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Fold maps with singular value sets of concentric spheres

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Abstract. In this paper, we study *fold maps* from C^∞ closed manifolds into Euclidean spaces whose singular value sets are disjoint unions of spheres embedded concentrically. We mainly study homology and homotopy groups of manifolds admitting such maps.

Key words: Singularities of differentiable maps; singular sets, fold maps. Differential topology.

1. Introduction

Fold maps are important in generalizing the theory of Morse functions. Studies of such maps were started by Whitney ([23]) and Thom ([20]) in the 1950's. A *fold map* is a C^∞ map whose singular points are of the form

$$(x_1, \ldots, x_m) \mapsto \left(x_1, \ldots, x_{n-1}, \sum_{j=n}^{m-i} {x_j}^2 - \sum_{j=m-i+1}^{m} {x_j}^2\right)$$

for two positive integers $m \geq n$ and an integer $0 \leq i \leq m-n+1$. A Morse function is naturally regarded as a fold map ($n = 1$).

Since around the 1990's, fold maps with additional conditions have been actively studied. For example, in [1], [3], [10], [12] and [14], *special generic* maps, which are fold maps whose singular points are of the form

$$(x_1, \ldots, x_m) \mapsto \left(x_1, \ldots, x_{n-1}, \sum_{j=n}^{m} {x_j}^2\right)$$

for two positive integers $m \geq n$, were studied. In [15], Sakuma studied *simple* fold maps, which are fold maps such that fibers of singular values do not have any connected component with more than one singular points (see also [9]). For example, special generic maps are simple. In [5] and [6],

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Kobayashi and Saeki investigated topology of *stable* maps into the plane including fold maps which are stable (for *stable* maps, see [4] for example). In [13], Saeki and Suzuoka found good properties of manifolds admitting stable maps whose regular fibers, or fibers of regular values, are disjoint unions of spheres. In [11], Saeki investigated Morse functions whose regular fibers are disjoint unions of spheres.

For a fold map from a closed C^∞ manifold of dimension m into a C^∞ manifold of dimension n (without boundary), the followings hold where $m \geq n \geq 1$.

(1) The singular set, or the set of all the singular points of the map, is a closed C^∞ $(n-1)$-submanifold of the source manifold.
(2) The restriction to the singular set is a C^∞ immersion of codimension 1.

In this paper, we introduce a new class of fold maps called *round* fold maps. A *round* fold map is a fold map into $\mathbb{R}^n (n \geq 2)$ satisfying the followings.

(1) The singular set is a disjoint union of standard spheres.
(2) The restriction to the singular set is a C^∞ embedding.
(3) The singular value set is a disjoint union of spheres embedded concentrically.

This class includes some special generic maps on homotopy spheres (see [10]) and some maps in [6] and [13], which are not special generic, for example. We can construct many round fold maps easily. We study manifolds admitting such maps in this paper.

This paper is organized as follows.

Section 2 is for preliminaries. We recall *fold maps*. We also recall *special generic* maps and *simple* fold maps. Finally we review the *Reeb space* of a smooth map, which is the space consisting of all the connected components of all the fibers of the smooth map.

In Section 3, we introduce *round* fold maps and examples (Example 2) and give a method of construction. We also introduce terminologies on structures of round fold maps and in Example 3, study structures of examples including ones mentioned in Example 2 in these terminologies. As a main theorem, we have relations between homology groups of a manifold which admits a round fold map of a certain structure and those of the inverse image of a ray in $\mathbb{R}^n$ called an *axis* (Theorem 1).

In Section 4, we study round fold maps whose regular fibers are disjoint unions of spheres. We have a lot of examples of such maps, as presented in Example 2 and Example 3. We then show that the homology groups of manifolds admitting round fold maps of certain structures with all the regular fibers homeomorphic to disjoint unions of spheres are torsion-free (Theorem 2) by applying our Theorem 1 and Corollary 3.17 of [11].

We next study homotopy groups of manifolds admitting such fold maps. According to Theorem 4.1 of [13], if there exists a stable map from a closed C^∞ manifold of dimension 4 into the plane whose regular fibers are disjoint unions of spheres, then the source manifold bounds a nice manifold. As a corollary to the theorem, it has been shown that the fundamental groups of such a manifold and the Reeb space agree (Corollary 4.8 in [13]). We generalize these results under some constraints (Lemma 1 and Corollary 4). After that, we prove Theorem 3. It states that some homotopy groups of manifolds admitting such round fold maps are determined by topological properties of the Reeb spaces. Furthermore, Theorem 3 is an extension of the last part of the proof of Theorem 7.1 of [6], which states that if we assume good conditions on the Reeb space of a simple fold map from a simply-connected manifold into the plane, then the source manifold is a homotopy sphere.

In Section 5, we study the homeomorphism types of manifolds admitting round fold maps as in the previous section for some cases (Corollary 5, Theorem 4 and Example 6).

Throughout this paper, we assume that M is a closed C^∞ manifold of dimension m, that N is a C^∞ manifold of dimension n without boundary, that $f : M \to N$ is a C^∞ map and that $m \geq n \geq 1$. We denote the *singular set* of f, or the set consisting of all the singular points of f, by $S(f)$.

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2. Preliminaries

2.1. Notations on topological spaces

We introduce notations on topological spaces which we use in this paper. For a topological space X, we denote the *identity map* on X by id_X. We denote the *interior*, and the *closure* of a subspace X of a topological space by Int$X$ and $\overline{X}$, respectively. For a manifold X, we denote the *boundary* of X by ∂X.

Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces. We denote the *disjoint union* of $\{X_\lambda\}_{\lambda \in \Lambda}$ by $\sqcup_{\lambda \in \Lambda} X_\lambda$. If Λ is a finite set consisting of all the integers not smaller than $l_1 \in \mathbb{Z}$ and not larger than $l_2 \in \mathbb{Z}$, then we also denote the disjoint union by $\sqcup_{k=l_1}^{l_2} X_k$ or $X_{l_1} \sqcup \cdots \sqcup X_{l_2}$.

For a family of maps $\{c_\lambda : X_\lambda \to Y_\lambda\}_{\lambda \in \Lambda}$, we denote the *disjoint union* of $\{c_\lambda\}_{\lambda \in \Lambda}$ by $\sqcup_{\lambda \in \Lambda} c_\lambda : \sqcup_{\lambda \in \Lambda} X_\lambda \to \sqcup_{\lambda \in \Lambda} Y_\lambda$. We use the notation $\sqcup_{k=l_1}^{l_2} c_k$ or $c_{l_1} \sqcup \cdots \sqcup c_{l_2}$ if Λ is a finite set consisting of all the integers not smaller than $l_1 \in \mathbb{Z}$ and not larger than $l_2 \in \mathbb{Z}$ as before.

Let X_1, X_2 be topological spaces. Let $A_i \subset X_i$ $(i = 1, 2)$ and $\phi : A_2 \to A_1$ be a homeomorphism. By glueing X_1 and X_2 together by ϕ, we obtain a topological space $X_1 \bigcup_\phi X_2$. We often omit ϕ of $X_1 \bigcup_\phi X_2$ and denote it by $X_1 \bigcup X_2$ in case we consider a natural identification.

Let $c_1 : X_1 \to X_2$ and $c_2 : X_2 \to Y_2$ be continuous maps. Let $A_i \subset X_i$, $B_i \subset Y_i$ and $c_i(A_i) \subset B_i$ $(i = 1, 2)$. If for a pair of homeomorphisms $(\phi_A : A_2 \to A_1, \phi_B : B_2 \to B_1)$, the relation $\phi_B \circ c_2|_{A_2} = c_1|_{A_1} \circ \phi_A$ holds, then by glueing X_1 and X_2 together by ϕ_A and by glueing Y_1 and Y_2 together by ϕ_B, two spaces $X_1 \bigcup_{\phi_A} X_2$ and $Y_1 \bigcup_{\phi_B} Y_2$ and a continuous map $c := c_1 \bigcup_{\phi_A, \phi_B} c_2 : X_1 \bigcup_{\phi_A} X_2 \to Y_1 \bigcup_{\phi_B} Y_2$ such that

$$c(\pi_{\phi_A}(x)) := \begin{cases} \pi_{\phi_B} \circ c_1(x) & x \in X_1 \\ \pi_{\phi_B} \circ c_2(x) & x \in X_2 \end{cases}$$

are obtained where $\pi_{\phi_A} : X_1 \sqcup X_2 \to X_1 \bigcup_{\phi_A} X_2$ and $\pi_{\phi_B} : Y_1 \sqcup Y_2 \to Y_1 \bigcup_{\phi_B} Y_2$ are the quotient maps. We often omit (ϕ_A, ϕ_B) and denote the map by $g_1 \bigcup g_2$ in case we consider natural identifications.

2.2. Fold maps

We recall *fold maps*, which are simplest generalizations of Morse functions. See also [4], [7] and [8] for example.

Definition 1 For a C^∞ map $f : M \to N$, a point $p \in M$ is said to be a *fold* point of f if at p, the map f has the normal form

$$f(x_1, \ldots, x_m) := \left(x_1, \ldots, x_{n-1}, \sum_{j=n}^{m-i} {x_j}^2 - \sum_{j=m-i+1}^{m} {x_j}^2 \right)$$

and f is said to be a *fold map* if all the singular points of f are fold.

If a point $p \in M$ is a fold point of f, then we can define $j := \min\{i, m - n + 1 - i\}$ uniquely in the previous definition. We call p a *fold point of index* j of f. We call a fold point of index 0 a *definite* fold point of f and we call f a *special generic* map if all the singular points are definite fold points. For special generic maps, see [1], [3], [10] and [14] for example. Let f be a fold map. Then the singular set $S(f)$ and the set of all the fold points of indices i (we denote the set of all such points by $F_i(f)$) are C^∞ $(n-1)$-submanifolds of M. The restriction $f|_{S(f)}$ is a C^∞ immersion.

A Morse function on a closed manifold is naturally regarded as a fold map ($n = 1$). A Morse function on a closed manifold which has just two singular points is regarded as a special generic map.

We introduce *simple* fibers of fold maps and *simple* fold maps.

Definition 2 (see *e.g.* [9] and [14]) For a fold map f and a singular value $p \in f(S(f))$, $f^{-1}(p)$ is said to be *simple* if each connected component of $f^{-1}(p)$ includes at most one singular point of f. f is said to be a *simple* fold map if for each $p \in f(S(f))$, $f^{-1}(p)$ is simple.

Example 1 (1) Morse functions on closed manifolds are simple if the values are always distinct at distinct singular points.
(2) A fold map $f : M \to \mathbb{R}^n$ is simple if $f|_{S(f)}$ is a C^∞ embedding.
(3) Special generic maps are simple.

