if and only if for any right ideal $J$ of $R$ with $J \subsetneq I$, there exists a nonzero element $x$ of $R$ such that $0 \neq xI \subseteq I$ and $xJ = 0$.*

*Proof.* (⇒) Let $I_R$ be coretractable. Let $J$ be any right ideal of $R$ with $J \subsetneq I$. By Lemma 2.4, $\mathcal{A}_J$ is nonempty. Then there exists a nonzero homomorphism $f : R \longrightarrow R$ such that $f(I) \subseteq I$, $J \subseteq \operatorname{Ker} f$ and $I \nsubseteq \operatorname{Ker} f$. Let $f(1) = x$. Then $x \neq 0, 0 \neq xI \subseteq I$ and $xJ = 0$.

(⇐) Let $L$ be a proper submodule of $I$. By hypothesis, there exists a nonzero element $x$ of $R$ such that $0 \neq xI \subseteq I$ and $xL = 0$. Define the homomorphism $f : R \longrightarrow R$, $f(r) = xr$. Since $f(1) = x \neq 0$, $f \neq 0$. Since $f(I) = xI$, then $f(I) \subseteq I$ and $I \nsubseteq \operatorname{Ker} f$. Since $xL = 0$, $L \subseteq \operatorname{Ker} f$. By Lemma 2.4, $I_R$ is coretractable. □

Let $M$ be a module. We say that $M$ is *small coretractable* if $Hom(M/N, M)$ is nonzero for all small submodules $N$ of $M$.

**Lemma 2.6** *Let $M$ be a module with projective cover $(P, \alpha)$. Then $M$ is small coretractable if and only if there exists a nonzero $f \in Hom(P, M)$ such that $P/\operatorname{Ker} f$ is a small coretractable module.*

*Proof.* Necessity: Since $P/\operatorname{Ker}\alpha \cong M$, this is clear.
Sufficiency: Let $K$ be a small submodule of $M$. Then $(\alpha^{-1}(K) + \operatorname{Ker} f)/\operatorname{Ker} f \ll P/\operatorname{Ker} f$. Since $P/\operatorname{Ker} f$ is small coretractable, there exists a nonzero homomorphism $\beta : P/(\alpha^{-1}(K) + \operatorname{Ker} f) \longrightarrow P/\operatorname{Ker} f$. Define the homomorphism $\eta : M/K \longrightarrow P/\alpha^{-1}(K)$ by $m + K \mapsto p + \alpha^{-1}(K)$ where $\alpha(p) = m, p \in P$ and $m \in M$. It follows that $Hom(M/K, M)$ is nonzero. □

Let $R$ be a ring. If every right $R$-module is coretractable, then $R$ is right and left perfect and right Kasch (see [2, Theorems 2.14 and 3.10]). Let $R$ be any ring. We will say that $R$ satisfies $(C)$ if every right $R$-module is coretractable. Now we give the following characterizations. Note that these characterizations are left and right symmetric by [14, Theorem 2.4].

**Theorem 2.7** *For a ring $R$ the following are equivalent*:

(1) *$R$ satisfies $(C)$.*

(2) *$R$ is right perfect and every right $R$-module is small coretractable.*
(3) *$R$ is right perfect and for every right $R$-module $M$, there exists a nonzero $f \in Hom(P, M)$ such that $P/\operatorname{Ker} f$ is a small coretractable module, where $P$ is the projective cover of $M$.*
(4) *$R$ is right perfect and for all right $R$-modules $M$ and $X$, $Hom(X, M) = 0$ if and only if $Hom(P, M) = 0$, where $P$ is the projective cover of $X$.*
(5) *All torsion theories on $R$ are cohereditary.*

*Proof.* (1) $\Rightarrow$ (2) is clear.

(2) $\Leftrightarrow$ (3) follows by Lemma 2.6.

(2) $\Rightarrow$ (4): Let $(P, \alpha)$ be the projective cover of $X$. Assume $Hom(P, M) \neq 0$. Then there exists a nonzero homomorphism $\beta$ from $P$ to $M$. Since $\operatorname{Ker}\alpha \ll P$, $(\operatorname{Ker}\alpha + \operatorname{Ker}\beta)/\operatorname{Ker}\beta \ll P/\operatorname{Ker}\beta$. Then there exists a nonzero homomorphism $\eta : P/(\operatorname{Ker}\alpha + \operatorname{Ker}\beta) \longrightarrow P/\operatorname{Ker}\beta$. Therefore $Hom(X, M) \neq 0$. The converse is easy.