2.3. Reeb spaces

We review the *Reeb space* of a map.

Definition 3 Let X, Y be topological spaces. For $p_1, p_2 \in X$ and for a map $c : X \to Y$, we define as $p_1 \sim_c p_2$ if and only if p_1 and p_2 are in the same connected component of $c^{-1}(p)$ for some $p \in Y$. The relation $\sim_c$ is an equivalence relation.

We denote the quotient space $X/\sim_c$ by W_c and call it the *Reeb space* of c.

We denote the induced quotient map from X into W_c by q_c. We define $\bar{c} : W_c \to Y$ so that $c = \bar{c} \circ q_c$. W_c is often homeomorphic to a polyhedron. For example, for a Morse function, the Reeb space is a graph and for a simple fold map, the Reeb space is homeomorphic to a polyhedron which is not so complex (see Proposition 1 later). For a special generic map, the Reeb space is homeomorphic to a C^∞ manifold (see Section 2 of [10]).

Here, we introduce terms on spheres and fiber bundles which are important in this paper.

An *almost-sphere* of dimension k means a C^∞ homotopy sphere given by glueing two k-dimensional standard closed discs together by a diffeomorphism between the boundaries.

We often use terminologies on (fiber) bundles in this paper (see also [18]). For a topological space X, an X-*bundle* is a bundle whose fiber is X. A bundle whose structure group is G is said to be a *trivial* bundle if it is equivalent to the product bundle as a bundle whose structure group is G. Especially, a trivial bundle whose structure group is a subgroup of the homeomorphism group of the fiber is said to be a *topologically trivial* bundle. In this paper, a C^∞ (PL) *bundle* means a bundle whose fiber is a C^∞ (resp. PL) manifold and whose structure group is a subgroup of the diffeomorphism group (resp. PL homeomorphism group) of the fiber. A *linear* bundle is a C^∞ bundle whose fiber is a standard disc or a standard sphere and whose structure group is a subgroup of an orthogonal group.

The following Proposition 1 is well-known and we omit the proof. See [6], [9] and [13] for example. This proposition is a basic tool in the proof of Theorem 4.1 of [13] and Lemma 1 of this paper, for example.

Proposition 1 *Let $f : M \to N$ be a special generic map or a simple fold map or a stable fold map from a closed C^∞ manifold M of dimension m into a C^∞ manifold N of dimension n. Then W_f has the structure of a polyhedron and the followings hold.*

(1) *$W_f - q_f(S(f))$ is uniquely given the structure of a C^∞ manifold such that $q_f|_{M-S(f)} : M - S(f) \to W_f - q_f(S(f))$ is a C^∞ submersion. Furthermore, for any compact C^∞ submanifold R of dimension n of any connected component of $W_f - q_f(S(f))$, R is a subpolyhedron of W_f and $q_f|_{{q_f}^{-1}(R)} : {q_f}^{-1}(R) \to R$ gives the structure of a C^∞ bundle whose fiber is a connected C^∞ manifold of dimension $m - n$.*

(2) *The restriction of q_f to the set $F_0(f)$ of all the definite fold points is injective.*

(3) *f is simple if and only if $q_f|_{S(f)} : S(f) \to W_f$ is injective.*

(4) *If f is simple, then for any connected component C of $S(f)$, $q_f(C)$ has a small regular neighborhood $N(q_f(C))$ in W_f such that ${q_f}^{-1}(N(q_f(C)))$*

has the structure of a C^∞ bundle over $q_f(C)$.

(5) *For any connected component C of $F_0(f)$, any small regular neighborhood of $q_f(C)$ has the structure of a trivial* PL $[0,1]$*-bundle over $q_f(C)$ such that $q_f(C)$ corresponds to the* 0*-section. We can take a small regular neighborhood $N(q_f(C))$ of $q_f(C)$ so that ${q_f}^{-1}(N(q_f(C)))$ has the structure of a linear D^{m-n+1}-bundle over $q_f(C)$. More precisely, the bundle structure is given by the composition of $q_f|_{{q_f}^{-1}(N(q_f(C))}$: ${q_f}^{-1}(N(q_f(C))) \to N(q_f(C))$ and the projection to $q_f(C)$.*

(6) *Let f be simple and $m-n \geq 1$. If $m-n=1$, then we also assume that M is orientable.*
Then for any connected component C of the set $F_1(f)$ of all the fold points of indice 1 *such that for any point $p \in q_f(C)$, any small neighborhood $N(p)$ of p and any point q in $N(p) - q_f(C)$, the inverse image ${q_f}^{-1}(q)$ is an almost-sphere, any small regular neighborhood $N(q_f(C))$ of $q_f(C)$ has the structure of a K-bundle over $q_f(C)$, where $K := \{r\exp(2\pi i\theta) \in \mathbb{C} \mid 0 \leq r \leq 1, \theta = 0, 1/3, 2/3\}$ with the structure group consisting of just two elements, where one element is defined as the identity transformation and the other element is defined as the transformation defined by $z \to \bar{z}$ ($\bar{z} \in K$ is the complex conjugation of $z \in K$), such that $q_f(C)$ corresponds to the* 0*-section. We can take the neighborhood $N(q_f(C))$ of $q_f(C)$ so that ${q_f}^{-1}(N(q_f(C)))$ has the structure of a C^∞ bundle over $q_f(C)$ with a fiber* PL *homeomorphic to S^{m-n+1} with the interior of a union of disjoint three $(m-n+1)$-dimensional standard closed discs removed. More precisely, the bundle structure is given by the composition of $q_f|_{{q_f}^{-1}(N(q_f(C)))} : {q_f}^{-1}(N(q_f(C))) \to N(q_f(C))$ and the projection to $q_f(C)$.*

3. Round fold maps

In this section, we introduce *round* fold maps.

3.1. The definition of a round fold map

We introduce two definitions of a round fold map and show equivalence of them.

First we recall C^∞ *equivalence* (see [4] for example). For two C^∞ maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, they are said to be C^∞ *equivalent* if there exist diffeomorphisms $\phi_X : X_1 \to X_2$ and $\phi_Y : Y_1 \to Y_2$ such that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi_X} & X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{\phi_Y} & Y_2 \end{array}$$

Definition 4 $f : M \to \mathbb{R}^n$ $(m \geq n \geq 2)$ is said to be a *round* fold map if f is C^∞ equivalent to a fold map $f_0 : M_0 \to \mathbb{R}^n$ on a closed C^∞ manifold M_0 such that the followings hold.

(1) The singular set $S(f_0)$ is a disjoint union of $l \in \mathbb{N}$ copies of $(n-1)$-dimensional standard spheres.
(2) The restriction $f_0|_{S(f_0)}$ is a C^∞ embedding.
(3) Let ${D^n}_r := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} {x_j}^2 \leq r\}$. Then $f_0(S(f_0)) = \sqcup_{j=1}^{l} \partial {D^n}_j$.

We call f_0 a *normal form* of f. We call a ray L from $0 \in \mathbb{R}^n$ an *axis* of f_0 and ${D^n}_{1/2}$ the *proper core* of f_0. Suppose that for a round fold map f, its normal form f_0 and diffeomorphisms $\Phi : M \to M_0$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$, the equation $\phi \circ f = f_0 \circ \Phi$ holds. Then for an axis L of f_0, we also call $\phi^{-1}(L)$ an *axis* of f and for the proper core ${D^n}_{1/2}$ of f_0, we also call $\phi^{-1}({D^n}_{1/2})$ a *proper core* of f.

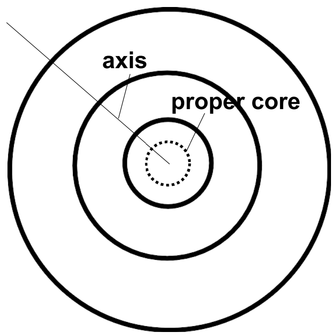

Figure 1. An axis and a proper core of a round fold map.

We introduce another definition.

Definition 5 Assume that $f : M \to \mathbb{R}^n$ is a fold map and that $m \geq n \geq 2$. f is said to be a *round* fold map if the followings hold.

(1) The singular set $S(f)$ is a disjoint union of finite copies of $(n-1)$-dimensional standard spheres.
(2) The restriction $f|_{S(f)}$ is a C^∞ embedding.
(3) We can denote by $\{U_0\} \sqcup \{U_\infty\} \sqcup \{U_\lambda\}_{\lambda \in \Lambda}$ the set of all the connected components of $\mathbb{R}^n - f(S(f))$ where Λ is a finite set that may possibly be empty so that the followings hold.
   (a) The closure $\overline{U_0}$ is diffeomorphic to D^n.
   (b) The closure $\overline{U_\infty}$ is diffeomorphic to $S^{n-1} \times [0, +\infty)$.
   (c) The closure $\overline{U_\lambda}$ is diffeomorphic to $S^{n-1} \times [0, 1]$.

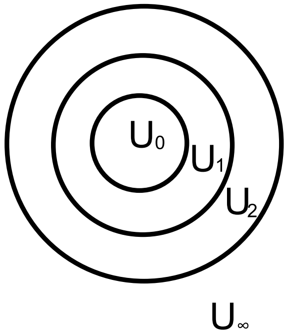

Figure 2. The image $\overline{U_0} \bigcup \overline{U_1} \bigcup \overline{U_2}$ of a round fold map such that $\Lambda = \{1, 2\}$ (Definition 5).

Note that $f(M) \bigcap U_\infty = \emptyset$ always holds in Definition 5.

Proposition 2 *Two definitions of a round fold map* (*Definition* 4 *and Definition* 5) *are equivalent.*

It immediately follows that if f is a fold map satisfying Definition 4, then it satisfies Definition 5. Hence to show this proposition, it suffices to prove that any fold map satisfying Definition 5 satisfies Definition 4.

Before a strict proof, we sketch the outline of the proof. First we decompose f into a singular part f_1 and a regular part f_2 (FIGURE 3). Second, we construct two C^∞ maps f_{01} and f_{02} so that f_i and f_{0i} are C^∞ equivalent $(i = 1, 2)$. Finally we construct f_0 by glueing f_{01} and f_{02} together so that f and f_0 are C^∞ equivalent and that f_0 is a normal form of f.

Proof of Proposition 2. Suppose that f is a fold map in the sense of Definition 5.

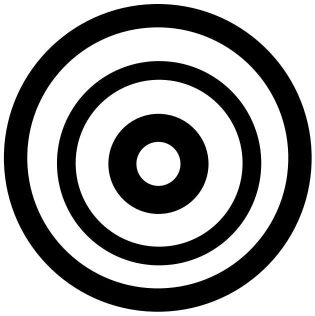

Figure 3. The image of the singular part f_1 and the image of the regular part f_2 of the round fold map f. (The image of the singular part is black and the image of the regular part is white.)

Let the singular value set $f(S(f))$ consist of $l \in \mathbb{N}$ connected components and we label all the connected components of the singular set of f by $\{S_r\}_{r=1}^{l}$ so that for $s, t \in \mathbb{N}$ and $1 \leq s, t \leq l$, $s < t$ holds if and only if $f(S_s)$ is in the bounded connected component of $\mathbb{R}^n - f(S_t)$.

Let A be a small C^∞ closed tubular neighborhood of $f(S(f))$ in $\mathbb{R}^n$. Note that A and $f(S(f)) \times [-1,1]$ are both diffeomorphic to a disjoint union of l copies of $S^{n-1} \times [-1,1]$. We put $f_1 := f|_{f^{-1}(A)} : f^{-1}(A) \to A$, and $f_2 := f|_{M - f^{-1}(\mathrm{Int}A)} : f^{-1}(\mathbb{R}^n - \mathrm{Int}A) \to \mathbb{R}^n - \mathrm{Int}A$. Hence we regard $f = f_1 \bigcup f_2$, where we identify $\partial f^{-1}(A)$ with $\partial f^{-1}(\mathbb{R}^n - \mathrm{Int}A)$ and ∂A with $\partial(\mathbb{R}^n - \mathrm{Int}A)$. Since A is so small that $\mathbb{R}^n - \mathrm{Int}A$ is diffeomorphic to the disjoint union of D^n, $S^{n-1} \times [0, +\infty)$ and $l-1$ copies of $S^{n-1} \times [0,1]$.

Set $M_1 := f^{-1}(A)$. There exist a diffeomorphism $\phi_1 : A \to \sqcup_{r=1}^{l} ({D^n}_{r+1/4} - \mathrm{Int}{D^n}_{r-1/4})$ and a C^∞ map $f_{01} : M_1 \to \sqcup_{r=1}^{l} ({D^n}_{r+1/4} - \mathrm{Int}{D^n}_{r-1/4})$ such that $\phi_1(f(S_r)) = \partial {D^n}_r$ holds for $1 \leq r \leq l$ and that the following diagram commutes.

$$\begin{array}{ccc} M_1 & \xrightarrow{\mathrm{id}_{M_1}} & M_1 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_{01}} \\ A & \xrightarrow{\phi_1} & \sqcup_{r=1}^{l} ({D^n}_{r+1/4} - \mathrm{Int}{D^n}_{r-1/4}) \end{array}$$

We label all the connected components of ∂A by P_r and Q_r ($r = 1, \ldots, l,$) so that $\{P_r\}_{r=1}^{l} \sqcup \{Q_r\}_{r=1}^{l}$ denotes the set of all the connected components of ∂A and that the followings hold for $s, t \in \mathbb{N}$, $1 \leq s, t \leq l$.

(1) $s < t$ holds if and only if P_s and Q_s are in the bounded connected component of $\mathbb{R}^n - P_t$ and $\mathbb{R}^n - Q_t$.
(2) P_s is in the bounded connected component of $\mathbb{R}^n - Q_s$.

We note that the disjoint union $P_r \sqcup Q_r$ is the boundary of a connected component of A.

Set $M_2 := f^{-1}(\mathbb{R}^n - \mathrm{Int}A)$. There exist a diffeomorphism $\phi_2 : \mathbb{R}^n - \mathrm{Int}A \to \sqcup_{r=1}^{l-1}({D^n}_{r+3/4} - \mathrm{Int}{D^n}_{r+1/4}) \sqcup {D^n}_{3/4} \sqcup (\mathbb{R}^n - \mathrm{Int}{D^n}_{l+1/4})$, and a C^∞ map $f_{02} : M_2 \to \sqcup_{r=1}^{l-1}({D^n}_{r+3/4} - \mathrm{Int}{D^n}_{r+1/4}) \sqcup {D^n}_{3/4} \sqcup (\mathbb{R}^n - \mathrm{Int}{D^n}_{l+1/4})$ such that $\phi_2(P_r) = \partial {D^n}_{r-1/4}$ and $\phi_2(Q_r) = \partial {D^n}_{r+1/4}$ hold and that the following diagram commutes.

$$\begin{array}{ccc} M_2 & \xrightarrow{\mathrm{id}_{M_2}} & M_2 \\ \downarrow f_2 & & \downarrow f_{02} \\ \mathbb{R}^n - \mathrm{Int}A & \xrightarrow{\phi_2} & \sqcup_{r=1}^{l-1}({D^n}_{r+3/4} - \mathrm{Int}{D^n}_{r+1/4}) \sqcup {D^n}_{3/4} \sqcup (\mathbb{R}^n - \mathrm{Int}{D^n}_{l+1/4}) \end{array}$$

Then for $\phi_0 := \phi_1|_{\partial A} \circ {\phi_2}^{-1}|_{\partial(\sqcup_{r=1}^{l-1}({D^n}_{r+3/4} - \mathrm{Int}{D^n}_{r+1/4}) \sqcup {D^n}_{3/4} \sqcup (\mathbb{R}^n - \mathrm{Int}{D^n}_{l+1/4}))}$, the map f is C^∞ equivalent to $f_0 := f_{01} \bigcup_{\mathrm{id}_{\partial M_2}, \phi_0} f_{02}$. In fact, by using ϕ_0, $\phi_1 \bigcup_{\mathrm{id}_{\partial(\mathbb{R}^n - \mathrm{Int}A)}, \phi_0} \phi_2$ is defined. We note that the following diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{id}_M} & M \\ \downarrow f & & \downarrow f_0 \\ \mathbb{R}^n & \xrightarrow{\phi_1 \bigcup_{\mathrm{id}_{\partial(\mathbb{R}^n - \mathrm{Int}A)}, \phi_0} \phi_2} & \mathbb{R}^n \end{array}$$

This means that f and $f_0 = f_{01} \bigcup_{\mathrm{id}_{\partial M_2}, \phi_0} f_{02}$ are C^∞ equivalent. By the construction and Definition 5, f_0 satisfies the followings.

(1) f_0 is a fold map.
(2) $f_0|_{S(f_0)}$ is a C^∞ embedding and $f_0(S(f_0)) = \sqcup_{r=1}^{l} \partial {D^n}_r$.

More precisely, (1) is derived from the fact that f is a fold map and (2) is derived from the definitions of diffeomorphisms ϕ_1 and ϕ_2.

This means that f_0 is a normal form of f and hence Definition 4 and Definition 5 are equivalent. □

Example 2 (1) In Section 5 of [10], special generic maps from C^∞ homotopy spheres into $\mathbb{R}^2$ are constructed and they are round. Each of them has the Reeb space homeomorphic to D^2.

(2) We may regard that Figure 7 (b) of [6] or Figure 4 of the present paper represents the Reeb space of a round fold map into the plane whose singular set consists of three connected components and whose source manifold is a homotopy sphere. Regular fibers of the map are disjoint unions of finite copies of spheres. The Reeb space is homeomorphic to a polyhedron represented as $D^2 \bigcup_{\phi} (S^1 \times [0,1])$ for a homeomorphism ϕ from a connected component of $\partial(S^1 \times [0,1])$ onto a circle in the interior of D^2.

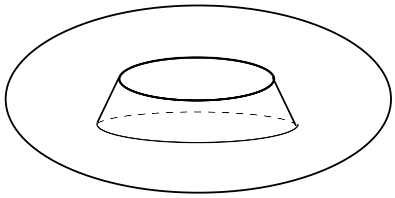

Figure 4. The Reeb space of the map in (2) of Example 2 ([6, Figure 7 (b)]).

(3) We may regard that Figure 8 of [13] or Figure 5 of the present paper represents the Reeb space of a round fold map into the plane whose singular set consists of two connected components and whose source manifold is a manifold having the structure of a C^∞ S^2-bundle over S^2. Regular fibers of the map are disjoint unions of finite copies of S^2. The Reeb space is given by glueing two copies of 2-dimensional closed discs by a homeomorphism from the boundary of one disc onto a circle in the interior of another disc.

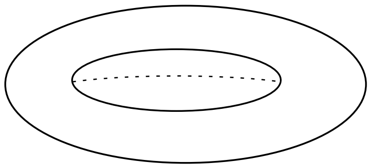

Figure 5. The Reeb space of the map in (3) of Example 2 ([13, Figure 8]).

We can construct a round fold map as in the following manner. We use this construction in the proceeding sections.

Before the construction, we introduce *good* Morse functions on compact C^∞ manifolds possibly with non-empty boundaries. A Morse function on a compact manifold with non-empty boundary is said to be *good* if on the boundary, it is constant and minimal, singular points of it are not on the boundary and at any two distinct singular points, the singular values are distinct. A Morse function on a closed or compact manifold without boundary is said to be *good* if at any two distinct singular points, the singular values are distinct.

Let $\bar{M}$ be a compact C^∞ manifold with non-empty boundary $\partial\bar{M}$. Then there exists a good Morse function $\tilde{f} : \bar{M} \to [a, +\infty)$, where a is the minimal value.

Let $\Phi : \partial(\bar{M} \times \partial(\mathbb{R}^n - \mathrm{Int}D^n)) \to \partial(\partial\bar{M} \times D^n)$ and $\phi : \partial(\mathbb{R}^n - \mathrm{Int}D^n) \to \partial D^n$ be diffeomorphisms. Let $p_1 : \partial\bar{M} \times \partial(\mathbb{R}^n - \mathrm{Int}D^n) \to \partial(\mathbb{R}^n - \mathrm{Int}D^n)$ and $p_2 : \partial\bar{M} \times \partial D^n \to \partial D^n$ be the canonical projections. Suppose that the following diagram commutes.

$$\begin{array}{ccc}
\partial\bar{M} \times \partial(\mathbb{R}^n - \mathrm{Int}D^n) & \xrightarrow{\Phi} & \partial\bar{M} \times \partial D^n \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\
\partial(\mathbb{R}^n - \mathrm{Int}D^n) & \xrightarrow{\phi} & \partial D^n
\end{array}$$

By using Φ, we construct $M := (\partial\bar{M} \times D^n)\bigcup_\Phi(\bar{M} \times \partial(\mathbb{R}^n - \mathrm{Int}D^n))$. Let $p : \partial\bar{M} \times D^n \to D^n$ be the canonical projection. Then a C^∞ map $f := p\bigcup_{\Phi,\phi}(\tilde{f} \times \mathrm{id}_{S^{n-1}})$ is a round fold map whose source manifold is M.