(4) $\Rightarrow$ (5): Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory on $R$, $N \leq M \in \mathcal{F}$ and $X \in \mathcal{T}$. Then $Hom(X, M) = 0$. By (4), $Hom(P, M) = 0$, where $P$ is the projective cover of $X$. Hence $Hom(X, M/N) = 0$, and so $M/N \in \mathcal{F}$.

(5) $\Rightarrow$ (1): Let $M$ be a nonzero right $R$-module and $N$ a proper submodule of $M$. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory cogenerated by $M$. Note that $M \in \mathcal{F}$. By (5), $M/N \in \mathcal{F}$. Thus $Hom(M/N, M) \neq 0$ (see [5, 7.2]). □

Given any ring $R$, we call a nonzero right $R$-module $M$ a *weak generator* for $Mod$-$R$ if, for each nonzero right $R$-module $X$, $Hom(M, X) \neq 0$.

**Theorem 2.8** *Let $R$ be a ring with Jacobson radical $J$ such that the ring $R/J$ is simple artinian. Then the following are equivalent*:

(1) *$R$ is right and left semi-artinian.*
(2) *every nonzero right (left) $R$-module is a weak generator for $Mod$-$R$.*
(3) *$R$ satisfies $(C)$.*
(4) *$R$ is right and left perfect.*

*Proof.* (1) $\Rightarrow$ (2): By [11, Variation of Corollary 3.6].

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are clear.

(3) $\Rightarrow$ (4): By [2, Theorem 3.10]. □

Note that $Hom_R(M, N) = Hom_{R/I}(M, N)$ for each ideal $I$ of $R$ and $M, N \in Mod$-$R/I$. Therefore the class of rings satisfying $(C)$ is closed under homomorphic images.

**Theorem 2.9** *Being coretractable is a Morita invariant property.*

*Proof.* Let $R$ and $S$ be two Morita equivalent rings. Assume that $F : Mod\text{-}R \longrightarrow Mod\text{-}S$ and $G : Mod\text{-}S \longrightarrow Mod\text{-}R$ are two category equivalences. Let $M_R$ be a coretractable object in $Mod\text{-}R$. Let $N$ be a proper submodule of $F(M)$. Now we have the exact sequence

$$0 \longrightarrow N \longrightarrow F(M) \longrightarrow F(M)/N \longrightarrow 0$$

in $Mod\text{-}S$. By [1, Proposition 21.4],

$$0 \longrightarrow G(N) \longrightarrow M \longrightarrow G(F(M)/N) \longrightarrow 0$$

is exact in $Mod\text{-}R$. Therefore $M/G(N) \cong G(F(M)/N)$. Since $M_R$ is coretractable, $Hom_R(M/G(N), M) \neq 0$. Hence

$$Hom_R(M/G(N), M) \cong Hom_S(F(M)/N, F(M))$$

implies that $F(M)$ is coretractable in $Mod\text{-}S$. □

The following corollary is well-known for right Kasch rings:

**Corollary 2.10** *Let $R_R$ be a coretractable module (namely, $R$ is right Kasch). Then the ring $M_n(R)$ of all $n \times n$ matrices with entries in $R$ is coretractable as a right module over itself (namely, it is right Kasch).*

**Corollary 2.11** *Let $R$ satisfy $(C)$. Then the ring $M_n(R)$ of all $n \times n$ matrices with entries in $R$ satisfies $(C)$.*

## 3. Mono-coretractable Modules

We call an $R$-module $M$ *mono-coretractable* if for every submodule $N$ of $M$ there is a monomorphism from $M/N$ to $M$. Mono-coretractable modules are defined as *co-epi-retractable* modules in [7]. Let $I$ be a nonzero proper ideal of a principal ideal domain $R$. Then it is easy to see that the $R$-module $R/I$ is mono-coretractable (see also [7, Corollary 1.5]). Firstly we give the following easy characterization (may be it is known):