If $\bar{M}$ is a compact C^∞ manifold without boundary, then there exists a good Morse function $\tilde{f} : \bar{M} \to [a, +\infty)$ such that $\tilde{f}(\bar{M}) \subset (a, +\infty)$. We are enough to consider $\tilde{f} \times \mathrm{id}_{S^{n-1}}$ and embed $[a, +\infty) \times S^{n-1}$ into $\mathbb{R}^n$ to construct a round fold map whose source manifold is $\bar{M} \times S^{n-1}$.

3.2. Round fold maps and homology groups of source manifolds

Throughout this subsection, let $f : M \to \mathbb{R}^n$ be a round fold map and let L be its axis. We study relations between homology groups of M and those of $f^{-1}(L)$.

Let f be a normal form of a round fold map and $P^{(1)} := {D^n}_{1/2}$. We set $E := f^{-1}(P^{(1)})$ and $E' := M - f^{-1}(\mathrm{Int}P^{(1)})$. We set $F := f^{-1}(p)$ for $p \in \partial P^{(1)}$. We put $P^{(2)} := \mathbb{R}^n - \mathrm{Int}P^{(1)}$. Let $f_1 := f|_E : E \to P^{(1)}$ if F is non-empty and let $f_2 := f|_{E'} : E' \to P^{(2)}$.

f_1 gives the structure of a trivial C^∞ bundle over $P^{(1)}$ and $f_1|_{\partial E}$: $\partial E \to \partial P^{(1)}$ gives the structure of a trivial C^∞ bundle over $\partial P^{(1)}$ if F is non-empty. $f_2|_{\partial E'} : \partial E' \to \partial P^{(2)}$ gives the structure of a trivial C^∞ bundle over $\partial P^{(2)}$.

We can give E' and $q_f(E')$ the structures of bundles over $\partial P^{(2)}$ as follows.

Since for $\pi_P(x) := (1/2)(x/|x|)$ $(x \in P^{(2)})$, $\pi_P \circ f|_{E'}$ is a proper C^∞ submersion, this map gives E' the structure of a C^∞ $f^{-1}(L)$-bundle over $\partial P^{(2)}$ (apply Ehresmann's fibration theorem [2]).

For $p \in q_f(E')$ and $p_1, p_2 \in {q_f}^{-1}(p)$, the equation $\pi_P \circ f(p_1) = \pi_P \circ f(p_2)$ holds. We can correspond $\pi_P \circ f(p_1) = \pi_P \circ f(p_2) = \pi_P \circ \bar{f}(p)$ to p. The resulting map from $q_f(E')$ into $\partial P^{(2)}$ gives the structure of a $\bar{f}^{-1}(L)$-bundle since $\pi_P \circ f|_{E'}$ gives the structure of a C^∞ bundle and $q_f(E')$ is the quotient space of E' by $\sim_f$.

For a round fold map f which is not a normal form, we can consider similar maps and similar structures of bundles.

Now, we introduce (algebraic) topological conditions for round fold maps as the following definition. Here we use the notations above.

Definition 6 Let $f : M \to \mathbb{R}^n$ be a round fold map, and let R be a commutative group.

(1) If the natural projection $\pi_P \circ f_2$ from the total space of the bundle E' onto the base space $\partial P^{(2)}$ ($\partial P^{(2)}$ is diffeomorphic to S^{n-1}) gives the structure of a topologically trivial bundle, then f is said to be *topologically trivial*.
(2) If the natural projection $\pi_P \circ \bar{f}_2$ from the total space of the bundle $q_{f_2}(E')$ onto the base space $\partial P^{(2)}$ ($\partial P^{(2)}$ is diffeomorphic to S^{n-1}) gives the structure of a topologically trivial bundle, then f is said to be *topologically quasi-trivial*.
(3) If $H_*(E'; R) \cong H_*(\partial P^{(2)} \times f^{-1}(L); R)$ $(\cong H_*(\partial P^{(2)}; R) \otimes H_*(f^{-1}(L); R))$, then f is said to be *homologically product about* R.
(4) Suppose that f is homologically product about R and that the following diagram commutes for the canonical projection $p : \partial P^{(2)} \times f^{-1}(L) \to \partial P^{(2)}$ and two maps π_P and f_2.

$$\begin{array}{ccc} H_k(E';R) & \xrightarrow{\cong} & H_k(\partial P^{(2)} \times f^{-1};R) \\ \downarrow {\scriptstyle (\pi_P \circ f_2)_*} & & \downarrow {\scriptstyle p_*} \\ H_k(\partial P^{(2)};R) & \xrightarrow{\cong} & H_k(\partial P^{(2)};R) \end{array}$$

Then f is *said to be homologically R-trivial.*

For any commutative group R, if f is topologically trivial, then f is homologically product about R and it is homologically R-trivial.

Example 3 (1) Let M be a C^∞ homotopy sphere of dimension m. Let $f : M \to \mathbb{R}^n$ be a round fold map such that $f(M)$ is diffeomorphic to D^n and that the singular set $S(f)$ is connected (f is special generic, too). Then f is topologically trivial since the natural projection from E' onto $\partial P^{(2)}$ as in Definition 6 gives the structure of a linear bundle and the restriction of the projection to $\partial E'$ gives the structure of a trivial C^∞ bundle. Furthermore, f is homologically R-trivial for any commutative group R. For special generic maps of homotopy spheres into Euclidean spaces, see [10].

(2) The round fold map in (2) of Example 2 is topologically trivial since the group of all the orientation-preserving homeomorphisms of a disc is connected.

(3) Topologically trivial round fold maps are easy to construct by the method in Subsection 3.1.

(4) Round fold maps whose fibers are always connected are topologically quasi-trivial. In fact the Reeb spaces are homeomorphic to D^n or $S^{n-1} \times [0,1]$.

(5) In [19], one can find a round fold map f from a closed C^∞ manifold M of dimension 4 into $\mathbb{R}^2$ whose singular set consists of three connected components such that the followings hold.
   - (a) We can denote by $\{U_0\} \sqcup \{U_\infty\} \sqcup \{U_1, U_2\}$ the set of all the connected components of $\mathbb{R}^n - f(S(f))$ as in Figure 2. For any $p \in U_0 \bigcup U_2$, $f^{-1}(p)$ is diffeomorphic to S^2 and for any $p \in U_1$, $f^{-1}(p)$ is diffeomorphic to two copies of S^2.
   - (b) For the closure $\bar{U_1}$ of U_1, $\bar{f}^{-1}(\bar{U_1}) \subset W_f$ is homeomorphic to a Klein Bottle.

   The map f is not topologically quasi-trivial.

(6) In [10], closed C^∞ manifolds which admit special generic maps into $\mathbb{R}^2$ are determined. $S^1 \times S^{m-1}$ is one of such manifolds and admits a special generic map which is round and whose Reeb space is homeomorphic to $S^1 \times [0,1]$. For some orientation reversing diffeomorphism τ, there also exists a round fold map from the non-orientable C^∞ S^{m-1}-bundle over S^1 whose monodromy is $[\tau] \in \pi_0(\mathrm{Diff}^\infty(S^{m-1}))$ into $\mathbb{R}^2$ which is special generic and whose Reeb space is $S^1 \times [0,1]$. This is not homologically product about $\mathbb{Z}$ and topologically trivial, either.

Theorem 1 *Let M be a closed and connected C^∞ manifold of dimension m, $f : M \to \mathbb{R}^n$ $(m > n \geq 2)$ be a round fold map and $f(M)$ be diffeomorphic to D^n. Let F be the fiber of a point in a proper core of f and R be a commutative group.*

(1) *Assume that F is a disjoint union of a finite number of R-homology $(m-n)$-spheres. Assume also that f is homologically product about R. Then, for an axis L, we have $H_k(M;R) \cong H_k(f^{-1}(L);R)$ for $k \leq \min\{n-2, m-n-1\}$.*
(2) *Assume that M is orientable. Assume also that F is an R-homology sphere and that f is homologically R-trivial. Then for an axis L, we have*

$$H_k(M;R) \cong H_k(f^{-1}(L);R) \quad (k \leq n-2)$$

and

$$H_{n-1}(M;R) \cong \begin{cases} \{0\} & n-1 > m-n \\ H_{n-1}(f^{-1}(L);R) & n-1 \leq m-n. \end{cases}$$

Furthermore, if $n-1 < m-n$, then for $n-1 < k \leq m-n$, we have

$$H_k(M;R) \cong (R \otimes H_{k-n+1}(f^{-1}(L);R)) \oplus H_k(f^{-1}(L);R).$$

This theorem is shown by just calculations of homology groups. Let $P^{(1)}$ be a proper core of the map f, $P^{(2)} := \mathbb{R}^n - \mathrm{Int}P^{(1)}$, $E := f^{-1}(P^{(1)})$ and $E' := f^{-1}(P^{(2)})$ as before.

Remark 1 Assume that the fiber F of a point in a proper core is empty in Theorem 1. Then $M = E'$ holds. If f is homologically product about a commutative group R, then $H_k(M;R) \cong H_k(E';R)$ is isomorphic to $H_k(f^{-1}(L);R)$ for $k < n-1$ and $H_k(M;R) \cong H_k(E';R)$ is isomorphic to $(H_{k-n+1}(f^{-1}(L);R) \otimes H_{n-1}(\partial P^{(2)};R)) \oplus (H_k(f^{-1}(L);R) \otimes H_0(\partial P^{(2)};R))$ for $n-1 \leq k \leq m$ by the definition of a round fold map which is homologically product about R. Note that by this definition, we do not need to assume that M is connected in this situation.

Proof of Theorem 1. The manifold M is assumed to be connected and it follows easily that two manifolds E' and $f^{-1}(L)$ are connected. So $H_0(M;R) \cong H_0(E';R) \cong H_0(f^{-1}(L);R) \cong R$ holds.

The following exact sequences hold where $i : \partial E \to E$, $j : E \to M$, $i' : \partial E' \to E'$ and $j' : E' \to M$ are natural inclusions.

$$\begin{array}{ccccccc}
\longrightarrow H_k(\partial E;R) & \xrightarrow{(i_*,{i'}_*)} & H_k(E;R) \oplus H_k(E';R) & \xrightarrow{j_*-{j'}_*} & H_k(M;R) \longrightarrow \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\longrightarrow H_k(\partial P^{(1)} \times F;R) & \longrightarrow & H_k(P^{(1)} \times F;R) \oplus H_k(E';R) & \longrightarrow & H_k(M;R) \longrightarrow
\end{array}$$

If $0 < k \leq \min\{n-2, m-n-1\}$, then it holds that $H_k(\partial P^{(1)} \times F;R) \cong \{0\}$ and $H_k(P^{(1)} \times F;R) \cong \{0\}$. So, by the exact sequences, if the map f is homologically product about R, then we have $H_k(M;R) \cong H_k(E';R) \cong H_k(\partial P^{(2)} \times f^{-1}(L);R) \cong H_k(f^{-1}(L);R)$ for $0 \leq k \leq \min\{n-2, m-n-1\}$. This completes the proof of (1).

Now we prove (2). Since the manifold M is assumed to be orientable, two manifolds E' and $f^{-1}(L)$ are orientable. The map f is assumed to be homologically R-trivial, so we have $H_k(M;R) \cong H_k(E';R) \cong H_k(f^{-1}(L);R)$ for $0 \leq k \leq \min\{n-2, m-n-1\}$ as the previous case.

Suppose $m-n < n-1$. Since E is diffeomorphic to $D^n \times F$, $\partial E = \partial E'$ is diffeomorphic to $\partial D^n \times F$ and F is an R-homology sphere of dimension $m-n$, the homomorphism $i_* : H_{m-n}(\partial E;R) \cong H_{m-n}(F;R) \cong R \to H_{m-n}(E;R) \cong H_{m-n}(\partial E;R) \cong R$ is an isomorphism. The homomorphism ${i'}_* : H_{m-n}(\partial E';R) \to H_{m-n}(E' : R)$ is 0 since a fiber F of the trivial bundle $\partial E'$ over $\partial P^{(2)}$ is an R-homology sphere of dimension $m-n$ bounding an orientable compact manifold $f^{-1}(L)$, which is a fiber of the bundle E' over $\partial P^{(2)}$. For $m-n < k < n-1$, it holds that $H_k(\partial P^{(1)} \times F;R) \cong \{0\}$ and $H_k(P^{(1)} \times F;R) \cong \{0\}$. Together with the result of (1), by the exact sequences and the definition of a homologically R-trivial round fold map, we have $H_k(M;R) \cong H_k(E';R) \cong H_k(\partial P^{(2)} \times f^{-1}(L);R) \cong H_k(f^{-1}(L);R)$ for $0 \leq k \leq n-2$.

The homomorphism $i_* : H_{n-1}(\partial E;R) \to H_{n-1}(E;R) \cong \{0\}$ is surjective. The image of the homomorphism ${i'}_* : H_{n-1}(\partial E';R) \cong H_{n-1}(\partial P^{(2)};R) \to H_{n-1}(E';R) \cong H_{n-1}(\partial P^{(2)} \times f^{-1}(L);R)$ is regarded as $H_{n-1}(\partial P^{(2)};R) \oplus \{0\} \subset H_{n-1}(\partial P^{(2)};R) \oplus H_{n-1}(f^{-1}(L);R) \cong H_{n-1}(\partial P^{(2)} \times f^{-1}(L);R)$ and the homomorphism is injective since f is homologically R-trivial. Since the homomorphism $i_* : H_{n-2}(\partial E;R) \cong H_{n-2}(F;R) \to H_{n-2}(E;R) \cong H_{n-2}(\partial E;R)$ is an isomorphism, by the exact sequences, we have $H_{n-1}(E';R)/{i'}_*(H_{n-1}(\partial E';R)) \cong H_{n-1}(M;R)$. Since the map f is homologically R-trivial and $f^{-1}(L)$ is a compact and connected manifold of dimension $m-n+1 \leq n-1$ with non-empty boundary, we have $H_{n-1}(E';R) \cong (H_{n-1}(\partial P^{(2)};R) \otimes H_0(f^{-1}(L);R)) \oplus (H_0(\partial P^{(2)};R) \otimes H_{n-1}(f^{-1}(L);R)) \cong H_{n-1}(\partial P^{(2)};R) \otimes H_0(f^{-1}(L);R) \cong H_{n-1}(\partial P^{(2)};R)$ and we may regard $H_{n-1}(E';R) = {i'}_*(H_{n-1}(\partial E';R))$. Thus, we have $H_{n-1}(M;R) \cong \{0\}$.

Now suppose $n-1 < m-n$. Since E is diffeomorphic to $D^n \times F$, $\partial E = \partial E'$ is diffeomorphic to $\partial D^n \times F$ and F is an R-homology sphere of dimension $m-n$, the homomorphism $i_* : H_{n-2}(\partial E;R) \to H_{n-2}(E;R)$ is an isomorphism and the homomorphism $i_* : H_{n-1}(\partial E;R) \cong R \to H_{n-1}(E;R) \cong \{0\}$ is zero. Since f is homologically R-trivial, the image of the homomorphism ${i'}_* : H_{n-1}(\partial E';R) \cong H_{n-1}(\partial P^{(2)};R) \to H_{n-1}(E';R) \cong H_{n-1}(\partial P^{(2)};R) \oplus H_{n-1}(f^{-1}(L);R)$ is regarded as $H_{n-1}(\partial P^{(2)};R) \oplus \{0\}$ and the homomorphism is injective. Thus, by the exact sequences, we have $H_{n-1}(M;R) \cong H_{n-1}(f^{-1}(L);R)$.

For $n-1 < k < m-n$, it holds that $H_k(\partial P^{(1)} \times F;R) \cong H_k(P^{(1)} \times F;R) \cong \{0\}$. The homomorphism $i_* : H_k(\partial E;R) \cong \{0\} \to H_k(E;R) \cong \{0\}$ is an isomorphism and the homomorphism ${i'}_* : H_k(\partial E';R) \cong \{0\} \to H_k(E';R)$ is zero.

Since F is an R-homology sphere of dimension $m-n$, the homomorphism $i_* : H_{m-n}(\partial E;R) \cong R \to H_{m-n}(E;R) \cong R$ is an isomorphism. Since F is an R-homology sphere bounding an orientable compact manifold $f^{-1}(L)$, we have that the image of the homomorphism ${i'}_* : H_{m-n}(\partial E';R) \to H_{m-n}(E';R)$ is $\{0\}$ as before. Thus, by the exact sequences and by the definition of a homologically R-trivial round fold map, we have $H_k(M;R) \cong H_k(E';R) \cong H_k(\partial P^{(2)} \times f^{-1}(L);R) \cong (R \otimes H_{k-n+1}(f^{-1}(L);R)) \oplus H_k(f^{-1}(L);R)$ for $n-1 < k \leq m-n$.

Now suppose $n-1 = m-n$. Then the homomorphism $i_* : H_{n-1}(\partial E;$

$R) \cong R \oplus R \to H_{n-1}(E;R) \cong R$ is surjective by the diffeomorphism types of E and $\partial E = \partial E'$. Since f is homologically R-trivial and F is an R-homology sphere of dimension $m-n$ bounding an orientable compact manifold $f^{-1}(L)$, the image of the homomorphism ${i'}_* : H_{n-1}(\partial E';R) \cong H_{n-1}(\partial P^{(2)};R) \oplus H_{n-1}(F;R) \to H_{n-1}(E';R) \cong H_{n-1}(\partial P^{(2)};R) \oplus H_{n-1}(f^{-1}(L);R)$ is regarded as $H_{n-1}(\partial P^{(2)};R) \times \{0\}$. For $0 < k \leq \min\{n-2, m-n-1\} = n-2 = m-n-1$, it holds that $H_k(\partial P^{(1)} \times F;R) \cong \{0\}$ since ∂E is diffeomorphic to $\partial D^n \times F$. Thus, by the exact sequences, we have $H_{n-1}(M;R) \cong H_{n-1}(f^{-1}(L);R)$.

This completes the proof. □

In the situation of Theorem 1, if M is orientable and $f^{-1}(L)$ is an R-homology disc, then $\partial f^{-1}(L)$ is an R-homology sphere and f is homologically R-trivial. We can apply (2) of Theorem 1 and we have $H_k(M;R) \cong H_k(f^{-1}(L);R) \cong \{0\}$ and $H^k(M;R) \cong H^k(f^{-1}(L);R) \cong \{0\}$ for $0 < k \leq \max\{n-1, m-n\}$. By virtue of Poincare duality theorem, M is an R-homology sphere of dimension m.

Conversely, in the situation of (2) of Theorem 1, if M is an R-homology sphere and $n-1 \geq m-n$, then $f^{-1}(L)$ is an R-homology disc.

Now we have the following corollaries.

Corollary 1 *In the situation of Theorem* 1*, we assume that M is orientable. If $f^{-1}(L)$ is an R-homology disc, then M is an R-homology sphere.*

Corollary 2 *In the situation of* (2) *of Theorem* 1*, if M is an R-homology sphere and $n-1 \geq m-n$, then $f^{-1}(L)$ is an R-homology disc.*

Example 4 By applying the construction in Subsection 3.1, we obtain a round fold map $f : M \to \mathbb{R}^n$ which is topologically trivial, homologically R-trivial and satisfies the assumption of Theorem 1. If in this mentioned construction, the compact C^∞ manifold $\bar{M}$ is an R-homology disc and orientable, then the resulting source manifold is an orientable R-homology sphere by Corollary 1.

4. Round fold maps whose regular fibers are disjoint unions of spheres

Theorem 7.1 or Corollary 7.3 of [6] is a proposition for simple fold maps whose regular fibers are disjoint unions of spheres. In [13], Saeki and Suzuoka studied stable maps including stable fold maps with such regular fibers; mainly ones from closed 4-dimensional manifolds into surfaces without boundaries.

In this section, we study round fold maps whose regular fibers are disjoint unions of spheres. A lot of maps in Example 2 and Example 3 are examples of such maps. Here we review some of the results of [6] and [13] (Propositions 5, 6 and 7) and show two theorems (Theorems 2 and 3).

Theorem 2 *Let M be a closed and orientable C^∞ manifold of dimension m and $f : M \to \mathbb{R}^n$ $(m > n \geq 2)$ be a round fold map which is homologically product about $\mathbb{Z}$.*

Furthermore, assume that for each regular value p, $f^{-1}(p)$ is a disjoint union of almost-spheres. Let F be the fiber of a point in a proper core of f and suppose that F is empty or connected. If F is non-empty, then we also assume that M is connected and that f is homologically $\mathbb{Z}$-trivial.

Then, the k-th homology group $H_k(M;\mathbb{Z})$ is torsion-free for any k.

To show that the homology groups are torsion-free, we need the following proposition, which is a part of Corollary 3.17 of [11].

Proposition 3 (Saeki, [11]) *Suppose that M is a closed and orientable C^∞ manifold of dimension $m > 1$. Let $f : M \to \mathbb{R}$ be a Morse function such that each regular fiber is a disjoint union of almost-spheres. Then, the k-th homology group $H_k(M;\mathbb{Z})$ is torsion-free for any k.*

Proof of Theorem 2. If F is empty, then we have $H_k(M;\mathbb{Z}) \cong H_k(f^{-1}(L);\mathbb{Z})$ $(0 \leq k < n-1)$ and $H_k(M;\mathbb{Z}) \cong H_k(S^{n-1} \times f^{-1}(L);\mathbb{Z}) \cong (H_{k-n+1}(f^{-1}(L);\mathbb{Z}) \otimes \mathbb{Z}) \oplus H_k(f^{-1}(L);\mathbb{Z})$ $(n-1 \leq k \leq m)$. Furthermore, $H_k(M;\mathbb{Z})$ is torsion-free for any k since $H_k(f^{-1}(L);\mathbb{Z})$ is torsion-free by Proposition 3.