**Lemma 3.1** *The following are equivalent for a module $M$ :*

(1) *$M$ is mono-coretractable.*

(2) *There exist monomorphisms $M \longrightarrow N$ and $N \longrightarrow M$ for some mono-coretractable module $N$.*
(3) *There exists a monomorphism from $M$ to $K$ for some mono-coretractable submodule $K$ of $M$.*

*Proof.* (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3): Let $\alpha : M \longrightarrow N$, $\beta : N \longrightarrow M$ be monomorphisms and $N$ a mono-coretractable module. Let $\operatorname{Im}\beta = K$. Now we have the monomorphism $\beta\alpha : M \longrightarrow K$. Since $N \cong K$, $K$ is a mono-coretractable submodule of $M$.

(3) $\Rightarrow$ (1): Let $\varphi : M \longrightarrow K$ be a monomorphism with $K$ a mono-coretractable submodule of $M$. Let $L$ be a submodule of $M$. Consider the monomorphism $\alpha : M/L \longrightarrow K/N$ defined by $\alpha(m+L) = \varphi(m)+N$, where $N = \varphi(L)$. Since $K$ is mono-coretractable, there exists a monomorphism $\theta : K/N \longrightarrow K$. Now we have the monomorphism $i\theta\alpha : M/L \longrightarrow M$, where $i : K \longrightarrow M$ is the inclusion map. Thus $M$ is mono-coretractable. □

Note that the Prüfer $p$-group $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}/\mathbb{Z}$ are noncosingular and mono-coretractable $\mathbb{Z}$-modules. But they are not discrete. Now we give the following:

**Proposition 3.2** *The following are equivalent for a noncosingular module $M$ :*

(1) *$M$ is semisimple.*
(2) *$M$ is discrete mono-coretractable.*

*Proof.* (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1): Let $N$ be a proper submodule of $M$. Then there exists a monomorphism $\alpha : M/N \longrightarrow M$. Since $M/N$ is noncosingular, $\alpha(M/N)$ is noncosingular and so it is a coclosed submodule of $M$. Since $M$ is lifting, $\alpha(M/N)$ is a direct summand of $M$, and since $M$ has $(D_2)$, $N$ is a direct summand of $M$. Thus $M$ is semisimple. □

Noncosingular condition in Proposition 3.2 is not superfluous:

**Example 3.3** It is easy to see that the $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$ is mono-coretractable. On the other hand, it is discrete, but not noncosingular and not semisimple.

**Proposition 3.4** *If $R$ is a ring such that every projective right $R$-module is mono-coretractable, then $R$ is a $QF$-ring.*

*Proof.* Let $X$ be an injective right $R$-module. Since every module is an epimorphic image of a free (projective) module, there exists an epimorphism $\alpha : P \longrightarrow X$ with $P$ projective. By hypothesis, $P$ is mono-coretractable and so $P/\operatorname{Ker}\alpha \cong A \leq P$ for some submodule $A$ of $P$. Since $P/\operatorname{Ker}\alpha$ is injective, $A$ is a direct summand of $P$. Therefore $A$ is projective. Thus $X$ is projective. Hence $R$ is a $QF$-ring. □

**Remark 3.5** (1) We should note that some of the dual results to the results in this paper can be found in [6], [8] and [13].

(2) There exist projective modules which are not mono-coretractable. For example, let $R$ be the ring $\left[\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right]$, where $F$ is any field. Then $R_R$ is not coretractable and so it is not mono-coretractable.

(3) Note that in [7, Corollary 1.9], it is proved that if $R_R$ and ${}_RR$ are mono-coretractable, then $R$ is a $QF$-ring. And it is given in Example 1.10 in [7] that there exists a $QF$-ring $R$ with $R_R$ not mono-coretractable. With the help of this example we show that any coretractable module need not be mono-coretractable.

**Example 3.6** For any division ring $K$, let $R$ be the 4-dimensional $K$-ring consisting of matrices of the form

$$\alpha = \begin{bmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{bmatrix}.$$

By [7, Example 1.10], $R_R$ is not mono-coretractable, but it is a $QF$-ring. By [9, Corollary, 19.17], $R$ is a cogenerator ring. Therefore $R_R$ is coretractable.

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