Let F be connected and non-empty. By the assumption that M is connected and orientable, we can apply Poincare duality theorem. By Proposition 3, $H_k(f^{-1}(L);\mathbb{Z})$ is torsion-free for any k. By (2) of Theorem 1, $H_k(M;\mathbb{Z})$ is torsion-free for $0 \leq k \leq \max\{n-1, m-n\}$. $H^k(M;\mathbb{Z})$ is also torsion-free for $0 \leq k \leq \max\{n-1, m-n\}$ by virtue of universal coefficient theorem. By virtue of Poincare duality theorem, $H_k(M;\mathbb{Z})$ is torsion-free for any k. □

For example, most of round fold maps in Example 3 satisfy the assumption of Theorem 2. It is also not so difficult to construct such round fold maps. In fact, we are enough to apply the method in Subsection 3.1.

The following proposition is a part of Theorem 4.1 of [13]. Note that as in Proposition 1, for a simple fold map $f : M \to N$, W_f is given the structure of a polyhedron.

Proposition 4 ([13]) *Let $f : M \to N$ be a simple fold map from a closed C^∞ manifold M of dimension 4 into a C^∞ manifold N of dimension 2 without boundary. For each regular value p, let $f^{-1}(p)$ be a disjoint union of finite copies of S^2.*

Then there exist a compact C^∞ manifold W of dimension 5 such that $\partial W = M$ and a continuous map $r : W \to W_f$ such that $r|_{\partial W}$ coincides with $q_f : M \to W_f$ and the followings hold.

(1) *For each $p \in W_f - q_f(S(f))$, $r^{-1}(p)$ is diffeomorphic to D^3.*
(2) *$\bar{f} \circ r$ is a C^∞ submersion.*
(3) *There exist a C^∞ triangulation of W and a triangulation of W_f such that r is a simplicial map.*
(4) *For each $p \in W_f$, $r^{-1}(p)$ collapses to a point and r is a homotopy equivalence.*
(5) *W collapses to a subpolyhedron ${W_f}'$ such that $r|_{{W_f}'} : {W_f}' \to W_f$ is a* PL *homeomorphism.*

As a corollary to Proposition 4, we have the following corollary.

Corollary 3 ([13]) *In the situation of Proposition 4, let M be connected and $i : M \to W$ be the natural inclusion. Then*

$$q_{f*} = r_* \circ i_* : \pi_k(M) \to \pi_k(W_f)$$

gives an isomorphism for $k = 0, 1$.

We note that Proposition 4 above is for simple fold maps from closed 4-dimensional manifolds into surfaces without boundaries. We can generalize it to the following Lemma 1.

Lemma 1 *Let M be a closed C^∞ manifold of dimension m, N be a C^∞ manifold without boundary of dimension n and $m > n \geq 1$. If $m - n = 1$, then we also assume that M is orientable.*

Let $f : M \to N$ be a simple fold map. We assume that for each regular value p, $f^{-1}(p)$ is a disjoint union of almost-spheres and that the indices of *all the fold points of f are 0 or 1.*

Then there exist a compact PL *$(m+1)$-manifold W such that $\partial W = M$ and a continuous map $r : W \to W_f$ such that $r|_{\partial W}$ coincides with $q_f : M \to W_f$. Furthermore, the followings hold.*

(1) *For each $p \in W_f - q_f(S(f))$, $r^{-1}(p)$ is* PL *homeomorphic to D^{m-n+1}.*
(2) *There exist a triangulation of W and a triangulation of W_f such that r is a simplicial map.*
(3) *For each $p \in W_f$, $r^{-1}(p)$ collapses to a point and r is a homotopy equivalence.*
(4) *W collapses to a subpolyhedron ${W_f}'$ such that $r|_{{W_f}'} : {W_f}' \to W_f$ is a* PL *homeomorphism.*

We prove the lemma by analogy of the proof of Theorem 4.1 of [13]. In the proof, we often apply Proposition 1 implicitly.

Proof of Lemma 1. We construct a compact manifold of dimension $m+1$ bounded by M.

STEP 1. Around a regular neighborhood of $q_f(F_0(f))$

$q_f(F_0(f))$ is the image of the set $F_0(f)$ of all the definite fold points of f. Let $N(q_f(F_0(f)))$ be a small regular neighborhood of $q_f(F_0(f))$.

We note that $N(q_f(F_0(f)))$ has the structure of a trivial bundle over $q_f(F_0(f))$ and that all the fibers are PL homeomorphic to $[0,1]$. We may assume that $q_f(F_0(f))$ corresponds to the 0-section ($\{0\} \subset [0,1]$).

For each $p \in q_f(F_0(f))$, set $K_p := {q_f}^{-1}(\{p\} \times [0,1])$ for a fiber $\{p\} \times [0,1]$ of the bundle $N(q_f(F_0(f)))$ over $q_f(F_0(f))$. K_p is diffeomorphic to D^{m-n+1}. We may assume that $q_f^{-1}(N(q_f(F_0(f))))$ has the structure of a C^∞ bundle over $q_f(F_0(f))$ and K_p is the fiber over a point $p \in q_f(F_0(f))$. We define a Morse function ${q_f}_p : K_p \to [0,1]$ with exactly one minimal point, which is the only singular point of the function, such that ${q_f}_p(K_p) = [0,1]$. Then ${q_f}_p$ ($p \in q_f(F_0(f))$) can be extended to a family of C^∞ submersions $\widetilde{{q_f}_p} : \widetilde{K_p} \to \{p\} \times [0,1]$ of $(m-n+2)$-dimensional submanifolds $\widetilde{K_p}$ ($p \in q_f(F_0(f))$) such that the followings hold.

(1) For $p \in q_f(F_0(f))$ and for $t \in (0,1]$, ${\widetilde{{q_f}_p}}^{-1}(p,t)$ is diffeomorphic to D^{m-n+1}.
(2) ${\widetilde{{q_f}_p}}^{-1}(p,0)$ is a point.

(3) $\widetilde{q_f}_p^{-1}(\{p\} \times [0,1]) = \widetilde{K_p}$ is a C^∞ manifold of dimension $m-n+2$ with corner along $\widetilde{q_f}_p^{-1}(p,1)$.
(4) $\widetilde{q_f}_p^{-1}(\{p\} \times [0,1]) = \widetilde{K_p}$ is diffeomorphic to D^{m-n+2} after the corner is smoothed.

Now we can construct a compact C^∞ manifold V_0 of dimension $m+1$ having the structure of a C^∞ D^{m-n+2}-bundle over a C^∞ manifold $q_f(F_0(f))$ such that the bundle $q_f^{-1}(N(q_f(F_0(f))))$ is a subbundle of the bundle V_0 and that the subbundle $q_f^{-1}(N(q_f(F_0(f))))$ is in the boundary of V_0. Note that $\widetilde{K_p}$ is the fiber over a point $p \in q_f(F_0(f))$. We also have a PL map $r_0 : V_0 \to N(q_f(F_0(f)))$ such that ${r_0}^{-1}(\{p\} \times [0,1]) = \widetilde{K_p}$ and $r_0|_{\widetilde{K_p}} = \widetilde{q_f}_p$ hold for all $p \in q_f(F_0(f))$ and that $r_0|_{{q_f}^{-1}(N(q_f(F_0(f))))} = q_f|_{{q_f}^{-1}(N(q_f(F_0(f))))}$ holds. Furthermore, there exists a PL submanifold $N'(q_f(F_0(f))) \subset V_0$ of dimension n such that the followings hold (Figure 6).

(1) $N'(q_f(F_0(f))) \bigcap \partial\widetilde{K_p}$ consists of two points in $\partial N'(q_f(F_0(f)))$. One of these points is in $q_f^{-1}(N(q_f(F_0(f))))$ and the other point is not in $q_f^{-1}(N(q_f(F_0(f))))$.
(2) V_0 collapses to $N'(q_f(F_0(f)))$.
(3) $N'(q_f(F_0(f)))$ has the structure of a subbundle of the bundle V_0.
(4) $r_0|_{N'(q_f(F_0(f)))} : N'(q_f(F_0(f))) \to N(q_f(F_0(f)))$ is a PL homeomorphism and a bundle isomorphism between the two PL bundles.

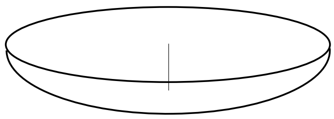

Figure 6. A fiber $\widetilde{K_p}$ over a point $p \in q_f(F_0(f))$ of the bundle V_0. (The segment in the center is a fiber over $p \in q_f(F_0(f))$ of the bundle $N'(q_f(F_0(f)))$. A fiber over p of the bundle ${q_f}^{-1}(N(q_f(F_0(f))))$ is the spherical part of the boundary of $\widetilde{K_p}$.)

STEP 2. Around a regular neighborhood of $q_f(F_1(f))$

$q_f(F_1(f))$ is the image of the set $F_1(f)$ of all the fold points of f whose indices are 1. Note that $q_f : F_1(f) \to W_f$ is injective since f is simple. Let $N(q_f(F_1(f)))$ be a small regular neighborhood of $q_f(F_1(f))$. By the assumptions on the pair (m, n) of dimensions and the orientability of M

and by Proposition 1 (6), $N(q_f(F_1(f)))$ has the structure of a PL K-bundle over $q_f(F_1(f))$ where $K := \{r\exp(2\pi i\theta) \in \mathbb{C} \mid 0 \leq r \leq 1, \theta = 0, 1/3, 2/3\}$. We may assume that $q_f(F_1(f))$ corresponds to the 0-section ($\{0\} \subset K$).

For each $p \in q_f(F_1(f))$, set $K_p := {q_f}^{-1}(\{p\} \times K)$ for a fiber $\{p\} \times K$ of the bundle $N(q_f(F_1(f)))$ over $q_f(F_1(f))$. K_p is PL homeomorphic to S^{m-n+1} with the interior of the union of three disjoint $(m-n+1)$-dimensional standard closed discs removed. We may assume that $q_f^{-1}(N(q_f(F_1(f))))$ has the structure of a C^∞ bundle over $q_f(F_1(f))$ and K_p is the fiber over a point $p \in q_f(F_1(f))$. We define a Morse function ${q_f}_p : K_p \to [0,1]$ with exactly one singular point such that ${q_f}_p(K_p) = [0,1]$, that ${{q_f}_p}^{-1}(0)$ is an almost-sphere, that ${{q_f}_p}^{-1}(1)$ is a disjoint union of two almost-spheres and that the singular value is $t_0 \in (0,1)$. Then ${q_f}_p$ ($p \in q_f(F_1(f))$) can be extended to a family of PL maps $\widetilde{{q_f}_p} : \widetilde{K_p} \to \{p\} \times K$ ($p \in q_f(F_1(f))$) such that the followings hold.

(1) For any $p \in q_f(F_1(f))$, $\widetilde{{q_f}_p}^{-1}(p,t)$ is PL homeomorphic to D^{m-n+1} for $t \in K - \{0\}$
(2) $\widetilde{{q_f}_p}^{-1}(\{p\} \times K) = \widetilde{K_p}$ is a PL manifold of dimension $m-n+2$ and PL homeomorphic to D^{m-n+2}.

Then, by an argument similar to those in the previous step, we can construct a compact PL manifold V_1 of dimension $m+1$ having the structure of a PL D^{m-n+2}-bundle over a C^∞ manifold $q_f(F_1(f))$ such that the bundle $q_f^{-1}(N(q_f(F_1(f))))$ (over $q_f(F_1(f))$) is a subbundle of the bundle V_1 and that the subbundle $q_f^{-1}(N(q_f(F_1(f))))$ is in the boundary of V_1. Note that $\widetilde{K_p}$ is the fiber over a point $p \in q_f(F_1(f))$. We also have a PL map $r_1 : V_1 \to N(q_f(F_1(f)))$ such that ${r_1}^{-1}(\{p\} \times K) = \widetilde{K_p}$ and $r_1|_{\widetilde{K_p}} = \widetilde{{q_f}_p}$ hold for all $p \in q_f(F_1(f))$ and that $r_1|_{{q_f}^{-1}(N(q_f(F_1(f))))} = q_f|_{{q_f}^{-1}(N(q_f(F_1(f))))}$ holds. We also have a subpolyhedron $N'(q_f(F_1(f))) \subset V_1$ of dimension n such that the followings hold (Figure 7).

(1) $N'(q_f(F_1(f))) \bigcap \partial\widetilde{K_p}$ consists of three points $(p,0)$, $(p, e^{(2/3)\pi i})$, $(p, e^{(4/3)\pi i}) \in \{p\} \times K$, the points are not in ${q_f}^{-1}(N(q_f(F_1(f))))$ and each connected component of $\partial\widetilde{K_p} - q_f^{-1}(N(q_f(F_1(f))))$ includes one of these three points.
(2) V_1 collapses to $N'(q_f(F_1(f)))$.
(3) $N'(q_f(F_1(f)))$ has the structure of a subbundle of the bundle V_1.

(4) $r_1|_{N'(q_f(F_1(f)))} : N'(q_f(F_1(f))) \to N(q_f(F_1(f)))$ is a PL homeomorphism and a bundle isomorphism between the two PL bundles.

Now we set the disjoint unions $V_S := V_0 \sqcup V_1$, $N_S := N(q_f(F_0(f))) \sqcup N(q_f(F_1(f))) \subset W_f$, $r_S := r_0 \sqcup r_1 : V_S \to N_S$ and $N'_S := N'(q_f(F_0(f))) \sqcup N'(q_f(F_1(f))) \subset V_S$.

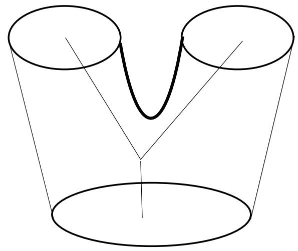

Figure 7. A fiber $\widetilde{K_p}$ over a point $p \in q_f(F_1(f))$ of the bundle V_1. (The Y-shaped graph in the center is a fiber over $p \in q_f(F_1(f))$ of the bundle $N'(q_f(F_1(f)))$. A fiber over p of the bundle ${q_f}^{-1}(N(q_f(F_1(f))))$ is in the boundary of $\widetilde{K_p}$.)

STEP 3. Around $R := W_f - \mathrm{Int}N_S$

Since each regular fiber of f is a disjoint union of almost-spheres, $q_f|_{{q_f}^{-1}(R)} : {q_f}^{-1}(R) \to R$ gives the structure of a bundle over R with a fiber PL homeomorphic to S^{m-n}. Here we define a bundle $r_R : V_R \to R$ whose fiber is D^{m-n+1} and which is an associated bundle of the bundle $q_f|_{{q_f}^{-1}(R)} : {q_f}^{-1}(R) \to R$. More precisely, we define the associated bundle so that the structure group is a group consisting of PL homeomorphisms satisfying the following; for any element r of the structure group, $r(0) = 0$ and for a PL homeomorphism r' on S^{m-n}, $r(x)/|x| = r'(x/|x|)$ $(x \neq 0)$. Let $R' \subset V_R$ be the 0-section of the associated bundle (the subbundle whose fiber is $\{0\} \subset D^{m-n+1}$).

Then it is easy to check that we can glue V_S and V_R together to give a compact PL $(m+1)$-manifold W, $r_S : V_S \to N_S$ and $r_R : V_R \to R$ together to give a PL map $r : W \to W_f$ and N'_S and R' together to give a polyhedron ${W_f}'$ of dimension n. From the construction, it is now easy to verify all the required conditions stated in Lemma 1. This completes the proof. □

Remark 2 If $m - n + 1$ is odd, then we don't need to assume that the indices of fold points are 0 or 1 in Lemma 1 as mentioned in Remark 7.2 of [6].

Corollary 4 *In the situation of Lemma 1, let M be connected and $i : M \to W$ be the natural inclusion. Then the homomorphism*

$$q_{f*} = r_* \circ i_* : \pi_k(M) \to \pi_k(W_f)$$

induced by q_f gives an isomorphism for $0 \leq k \leq m - n - 1$.

Proof. Since we have $q_f = r \circ i$ and r is a homotopy equivalence, we have only to show that the homomorphism $i_* : \pi_k(M) \to \pi_k(W)$ $(0 \leq k \leq m-n-1)$ induced by i is an isomorphism. Since W collapses to a polyhedron of dimension n, it admits a PL handlebody decomposition consisting of handles whose indices are not larger than n. Dualizing the handles, we see that W is obtained from $M \times [0, 1]$ by attaching handles whose indices are not smaller than $m - n + 1$ along $M \times \{1\}$. Hence, $i_* : \pi_k(M) \to \pi_k(W)$ $(0 \leq k \leq m - n - 1)$ is an isomorphism. This completes the proof. □

Now we introduce Theorem 7.1 (or Corollary 7.3) of [6] with some arrangements.

Proposition 5 ([6]) *Let $m \geq 4$, m be even and M be a closed C^∞ manifold of dimension m. Assume that $f : M \to \mathbb{R}^2$ is a simple fold map and that regular fibers of the map are disjoint unions of standard spheres. Assume also that $\pi_1(M) \cong \{0\}$ and $H_2(W_f; \mathbb{Z}) \cong \{0\}$ hold.*

Then M is a C^∞ homotopy sphere of dimension m.

The proof in [6] is summarized as follows in the terminologies of the present paper.

First, by certain operations (*R-operations*), we transform the given map without changing the diffeomorphism type of the source manifold so that the source manifold is represented as the connected sum of a finite number of the source manifolds of three types of maps; two of them are round fold maps whose Reeb spaces are homeomorphic to either D^2 or the one in Figure 4 and the third one is a simple fold map which is not round (for its Reeb space, refer to Figure 7 (c) of [6]). Then we prove that the source manifolds of these three types of maps are homotopy spheres.

In other words, the following was shown in the last part of the proof.

Proposition 6 *Let $m \geq 4$, m be even and M be a closed C^∞ manifold of dimension m. Assume that $f : M \to \mathbb{R}^2$ is a simple fold map and that regular fibers of the map are disjoint unions of standard spheres. Assume* *also that the Reeb space W_f is homeomorphic to either of the followings.*

(1) D^2.

(2) $D^2 \bigcup_{\phi} (S^1 \times K)$, *where* $K := \{r \exp(2\pi i\theta) \in \mathbb{C} \mid 0 \leq r \leq 1, \theta = 0, 1/3, 2/3\}$ *and* ϕ *is a homeomorphism from* $S^1 \times \{1\} (\subset S^1 \times K)$ *onto* ∂D^2.

Then M is a homotopy sphere of dimension m.

The following result, or Theorem 3 can be regarded as an extension of Proposition 6 and states that by studying their Reeb spaces, we can know some homotopy groups of manifolds admitting round fold maps whose regular fibers are disjoint unions of spheres, under some constraints.

Theorem 3 *Let M be a closed and connected C^∞ manifold of dimension m, $f : M \to \mathbb{R}^n$ be a round fold map and $m > n \geq 2$. If $m - n = 1$, then we also assume that M is orientable.*

We assume furthermore that $f^{-1}(p)$ is a disjoint union of almost-spheres for each regular value p and that the indices of all the fold points of f are 0 *or* 1.

Let L be an axis of f and $f_L := f|_{f^{-1}(L)}$. We denote by l_1 the number of loops of the Reeb space W_{f_L} of f_L (in other words, let $H_1(W_{f_L}; \mathbb{Z}) \cong \mathbb{Z}^{l_1}$). We denote by l_2 the number of connected components of the fiber of a point in a proper core of f.

Then there exist a PL *manifold W and a homotopy equivalence $r : W \to W_f$ as in Lemma* 1*. Furthermore, for the inclusion $i : M \to W$, $q_f = r \circ i$ gives an isomorphism of homotopy groups $\pi_k(M) \cong \pi_k(W_f)$ for $0 \leq k \leq m - n - 1$ and we have the following list where we denote the free group of rank r by F_r.*

(1) *When $n \geq 3$ and $m \geq 2n$ hold, we have the followings.*

$$\pi_k(M) \cong \pi_k(W_f) \cong \begin{cases} F_{l_1} & k = 1 \\ \{0\} & 2 \leq k < n - 1 \end{cases}$$

$$\pi_{n-1}(M) \cong \pi_{n-1}(W_f) \cong \begin{cases} \mathbb{Z} & l_2 = 0 \\ \{0\} & l_2 \neq 0 \end{cases}$$

(2) *When $n \geq 3$, $n < m \leq 2n-1$ and $m-n \geq 2$ hold, we have the following.*

$$\pi_k(M) \cong \pi_k(W_f) \cong \begin{cases} F_{l_1} & k=1 \\ \{0\} & 2 \leq k \leq m-n-1 \end{cases}$$

(3) *When $m \geq 4$ and $n = 2$ hold, we have the followings.*

(a) *If f is topologically quasi-trivial and $l_2 = 0$ holds, then we have the following.*

$$\pi_k(M) \cong \pi_k(W_f) \cong \begin{cases} \mathbb{Z} \times F_{l_1} & k=1 \\ \{0\} & 2 \leq k \leq m-3 \end{cases}$$

(b) *If f is topologically quasi-trivial and $l_2 \neq 0$ holds, then we have the following.*

$$\pi_1(M) \cong \pi_1(W_f) \cong F_{l_1}$$

Proof. The existence of W and r and the fact that q_f induces isomorphisms of homotopy groups easily follow from Lemma 1 and Corollary 4.

First we prove (1) and (2) of the list.

By the assumption, $n \geq 3$ holds. Since the boundary of a proper core of the map or S^{n-1} is simply-connected, the natural bundle $\bar{f}^{-1}(L)$ over the boundary of the proper core, whose fiber is a polyhedron of dimension 1 or a graph, is trivial. Thus, f is topologically quasi-trivial.

Suppose $l_2 = 0$. Then W_f is PL homeomorphic to $S^{n-1} \times T$, where T is a connected graph with l_1 loops. We have $\pi_1(W_f) \cong \pi_1(S^{n-1} \times T) \cong \pi_1(S^{n-1}) \oplus \pi_1(T) \cong F_{l_1}$, $\pi_k(W_f) \cong \pi_k(S^{n-1} \times T) \cong \pi_k(S^{n-1}) \oplus \pi_k(T) \cong \{0\}$ $(1 < k < n-1)$ and $\pi_{n-1}(W_f) \cong \pi_{n-1}(S^{n-1} \times T) \cong \pi_{n-1}(S^{n-1}) \oplus \pi_{n-1}(T) \cong \mathbb{Z}$.

Suppose $l_2 \geq 1$. Then W_f is represented as $A \bigcup_{\psi} B$, where A is a disjoint union of l_2 copies of D^n, B is the product of S^{n-1} and a connected graph T with l_1 loops and ψ is a homeomorphism from $S^{n-1} \times \Lambda \subset B$ onto ∂A, where Λ is a set consisting of l_2 degree 1 vertices of the previous graph. Then (W_f, B) is $(n-1)$-connected and we have $\pi_k(B) \cong \pi_k(W_f)$ for $0 \leq k \leq n-2$ by virtue of the homotopy exact sequence. It also follows that the inclusion from B into W_f induces a surjective homomorphism from $\pi_{n-1}(B) \cong \mathbb{Z}$ onto $\pi_{n-1}(W_f)$ which is zero. Thus we have $\pi_1(B) \cong \pi_1(W_f) \cong \pi_1(S^{n-1} \times T) \cong \pi_1(T) \cong F_{l_1}$, $\pi_k(B) \cong \pi_k(W_f) \cong \pi_k(S^{n-1} \times T) \cong \pi_k(S^{n-1}) \oplus \pi_k(T) \cong \{0\}$ $(1 < k < n-1)$ and $\pi_{n-1}(W_f) \cong \{0\}$.

This completes the proof of (1) and (2) of the list.

We prove (3) of the list (the case where $n = 2$).

Suppose $l_2 = 0$. By the extra assumption in (a) of the list, f is topologically quasi-trivial and W_f is PL homeomorphic to $S^1 \times T$, where T is a connected graph with l_1 loops. We have $\pi_1(M) \cong \pi_1(W_f) \cong \pi_1(S^1) \oplus \pi_1(T) \cong \mathbb{Z} \times F_{l_1}$ and $\pi_k(M) \cong \pi_k(W_f) \cong \pi_k(S^1) \oplus \pi_k(T) \cong \{0\}$ $(1 < k < m - 3)$.

Suppose $l_2 \geq 1$. By the extra assumption in (b) of the list, f is topologically quasi-trivial. Then W_f is represented as $A \bigcup_{\psi} B$, where A is a disjoint union of l_2 copies of D^n, B is the product of S^{n-1} and a connected graph T with l_1 loops and ψ is a homeomorphism from $S^{n-1} \times \Lambda \subset B$ onto ∂A, where Λ is a set consisting of l_2 degree 1 vertices of the previous graph. We have $\pi_1(M) \cong \pi_1(W_f) \cong \pi_1(T) \cong F_{l_1}$ by applying van Kampen's theorem. □

Example 5 We can construct a round fold map satisfying the assumption of Theorem 3 as in the following by applying the method in Subsection 3.1.

Let m and $n \geq 2$ be integers satisfying $2 \leq m-n+1 \leq 3$ or $m-n+1 \geq 6$. Let $\bar{M}$ be a compact C^∞ manifold homeomorphic to S^{m-n+1} with the interior of a disjoint union of a finite number of $(m-n+1)$-dimensional closed standard discs removed. For diffeomorphisms $\Phi : \partial\bar{M} \times S^{n-1} \to \partial\bar{M} \times \partial D^n$ and $\phi : S^{n-1} \to \partial D^n$ and the canonical projections $p_1 : \partial\bar{M} \times S^{n-1} \to S^{n-1}$ and $p_2 : \partial\bar{M} \times \partial D^n \to \partial D^n$, we assume that the following diagram commutes. There exists a good Morse function $\tilde{f} : \bar{M} \to [a, +\infty)$ such that regular fibers are disjoint unions of almost-spheres, where a is the minimal value (see Theorem 6.1 of [17] for example). Hence we can construct a closed C^∞ manifold M and a round fold map $f : M \to \mathbb{R}^n$ which satisfies the assumption of Theorem 3.

$$\begin{array}{ccc} \partial\bar{M} \times S^{n-1} & \xrightarrow{\Phi} & \partial\bar{M} \times \partial D^n \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ S^{n-1} & \xrightarrow{\phi} & \partial D^n \end{array}$$

The construction above still works even if we add extra singular points to $\tilde{f}$ by generating a cancelling pair of an $(m-n)$-handle and an $(m-n+1)$-handle. Note that regular fibers of $\tilde{f}$ are still disjoint unions of almost-spheres.

Remark 3 If $m-n+1=4,5$, then in Example 5, it is difficult to know whether we can obtain the Morse function $\tilde{f}$ on $\tilde{M}$ to perform the construction. We can indeed obtain the function $\tilde{f}$ if $\bar{M}$ is diffeomorphic to S^{m-n+1} with the interior of a disjoint union of a finite number of of $(m-n+1)$-dimensional closed standard discs removed (see [17]).

5. The homeomorphism and diffeomorphism types of manifolds admitting round fold maps

In this section, we study the homeomorphism and diffeomorphism types of the source manifolds of round fold maps satisfying the assumption of Theorem 3. Here we denote the h-cobordism group of k-dimensional C^∞ oriented homotopy spheres by Θ_k.

The following propositions are well-known.

Proposition 7 ([16], [17]) *If X is a closed and simply-connected manifold of dimension $k \neq 0,4$ which is the boundary of a contractible* PL *manifold of dimension $k+1$, then X is* PL *homeomorphic to S^k.*

Proposition 8 ([21]) *Let X be a closed C^∞ oriented manifold of dimension $2k$ having the same homotopy type as that of a connected sum of finite copies of $S^k \times S^k$. If $k \equiv 3,5,6,7 \bmod 8$, then X is diffeomorphic to a connected sum of finite copies of $S^k \times S^k$ and an oriented almost-sphere of dimension $2k$. If $k \equiv 3,5,6,7 \bmod 8$ and $\Theta_{2k} \cong \{0\}$ (e.g. $k=3,6$), then X is diffeomorphic to a connected sum of finite copies of $S^k \times S^k$.*

We have the following corollary to Theorem 3.

Corollary 5 *Let M be a closed and connected C^∞ manifold of dimension m. Suppose that there exists a round fold map $f : M \to \mathbb{R}^n$ $(m \geq n \geq 3)$ such that the followings hold.*

(1) *The indices of all the fold points are 0 or 1.*
(2) *Regular fibers are disjoint unions of almost-spheres.*
(3) $\pi_1(W_f) \cong \{0\}$.
(4) *The fiber of a point in a proper core is non-empty and connected.*

We also assume that $m-n \geq 2$. Then M is a homotopy sphere.

Proof. By Corollary 4, we have $\pi_1(M) \cong \pi_1(W_f) \cong \{0\}$. Furthermore, M is the boundary of a PL manifold simple homotopy equivalent to W_f. As mentioned in the proof of Theorem 3, W_f is represented as $D^n \bigcup_\psi P$, where P is the product of S^{n-1} and a connected graph without loops and ψ is a homeomorphism from $S^{n-1} \times \{*\} \subset P$ onto ∂D^n, where $*$ is a point in the previous graph, so W_f is contractible. Thus, by Proposition 7, M is a homotopy sphere. □

We also have the following theorem.

Theorem 4 *Let M be a closed and connected C^∞ oriented manifold of dimension m. Suppose that there exists a round fold map $f : M \to \mathbb{R}^n$ $(n \geq 2)$. Let $m = 2n$ and $n \equiv 3, 5, 6, 7 \bmod 8$. We also assume that the followings hold.*

(1) *The indices of all the fold points of f are 0 or 1.*
(2) *Regular fibers of f are always disjoint unions of almost-spheres.*
(3) $\pi_1(W_f) \cong \{0\}$.
(4) *The fiber of a point in a proper core of f is non-empty.*

Then M is $(n-1)$-connected. If M has the same homotopy type as that of a connected sum of finite copies of $S^n \times S^n$, then M is diffeomorphic to a connected sum of finite copies of $S^n \times S^n$ and an oriented almost-sphere of dimension m. If $\Theta_{2n} \cong \{0\}$ (e.g. $n = 3, 6$) and M has the same homotopy type as that of a connected sum of finite copies of $S^n \times S^n$, then M is diffeomorphic to a connected sum of finite copies of $S^n \times S^n$.

Proof. f satisfies the assumption of Theorem 3. By applying Theorem 3, we have $\pi_1(M) \cong \pi_1(W_f) \cong \{0\}$ and by (1) of Theorem 3, we have $\pi_k(M) \cong \pi_k(W_f) \cong \{0\}$ for $2 \leq k \leq n-1$, so M is $(n-1)$-connected. From Proposition 8, the result follows. □

Example 6 It is known that any closed and 2-connected manifold of dimension 6 has a C^∞ differentiable structure and that the resulting C^∞ manifold is always diffeomorphic to a connected sum of finite copies of $S^3 \times S^3$ ([22]). So in the situation of Theorem 4, if $m = 6$, then M is diffeomorphic to a connected sum of finite copies of $S^3 \times S^3$ without the assumption about the homotopy type.

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